Logic Systems

Lattices, classical logic and quantum logic
Logic – Lattice structure

- A lattice is a set of elements $a$, $b$, $c$, ... that is closed for the connections $\cap$ and $\cup$. These connections obey:

  - The set is partially ordered. With each pair of elements $a$, $b$ belongs an element $c$, such that $a \subseteq c$ and $b \subseteq c$.
  - The set is a $\cap$ half lattice if with each pair of elements $a$, $b$ an element $c$ exists, such that $c = a \cap b$.
  - The set is a $\cup$ half lattice if with each pair of elements $a$, $b$ an element $c$ exists, such that $c = a \cup b$.
  - The set is a lattice if it is both a $\cap$ half lattice and a $\cup$ half lattice.
Partially ordered set

- The following relations hold in a lattice:

\[
\begin{align*}
    a \cap b &= b \cap a \\
    (a \cap b) \cap c &= a \cap (b \cap c) \\
    a \cap (a \cup b) &= a \\
    a \cup b &= b \cup a \\
    (a \cup b) \cup c &= a \cup (b \cup c) \\
    a \cup (a \cap b) &= a
\end{align*}
\]

- has a partial order inclusion \(\subset\):

\[
    a \subset b \iff a \subset b = a
\]

- A complementary lattice contains two elements \(n\) and \(e\) with each element \(a\) an complementary element \(a'\)

\[
\begin{align*}
    a \cap a' &= n \\
    a \cap n &= n \\
    a \cap e &= a \\
    a \cup a' &= e \\
    a \cup e &= e \\
    a \cup n &= a
\end{align*}
\]
Orthocomplemented lattice

Contains with each element $a$ an element $a''$ such that:

\[ a \cup a'' = e \]
\[ a \cap a'' = n \]
\[ (a'')'' = a \]
\[ a \subset b \iff b'' \subset a'' \]

**Distributive lattice**

\[ a \land (b \lor c) = (a \land b) \lor (a \land c) \]
\[ a \lor (b \land c) = (a \lor b) \land (a \lor c) \]

**Modular lattice**

\[ (a \land b) \lor (a \land c) = a \land (b \lor (a \land c)) \]

Classical logic is an orthocomplemented modular lattice
Weak modular lattice

- There exists an element $d$ such that

\[ a \subset c \iff (a \cup b) \cap c = a \cup (b \cap c) \cup (d \cap c) \]

- where $d$ obeys:

\[
\begin{align*}
(a \cup b) \cap d &= d \\
a \cap d &= n \\
b \cap d &= n \\
[(a \subset g) \text{ and } (b \subset g)] &\iff d \subset g
\end{align*}
\]
Atoms

- In an atomic lattice

\[ \exists p \in L \forall x \in L \{ x \subset p \Rightarrow x = n \} \]

\[ \forall a \in L \forall x \in L \{ (a < x < a \cap p) \Rightarrow (x = a \text{ or } x = a \cap p) \} \]

\( p \) is an atom
Logics

• Classical logic has the structure of an orthocomplemented distributive modular and atomic lattice.
• Quantum logic has the structure of an orthocomplemented weakly modular and atomic lattice.
• Also called orthomodular lattice.
Hilbert space

• The set of closed subspaces of an infinite dimensional separable Hilbert space forms an orthomodular lattice

• Is lattice isomorphic to quantum logic
Hilbert logic

- Add linear propositions
  - Linear combinations of atomic propositions
- Add relational coupling measure
  - Equivalent to inner product of Hilbert space
- Close subsets with respect to relational coupling measure

- Propositions $\iff$ subspaces
- Linear propositions $\iff$ Hilbert vectors
Superposition principle

Linear combinations of linear propositions are again linear propositions that belong to the same Hilbert logic system
Isomorphism

• Lattice isomorphic
  • Propositions $\iff$ closed subspaces

• Topological isomorphic
  • Linear atoms $\iff$ Hilbert base vectors
Navigate

To start of Hilbert Book slides:  
http://vixra.org/abs/1302.0125

To Hilbert Book slide, part 2:  
http://vixra.org/abs/1302.0121

To Hilbert Book Model slides, part 3  
http://vixra.org/abs/1309.0018

To Hilbert Book Model slides, part 4:  
http://vixra.org/abs/1309.0017

To “Physics of the Hilbert Book Model”  
http://vixra.org/abs/1307.0106