INTEGRAL MEAN ESTIMATES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. Let \( P(z) \) be a polynomial of degree \( n \) having all zeros in \( |z| \leq k \) where \( k \leq 1 \), then it was proved by Dewan et al \[6\] that for every real or complex number \( \alpha \) with \( |\alpha| \geq k \) and each \( r \geq 0 \)

\[
n(|\alpha| - k) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + ke^{i\theta}|^r d\theta \right\}^{\frac{1}{r}} \text{Max}_|z|=1 |D_\alpha P(z)|.
\]

In this paper, we shall present a refinement and generalization of above result and also extend it to the class of polynomials \( P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu} \), \( 1 \leq \mu \leq n \), having all its zeros in \( |z| \leq k \) where \( k \leq 1 \) and thereby obtain certain generalizations of above and many other known results.

1. Introduction and statement of results

Let \( P(z) \) be a polynomial of degree \( n \). It was shown by Turán \[12\] that if \( P(z) \) has all its zeros in \( |z| \leq 1 \), then

\[
n \text{Max}_|z|=1 |P(z)| \leq 2 \text{Max}_|z|=1 |P'(z)|.
\]

Inequality (1.1) is best possible with equality holds for \( P(z) = \alpha z^n + \beta \) where \( |\alpha| = |\beta| \). The above inequality (1.1) of Turán \[12\] was generalized by Malik \[10\], who proved that if \( P(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \), where \( k \leq 1 \), then

\[
\text{Max}_|z|=1 |P'(z)| \geq \frac{n}{1 + k^{n-1}} \text{Max}_|z|=1 |P(z)|.
\]

where as for \( k \geq 1 \), Govil \[7\] showed that

\[
\text{Max}_|z|=1 |P'(z)| \geq \frac{n}{1 + k^n} \text{Max}_|z|=1 |P(z)|.
\]

Both the above inequalities (1.2) and (1.3) are best possible, with equality in (1.2) holding for \( P(z) = (z + k)^{n} \), where \( k \geq 1 \). While in (1.3) the equality holds for the polynomial \( P(z) = \alpha z^n + \beta k^n \) where \( |\alpha| = |\beta| \).

As a refinement of (1.2), Aziz and Shah \[4\] proved if \( P(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \), where \( k \leq 1 \), then

\[
\text{Max}_|z|=1 |P'(z)| \geq \frac{n}{1 + k} \left\{ \text{Max}_|z|=1 |P(z)| + \frac{1}{k^{n-1}} \text{Min}_|z|=1 |P(z)| \right\}.
\]

Let \( D_\alpha P(z) \) denotes the polar derivative of the polynomial \( P(z) \) of degree \( n \) with respect to the point \( \alpha \), then

\[
D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).
\]

The polynomial \( D_\alpha P(z) \) is a polynomial of degree at most \( n - 1 \) and it generalizes the ordinary derivative in the sense that

\[
\lim_{\alpha \to \infty} \left[ \frac{D_\alpha P(z)}{\alpha} \right] = P'(z).
\]
Aziz and Rather \cite{2} extends \cite{12} to polar derivatives of a polynomial and proved that if all the zeros of \( P(z) \) lie in \( |z| \leq k \) where \( k \leq 1 \) then for every real or complex number \( \alpha \) with \( |\alpha| \geq k \),

\[
\max_{|z|=1} |D_\alpha P(z)| \geq n \left( \frac{|\alpha| - k}{1 + k} \right) \max_{|z|=1} |P(z)|. \tag{1.5}
\]

For the class of polynomials \( P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}, 1 \leq \mu \leq n, \) of degree \( n \) having all its zeros in \( |z| \leq k \) where \( k \leq 1 \), Aziz and Rather \cite{3} proved that if \( \alpha \) is real or complex number with \( |\alpha| \geq k^2 \) then

\[
\max_{|z|=1} |D_\alpha P(z)| \geq n \left( \frac{|\alpha| - k^2}{1 + k^2} \right) \max_{|z|=1} |P(z)|. \tag{1.6}
\]

Malik \cite{10} obtained a generalization of \cite{11} in the sense that the left-hand side of \cite{11} is replaced by a factor involving the integral mean of \( |P(z)| \) on \( |z| = 1 \). In fact he proved that if \( P(z) \) has all its zeros in \( |z| \leq 1 \), then for each \( q > 0 \),

\[
n \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^q \, d\theta \right\}^{1/q} \leq \left\{ \int_{0}^{2\pi} |1 + e^{i\theta}|^q \, d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)|. \tag{1.7}
\]

If we let \( q \) tend to infinity in \cite{17}, we get \cite{11}.

The corresponding generalization of \cite{12} which is an extension of \cite{17} was obtained by Aziz \cite{1} by proving that if \( P(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \) where \( k \geq 1 \), then for each \( q \geq 1 \)

\[
n \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^q \, d\theta \right\}^{1/q} \leq \left\{ \int_{0}^{2\pi} |1 + ke^{i\theta}|^q \, d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)|. \tag{1.8}
\]

The result is best possible and equality in \cite{15} holds for the polynomial \( P(z) = \alpha z^n + \beta k^m \) where \( |\alpha| = |\beta| \).

As a generalization of inequality \cite{15} Dewan et al \cite{6} obtained an \( L^p \) inequality for the polar derivative of a polynomial and proved the following:

\textbf{Theorem 1.1.} If \( P(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \), where \( k \leq 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq k \) and for each \( r > 0 \),

\[
n(|\alpha| - k) \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^r \, d\theta \right\}^{1/r} \leq \left\{ \int_{0}^{2\pi} |1 + ke^{i\theta}|^r \, d\theta \right\}^{1/r} \max_{|z|=1} |D_\alpha P(z)|. \tag{1.9}
\]

In this paper, we consider the class of polynomials \( P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}, \) \( 1 \leq \mu \leq n, \) having all its zeros in \( |z| \leq k \) where \( k \leq 1 \) and establish some improvements and generalizations of inequalities \cite{11}, \cite{12}, \cite{15}, \cite{18} and \cite{19}.

In this direction, we first present the following interesting results which yields \cite{19} as a special case.

\textbf{Theorem 1.2.} If \( P(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \), where \( k \leq 1 \), then for every real or complex \( \alpha, \beta \) with \( |\alpha| \geq k, |\beta| \leq 1 \) and for each \( r > 0, p > 1, q > 1 \) with \( p^{-1} + q^{-1} = 1 \), we have

\[
n(|\alpha| - k) \left\{ \int_{0}^{2\pi} |P(e^{i\theta}) + \beta \frac{m}{k^{n-1}}|^r \, d\theta \right\}^{1/r} \leq \left\{ \int_{0}^{2\pi} |1 + ke^{i\theta}|^p \, d\theta \right\}^{1/p} \left\{ \int_{0}^{2\pi} |D_\alpha P(e^{i\theta})|^q \, d\theta \right\}^{1/q} \tag{1.10}
\]

where \( m = \min_{|z|=k} |P(z)|. \)
If we take $\beta = 0$, we get the following result.

**Corollary 1.3.** If $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex $\alpha$, with $|\alpha| \geq k$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have

$$n(|\alpha| - k) \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^{p} \, d\theta \right\}^{\frac{1}{p}} \leq \left\{ \int_{0}^{2\pi} |1 + ke^{i\theta}|^{pr} \, d\theta \right\}^{\frac{1}{p}} \left\{ \int_{0}^{2\pi} |D_{\alpha}P(e^{i\theta})|^{qr} \, d\theta \right\}^{\frac{1}{q}}. \quad (1.11)$$

**Remark 1.4.** Theorem 1.1 follows from (1.11) by letting $q \to \infty$ (so that $p \to 1$) in Corollary 1.3. If we divide both sides of inequality (1.11) by $|\alpha|$ and make $\alpha \to \infty$, we get (1.5).

Dividing the two sides of (1.10) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following result.

**Corollary 1.5.** If $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex $\beta$ with $|\beta| \leq 1$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have

$$n \left\{ \int_{0}^{2\pi} |P(e^{i\theta}) + \beta \frac{m}{k^{n-1}}|^{r} \, d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_{0}^{2\pi} |1 + ke^{i\theta}|^{pr} \, d\theta \right\}^{\frac{1}{p}} \left\{ \int_{0}^{2\pi} |P'(e^{i\theta})|^{qr} \, d\theta \right\}^{\frac{1}{q}} \quad (1.12)$$

where $m = \text{Min}_{|z|=k} |P(z)|$.

If we let $q \to \infty$ in (1.12), we get the following corollary.

**Corollary 1.6.** If $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex $\beta$ with $|\beta| \leq 1$ and for each $r > 0$, we have

$$n \left\{ \int_{0}^{2\pi} |P(e^{i\theta}) + \beta \frac{m}{k^{n-1}}|^{r} \, d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_{0}^{2\pi} |1 + ke^{i\theta}|^{pr} \, d\theta \right\}^{\frac{1}{p}} \text{Max}_{|z|=1} |P'(z)|, \quad (1.13)$$

where $m = \text{Min}_{|z|=k} |P(z)|$.

**Remark 1.7.** If we let $r \to \infty$ in (1.13) and choosing argument of $\beta$ suitably with $|\beta| = 1$, we obtain (1.4).

Next, we extend (1.3) to the class of polynomials $P(z) = a_{n}z^{n} + \sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}$, $1 \leq \mu \leq n$, having all its zeros in $|z| \leq k$, $k \leq 1$ and thereby obtain the following result.

**Theorem 1.8.** If $P(z) = a_{n}z^{n} + \sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having all its zeros in $|z| \leq k$ where $k \leq 1$, then for every real or complex $\alpha$ with $|\alpha| \geq k^{\mu}$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have

$$n(|\alpha| - k^{\mu}) \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^{r} \, d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_{0}^{2\pi} |1 + k^{\mu}e^{i\theta}|^{pr} \, d\theta \right\}^{\frac{1}{p}} \left\{ \int_{0}^{2\pi} |D_{\alpha}P(e^{i\theta})|^{qr} \, d\theta \right\}^{\frac{1}{q}} \quad (1.14)$$

**Remark 1.9.** We let $r \to \infty$ and $p \to \infty$ (so that $q \to 1$) in (1.14), we get inequality (1.6).

If we divide both sides of (1.14) by $|\alpha|$ and make $\alpha \to \infty$, we get the following result.

**Corollary 1.10.** If $P(z) = a_{n}z^{n} + \sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having all its zeros in $|z| \leq k$ where $k \leq 1$, then for for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have

$$n \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^{r} \, d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_{0}^{2\pi} |1 + k^{\mu}e^{i\theta}|^{pr} \, d\theta \right\}^{\frac{1}{p}} \left\{ \int_{0}^{2\pi} |P'(e^{i\theta})|^{qr} \, d\theta \right\}^{\frac{1}{q}} \quad (1.15)$$
Letting \( q \to \infty \) (so that \( p \to 1 \)) in (1.14), we get the following result:

**Corollary 1.11.** If \( P(z) = a_n z^n + \sum_{\nu=p}^{n} a_{n-\nu} z^{n-\nu}, 1 \leq \mu \leq n \), where \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \), where \( k \leq 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq k^{\mu} \) and for each \( r > 0 \),

\[
    n(|\alpha| - k^{\mu}) \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_{0}^{2\pi} \left| 1 + k^{\mu} e^{i\theta} \right|^r d\theta \right\}^{\frac{1}{r}} \left( \frac{1}{n} \right) \max_{|z|=1} |D_{\alpha} P(z)|. \tag{1.16}
\]

As a generalization of Theorem 1.8, we present the following result:

**Theorem 1.12.** If \( P(z) = a_n z^n + \sum_{\nu=p}^{n} a_{n-\nu} z^{n-\nu} \) where \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \) where \( k \leq 1 \), then for for each \( r > 0 \), \( p > 1 \), \( q > 1 \) with \( p^{-1} + q^{-1} = 1 \), we have

\[
    n(|\alpha| - k^{\mu}) \left\{ \int_{0}^{2\pi} |P(e^{i\theta}) + \beta m|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_{0}^{2\pi} \left| 1 + k^{\mu} e^{i\theta} \right|^r d\theta \right\}^{\frac{1}{r}} \left\{ \int_{0}^{2\pi} |D_{\alpha} P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}}. \tag{1.17}
\]

where \( m = \min_{|z|=k} |P(z)|. \)

If we divide both sides by \( |\alpha| \) and make \( \alpha \to \infty \), we get the following result:

**Corollary 1.13.** If \( P(z) = a_n z^n + \sum_{\nu=p}^{n} a_{n-\nu} z^{n-\nu} \) where \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \) where \( k \leq 1 \), then for each \( r > 0 \), \( p > 1 \), \( q > 1 \) with \( p^{-1} + q^{-1} = 1 \), we have

\[
    n \left\{ \int_{0}^{2\pi} |P(e^{i\theta}) + \beta m|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_{0}^{2\pi} \left| 1 + k^{\mu} e^{i\theta} \right|^r d\theta \right\}^{\frac{1}{r}} \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}} \tag{1.18}
\]

where \( m = \min_{|z|=k} |P(z)|. \)

Letting \( q \to \infty \) (so that \( p \to 1 \)) in (1.14), we get the following result:

**Corollary 1.14.** If \( P(z) = a_n z^n + \sum_{\nu=p}^{n} a_{n-\nu} z^{n-\nu} \) where \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \) where \( k \leq 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq k^{\mu} \) and for each \( r > 0 \),

\[
    n(|\alpha| - k^{\mu}) \left\{ \int_{0}^{2\pi} |P(e^{i\theta}) + \beta m|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_{0}^{2\pi} \left| 1 + k^{\mu} e^{i\theta} \right|^r d\theta \right\}^{\frac{1}{r}} \left( \frac{1}{n} \right) \max_{|z|=1} |D_{\alpha} P(z)|. \tag{1.19}
\]

where \( m = \min_{|z|=k} |P(z)|. \)

2. Lemmas

For the proofs of the theorems, we need the following Lemmas:

**Lemma 2.1.** If \( P(z) \) is a polynomial of degree almost \( n \) having all its zeros in \( |z| \leq k \) \( k \leq 1 \) then for \( |z| = 1 \),

\[
    |Q'(z)| + \frac{nm}{k^{n-1}} \leq k|P'(z)|, \tag{2.1}
\]

where \( Q(z) = z^n P'(1/z) \) and \( m = \min_{|z|=1} |P(z)|. \)

The above Lemma is due to Govil and McTume [8].
Proof of Theorem 1.2. Let $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu}z^{\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree $n$, which does not vanish for $|z| < k$, where $k \geq 1$ then for $|z| = 1$,

$$k^\mu |P(z)| \leq |Q'(z)|,$$

(2.2)

where $Q(z) = z^n P(1/z)$.

The above Lemma is due to Chan and Malik [5]. By applying Lemma 2.2 to the polynomial $z^n P(1/z)$, one can easily deduce:

Lemma 2.3. Let $P(z) = a_n z^n + \sum_{\nu=0}^{n} a_{n-\nu}z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree $n$, having all its zeros in $|z| \leq k$, where $k \leq 1$ then for $|z| = 1$

$$k^\mu |P'(z)| \geq |Q'(z)|,$$

(2.3)

where $Q(z) = z^n P(1/z)$.

3. Proof of Theorems

Proof of Theorem 1.2. Let $Q(z) = z^n P(1/z)$ then $P(z) = z^n Q(1/z)$ and it can be easily verified that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)| \text{ and } |P'(z)| = |nQ(z) - zQ'(z)|.$$  

(3.1)

By Lemma 2.1, we have for every $\beta$ with $|\beta| \leq 1$ and $|z| = 1$,  

$$\left| Q'(z) + \beta \frac{nmz^{n-1}}{k^{n-1}} \right| \leq |Q'(z)| + \frac{nm}{k^{n-1}} \leq k|P'(z)|.$$  

(3.2)

Using (3.1) in (3.2), for $|z| = 1$ we have

$$\left| Q'(z) + \beta \frac{nmz^{n-1}}{k^{n-1}} \right| \leq k|nQ(z) - zQ'(z)|.$$  

(3.3)

By Lemma 2.3 with $\mu = 1$, for every real or complex number $\alpha$ with $|\alpha| \geq k$ and $|z| = 1$, we have

$$|D_\alpha P(z)| \geq |\alpha||P'(z)| - |Q'(z)| \geq (|\alpha| - k)|P'(z)|.$$  

(3.4)

Since $P(z)$ has all its zeros in $|z| \leq k \leq 1$, it follows by Gauss-Lucas Theorem that all the zeros of $P'(z)$ also lie in $|z| \leq k \leq 1$. This implies that the polynomial

$$z^{n-1} P'(1/z) \equiv nQ(z) - zQ'(z)$$

does not vanish in $|z| < 1$. Therefore, it follows from (3.3) that the function

$$w(z) = \frac{z \left( Q'(z) + \beta \frac{nmz^{n-1}}{k^{n-1}} \right)}{k(nQ(z) - zQ'(z))}$$

is analytic for $|z| \leq 1$ and $|w(z)| \leq 1$ for $|z| = 1$. Furthermore, $w(0) = 0$. Thus the function $1 + kw(z)$ is subordinate to the function $1 + kz$ for $|z| \leq 1$. Hence by a well known property of subordination [9], we have

$$\int_0^{2\pi} \left| 1 + kw(e^{i\theta}) \right|^r \, d\theta \leq \int_0^{2\pi} \left| 1 + ke^{i\theta} \right|^r \, d\theta, \quad r > 0.$$  

(3.5)

Now

$$1 + kw(z) = \frac{n \left( Q(z) + \beta \frac{nmz^{n}}{k^{n-1}} \right)}{nQ(z) - zQ'(z)},$$

and

$$|P'(z)| = |z^{n-1} P'(1/z)| = |nQ(z) - zQ'(z)|,$$  

for $|z| = 1$, for $|z| = 1$.
therefore for \( |z| = 1 \),
\[
\left| n\frac{Q(z)}{1 + kw(z)} + \beta \frac{m z^n}{k^{n-1}} \right| = |1 + kw(z)||nQ(z) - zQ'(z)| = |1 + kw(z)||P'(z)|.
\]
equivalently,
\[
n \left| z^n \frac{P(1/z)}{1 + kw(z)} + \beta \frac{m z^n}{k^{n-1}} \right| = |1 + kw(z)||P'(z)|.
\]
This implies
\[
n \left| P(z) + \beta \frac{m z^n}{k^{n-1}} \right| = |1 + kw(z)||P'(z)| \quad \text{for} \quad |z| = 1.
\] (3.6)

From (3.4) and (3.6), we deduce that for \( r > 0 \),
\[
n^r (|\alpha| - k)^r \int_0^{2\pi} \left| P(e^{i\theta}) + \beta \frac{m z^n}{k^{n-1}} \right|^r \, d\theta \leq \left( \int_0^{2\pi} |1 + kw(e^{i\theta})|^{pr} \, d\theta \right)^{1/p} \left( \int_0^{2\pi} |D_{\alpha} P(e^{i\theta})|^{qr} \, d\theta \right)^{1/q},
\]
equivalently,
\[
n(|\alpha| - k^r) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) + \beta \frac{m z^n}{k^{n-1}} \right|^r \, d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + kw(e^{i\theta})|^{pr} \, d\theta \right\}^{\frac{1}{p}} \left\{ \int_0^{2\pi} |D_{\alpha} P(e^{i\theta})|^{qr} \, d\theta \right\}^{\frac{1}{q}}
\]
which proves the desired result. \( \square \)

**Proof of Theorem 1.8** Since \( P(z) \) has all its zeros in \( |z| \leq k \), therefore, by using Lemma 2.3 we have for \( |z| = 1 \),
\[
|Q'(z)| \leq k^\mu |nQ(z) - zQ'(z)|.
\] (3.7)

Now for every real or complex number \( \alpha \) with \( |\alpha| \geq k^\mu \), we have
\[
|D_{\alpha} P(z)| = |nP(z) + (\alpha - z)P'(z)|
\]
\[
\geq |\alpha||P'(z)| - |nP(z) - zP'(z)|,
\]
by using (3.1) and Lemma 2.3 for \( |z| = 1 \), we get
\[
|D_{\alpha} P(z)| \geq |\alpha||P'(z)| - |Q'(z)|
\]
\[
\geq (|\alpha| - k^\mu)|P'(z)|.
\] (3.8)

Since \( P(z) \) has all its zeros in \( |z| \leq k \leq 1 \), it follows by Gauss-Lucas Theorem that all the zeros of \( P'(z) \) also lie in \( |z| \leq k \leq 1 \). This implies that the polynomial
\[
z^{n-1}P(1/z) \equiv nQ(z) - zQ'(z)
\]
does not vanish in \( |z| < 1 \). Therefore, it follows from (3.7) that the function
\[
w(z) = \frac{zQ'(z)}{k^\mu (nQ(z) - zQ'(z))}
\]
is analytic for \( |z| \leq 1 \) and \( |w(z)| \leq 1 \) for \( |z| = 1 \). Furthermore, \( w(0) = 0 \). Thus the function
\[
1 + k^\mu w(z)
\]
is subordinate to the function \( 1 + k^\mu z \) for \( |z| \leq 1 \). Hence by a well known property of subordination [9], we have
\[
\left| \int_0^{2\pi} |1 + k^\mu w(e^{i\theta})|^{r} \, d\theta \right| \leq \left| \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{r} \, d\theta \right|, \quad r > 0.
\] (3.9)
Now
\[ 1 + k^\mu w(z) = \frac{nQ(z)}{nQ(z) - zQ'(z)}, \]
and
\[ |P'(z)| = |z^{n-1}P'(1/z)| = |nQ(z) - zQ'(z)|, \]
for \(|z| = 1\), therefore, for \(|z| = 1\),
\[ n|Q(z)| = |1 + k^\mu w(z)||nQ(z) - zQ'(z)| = |1 + k^\mu w(z)||P'(z)|. \tag{3.10} \]
From (3.8) and (3.10), we deduce that for \(r > 0\),
\[ n^r(|\alpha| - k^\mu)^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \leq \int_0^{2\pi} |1 + k^\mu w(e^{i\theta})|^r |D_{\alpha} P(e^{i\theta})|^r d\theta. \]
This gives with the help of Hölder’s inequality and (3.9), for \(p > 1, q > 1\) with \(p^{-1} + q^{-1} = 1\),
\[ n^r(|\alpha| - k^\mu)^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \leq \left( \int_0^{2\pi} |1 + k^\mu w(e^{i\theta})|^p |D_{\alpha} P(e^{i\theta})|^q d\theta \right)^{1/p}, \]
equivalently,
\[ n(|\alpha| - k^\mu) \left( \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right)^{\frac{1}{r}} \leq \left( \int_0^{2\pi} |1 + k^\mu w(e^{i\theta})|^p |D_{\alpha} P(e^{i\theta})|^q d\theta \right)^{\frac{1}{pq}}, \]
which proves the desired result.

**Proof of Theorem 1.12** Let \(m = \text{Min}_{|z|=\bar{k}}|P(z)|\), so that \(m \leq |P(z)|\) for \(|z| = k\). If \(P(z)\) has a zero on \(|z| = k\) then \(m = 0\) and result follows from Theorem 1.8. Henceforth we suppose that all the zeros of \(P(z)\) lie in \(|z| < k\). Therefore for every \(\beta\) with \(|\beta| < 1\), we have \(|m\beta| < |P(z)|\) for \(|z| = k\). Since \(P(z)\) has all its zeros in \(|z| < k\), it follows by Rouche’s theorem that all the zeros of \(F(z) = P(z) + \beta m\) lie in \(|z| < k\). If \(G(z) = z^nF(1/z) = Q(z) + \beta mz^n\), then by applying Lemma 2.3 to polynomial \(F(z) = P(z) + \beta m\), we have for \(|z| = 1\),
\[ |G'(z)| \leq k^\mu |F'(z)|. \]
This gives
\[ |Q'(z) + nm\beta z^{n-1}| \leq k^\mu |P'(z)|. \tag{3.11} \]
Using (3.1) in (3.11), for \(|z| = 1\) we have
\[ |Q'(z) + nm\beta z^{n-1}| \leq k^\mu |nQ(z) - zQ'(z)| \tag{3.12} \]
Since \(P(z)\) has all its zeros in \(|z| < k\), it follows by Gauss-Lucas Theorem that all the zeros of \(P'(z)\) also lie in \(|z| < k\). This implies that the polynomial
\[ z^{n-1}P'(1/z) \equiv nQ(z) - zQ'(z) \]
does not vanish in \(|z| < 1\). Therefore, it follows from (3.12) that the function
\[ w(z) = \frac{z(Q'(z) + nm\beta z^{n-1})}{k^\mu (nQ(z) - zQ'(z))} \]
is analytic for \(|z| \leq 1\) and \(|w(z)| \leq 1\) for \(|z| = 1\). Furthermore, \(w(0) = 0\). Thus the function \(1 + k^\mu w(z)\) is subordinate to the function \(1 + k^\mu z\) for \(|z| \leq 1\). Hence by a well known property of subordination [9], we have
\[ \int_0^{2\pi} |1 + k^\mu w(e^{i\theta})|^r d\theta \leq \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^r d\theta, \quad r > 0. \tag{3.13} \]
Now
\[ 1 + k^\mu w(z) = \frac{n(Q(z) + m\beta z^n)}{nQ(z) - zQ'(z)}, \]
and
\[ |P'(z)| = |z^{-1}P'(1/z)| = |nQ(z) - zQ'(z)|, \]
therefore, for \(|z| = 1,\)
\[ n(Q(z) + m\beta z^n) = |1 + k^\mu w(z)||nQ(z) - zQ'(z)| = |1 + k^\mu w(z)||P'(z)|. \]
This implies
\[ n|G(z)| = |1 + k^\mu w(z)||nQ(z) - zQ'(z)| = |1 + k^\mu w(z)||P'(z)|. \quad (3.14) \]
Since \(|F(z)| = |G(z)|\) for \(|z| = 1,\) therefore, from (3.14) we get
\[ n|P(z) + \beta m| = |1 + k^\mu w(z)||P'(z)| \quad \text{for} \quad |z| = 1. \quad (3.15) \]
From (3.8) and (3.15), we deduce that for \(r > 0,\)
\[ n^r (|r| - k^\mu)^r 2\pi 0 \int |P(e^{i\theta}) + \beta m|^r d\theta \leq 2\pi 0 \int |1 + k^\mu e^{i\theta}|^r |D_\alpha P(e^{i\theta})|^r d\theta. \]
This gives with the help of Hölder’s inequality in conjunction with (3.13) for \(p > 1, q > 1\) with \(p^{-1} + q^{-1} = 1,\)
\[ n^r (|r| - k^\mu)^r 2\pi 0 \int |P(e^{i\theta}) + \beta m|^r d\theta \leq \left( 2\pi 0 \int |1 + k^\mu e^{i\theta}|^p d\theta \right)^{1/p} \left( 2\pi 0 \int |D_\alpha P(e^{i\theta})|^q d\theta \right)^{1/q}, \]
equivalently,
\[ n(|r| - k^\mu) \left( 2\pi 0 \int |P(e^{i\theta}) + \beta m|^r d\theta \right)^{1/r} \leq \left( 2\pi 0 \int |1 + k^\mu e^{i\theta}|^p d\theta \right)^{1/p} \left( 2\pi 0 \int |D_\alpha P(e^{i\theta})|^q d\theta \right)^{1/q}, \]
which proves the desired result. \(\square\)

\textbf{References}