Octonion in Superstring Theory

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Abstract

In this paper, we have made an attempt to discuss the role of octonions in superstring theory (i.e. a theory of everything to describe the unification of all four types of forces namely gravitational, electromagnetic, weak and strong) where, we have described the octonion representation of the superstring (SS) theory as the combination of four complex ($\mathbb{C}$) spaces namely associated with the gravitational (G-space), electromagnetic (EM-space), weak (W-space) and strong (S-space) interactions. We have discussed the octonionic differential operator, octonionic valued potential wave equation, octonionic field equation and other various quantum equations of superstring theory in simpler, compact and consistent manner. Consequently, the generalized Dirac-Maxwell's equations are studied with the preview of superstring theory by means of octonions.

Key Words: octonions, wave equations, Maxwell's equations, superstring theory.

1 Introduction

Physics is a natural science that involves [1] the study of matter and its motion through space and time, along with related concepts such as energy and force. The search for unity and simplicity has been the theme of physics ever since Newton first showed that celestial and terrestrial mechanics could be unified . The 20^{th} century has been a time for tremendous format and change in our understanding new phenomena or adding new features to the existing theories [2]. Particle physics [1,2] is the branch of physics which deals with the study of matter, energy, space and time. Its objectives are to identify the most simple objects out of which all matter is composed and to understand the forces which cause them to interact and combine to make more complex things. Now a days particle physics is popularly known as high energy physics [1-3], which is the theory of basic structure of matter and its forces. The physics of elementary particles is currently described in terms of very successful theory called standard model [1,4]. It describes all known elementary particles and their interactions except gravitational interactions. The standard model accommodates the quarks and the leptons, which are constituents of matter, the vector particles that mediate the strong and electroweak forces and Higgs bosons, which is expected to account for the masses of particles. The standard model (SM) also describes [5] the unified picture of strong and electro-weak interactions within the framework of a $SU(3) \times SU(2) \times U(1)$ non-Abelian gauge theory.

Physics beyond the Standard Model [6,7] refers to the theoretical developments needed to explain the deficiencies of the Standard Model, such as the origin of mass, the strong CP problem, neutrino oscillations, matter–antimatter asymmetry, and the nature of dark matter and dark energy [6,8]. Another problem lies within the mathematical framework of the Standard Model itself. The Standard Model is inconsistent with that of general relativity to the point that one or both theories break down in their descriptions under certain conditions (for example within known space-time singularities like the Big Bang and black hole event horizons).

A Grand Unified Theory (GUT) [6,7], is a model in particle physics in which at high energy, the three gauge interactions of the Standard Model which define the electromagnetic, weak, and strong interactions, are merged into one single interaction characterized by one larger gauge symmetry and thus one unified coupling constant. In contrast, the experimentally verified Standard Model of particle physics is based on three independent interactions, symmetries and coupling constants. The standard model has three gauge symmetries [9]; the colour $SU(3)$, the weak isospin $SU(2)$, and the hypercharge $U(1)$ symmetry, corresponding to the three fundamental forces. Due to renormalization the coupling constants of each of these symmetries vary with the energy at which they are measured. Around 10^{16} GeV these couplings become approximately equal. This has led to speculation that above this energy the three gauge symmetries of the standard model are unified in one single gauge symmetry with a simple group gauge group, and just one coupling constant. Below this energy the symmetry is spontaneously broken to the standard model symmetries [10]. Unifying gravity with the other three interactions would provide a theory of everything (TOE), rather than a GUT. Nevertheless, GUTs are often seen as an intermediate step towards a TOE. The new particles predicted by models of grand unification cannot be observed directly at particle colliders because their masses are expected to be of the order of the so-called GUT scale, which is predicted to be just a few orders of magnitude below the Planck scale and thus far beyond the reach of currently foreseen collision experiments. Instead, effects of grand unification might be detected through indirect observations such as proton decay, electric dipole moments of elementary particles, or the properties of neutrinos [11]. Some grand unified theories predict the existence of magnetic monopoles. Popular choices for the unifying group are the special unitary group in five dimensions SU(5) and the special orthogonal group in ten dimensions SO(10) [12]. String theory [13] is one such reinvention, and many theoretical physicists think that such theories are the next theoretical step toward a true Theory of Everything. Theories of quantum gravity such as loop quantum gravity and others are thought by some to be promising candidates to the mathematical unification of quantum field theory and general relativity, requiring less drastic changes to existing theories [14]. However recent work places stringent limits on the putative effects of quantum gravity on the speed of light, and disfavors some current models of quantum gravity [15].

On the other hand, there has been a revival in the formulation of natural laws so that there exists [16] four-division algebras consisting the algebra of real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) and Octonions ($\mathcal{O}$). All four algebra's are alternative with totally anti symmetric associators. Quaternions [17, 18] were very first example of hyper complex numbers have been widely used [19-25] to the various applications of mathematics and physics. Since octonions [26] share with complex numbers and quaternions, many attractive mathematical properties, one might except that they would be equally as useful as others. Octonion [26] analysis has been widely discussed by Baez [27]. It has also played an important role in the context of various physical problems [28-31] of higher dimensional supersymmetry, super gravity and super strings etc. In recent years, it has also drawn interests of many [32-35] towards the developments of wave equation and octonion form of Maxwell's equations. Chanyal et al. [36-38] have also studied generalized octonion electrodynamics, generalized split octonion electrodynamics and octonion quantum chromodynamics and obtained the corresponding field equations (Maxwell's equations) and equation of motion in compact and simpler formulation.

2 Mathematical Preliminaries

An octonions (Cayley numbers) is the family of hypercomplex numbers and have eight dimensions. Thus, octonion $x \in \mathcal{O}$ is expressed [39,40] as

$$\begin{aligned} x = & e_0x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7 \\ = & e_0x_0 + \sum_{A=1}^{7} e_Ax_A \end{aligned} \tag{1}$$

where $e_A(A = 1, 2, ..., 7)$ are imaginary octonion units and e_0 is the multiplicative unit element. Set of octets $(e_0,\, e_1,\, e_2, e_3, e_4, e_5, e_6, e_7)$ are known as the octonion basis elements and satisfy the following multiplication rules

$$\begin{aligned} e_0 = 1;\; e_0e_A = e_Ae_0 = e_A; \\ e_Ae_B = -\delta_{AB}e_0 + f_{ABC}e_C.\; (A, B, C = 1, 2, ..., 7). \end{aligned} \tag{2}$$

The structure constants f_{ABC} is completely antisymmetric and takes the value 1 for following combinations,

$$f_{ABC} = +1; \forall (ABC) = (123), (471), (257), (165), (624), (543), (736). \tag{3}$$

It is to be noted that the summation convention is used for repeated indices. Here the octonion algebra $\mathcal{O}$ is described over the algebra of real numbers having the vector space of dimension 8. Octonion algebra is non associative and multiplication rules for its basis elements given by equations (2) and (3) are then generalized in the following table:

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_7	$-e_6$	e_5	$-e_4$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	$-e_5$	e_4	e_7	$-e_6$
e_4	$-e_7$	$-e_6$	e_5	-1	$-e_3$	e_2	e_1
e_5	e_6	$-e_7$	$-e_4$	e_3	-1	$-e_1$	e_2
e_6	$-e_5$	e_4	$-e_7$	$-e_2$	e_1	-1	e_3
e_7	e_4	e_5	e_6	$-e_1$	$-e_2$	$-e_3$	-1

Table 1: Octonion Multiplication table

As such we may write the following relations among octonion basis elements

$$\begin{aligned} [e_A,\ e_B] &= 2\, f_{ABC}\, e_C; \\ \{e_A,\ e_B\} &= -2\,\delta_{AB} e_0; \\ e_A(\, e_B\, e_C) &\neq (e_A\, e_B\,)\, e_C. \end{aligned} \tag{4}$$

Octonion conjugate is defined as

$$\begin{aligned} \overline{x} =& e_0x_0 - e_1x_1 - e_2x_2 - e_3x_3 - e_4x_4 - e_5x_5 - e_6x_6 - e_7x_7 \\ =& e_0x_0 - \sum_{A=1}^{7} e_A x_A \end{aligned} \tag{5}$$

where we have used the conjugates of basis elements as $\overline{e_0} = e_0$ and $\overline{e_A} = -e_A$. Hence an octonion can be decomposed in terms of its scalar $(Sc(x))$ and vector $(Vec(x))$ parts as

$$\begin{aligned} Sc(x) &= \frac{1}{2}(\, x\, +\, \overline{x}\,); \\ Vec(x) &= \frac{1}{2}(\, x\, -\, \overline{x}\,) = \sum_{A=1}^{7} e_A x_A. \end{aligned} \tag{6}$$

Conjugates of product of two octonions and its own are described as

$$\overline{(x\, y)} = \ \overline{y}\ \overline{x}; \qquad \overline{(\overline{x})} = x. \tag{7}$$

while the scalar product of two octonions is defined as

$$\langle x\, ,\, y\rangle \ = \tfrac{1}{2}(x\, \overline{y} + y\, \overline{x}) = \tfrac{1}{2}(\overline{x}\, y + \overline{y}\, x) = \ \sum_{\alpha=0}^{7} x_\alpha\, y_\alpha. \tag{8}$$

The norm $N(x)$ and inverse x^{-1} (for a nonzero x) of an octonion are respectively defined as

$$N(x) = x\,\overline{x} = \overline{x}\,x = \sum_{\alpha=0}^{7} x_{\alpha}^{2}.e_{0};$$
$$x^{-1} = \frac{\overline{x}}{N(x)} \Longrightarrow x\,x^{-1} = x^{-1}\,x = 1. \tag{9}$$

The norm $N(x)$ of an octonion x is zero if $x = 0$, and is always positive otherwise. It also satisfies the following property of normed algebra

$$N(x\,y) = \;\; N(x)\,N(y) = \;\; N(y)\,N(x). \tag{10}$$

Equation (4) shows that octonions are not associative in nature and thus do not form the group in their usual form. Non - associativity of octonion algebra $\mathcal{O}$ is provided by the associator $(x, y, z) = (xy)z - x(yz)\ \forall x, y, z \in \mathcal{O}$ defined for any three octonions. If the associator is totally antisymmetric for exchanges of any three variables, i.e. $(x, y, z) = -(z, y, x) = -(y, x, z) = -(x, z, y)$, then the algebra is called alternative.

3 Octonion Wave Equation

In order to consider the properties of octonions [41,42] and its eight dimensional connection, we may now write the octonion differential operator D [43,44] as

$$\begin{aligned} D =& e_0D_0 + e_1D_1 + e_2D_2 + e_3D_3 + e_4D_4 + e_5D_5 + e_6D_6 + e_7D_7 \\ =& \sum_{\mu=0}^{7} e_{\mu}D_{\mu}, \qquad (\mu = 0, 1, 2, 3, ..., 7) \end{aligned} \tag{11}$$

where D_{μ} are described as the components of a differential operator in an eight dimensional representation. Here we may consider the eight dimensional space as the combination of two (external and internal) four dimensional spaces. As such, a function of an octonion variable may be described as

$$\mathcal{F}(X) \;\; = \textstyle\sum_{\mu=0}^{7} \;\; e_{\mu}f_{\mu}(X) = f_0 + e_1f_1 + e_2f_2 + ... + e_7f_7, \tag{12}$$

where f_{μ} are scalar functions. Since octonions are neither commutative nor associative, one has to be very careful to multiply the octonion either from left or from right in terms of regularity conditions [45]. As such, a function $\mathcal{F}(X)$ of an octonion variable $X=\sum_{\mu=0}^{7}e_{\mu}X_{\mu}$ is left regular at X if and only if $\mathcal{F}(X)$ satisfies the condition

$$D\mathcal{F}(X) = 0. \tag{13}$$

Similarly, a function $G(X)$ is a right regular if and only if

$$G(X)D = 0, \tag{14}$$

where $G(X)=g_0+g_1e_1+g_2e_2+.....+g_7e_7$. Then we get

$$D\mathcal{F} = I = I_0+I_1e_1+I_2e_2+I_3e_3+I_4e_4+I_5e_5+I_6e_6+I_7e_7, \tag{15}$$

here

$$\begin{aligned}
I_0 &= \partial_0f_0-\partial_1f_1-\partial_2f_2-\partial_3f_3-\partial_4f_4-\partial_5f_5-\partial_6f_6-\partial_7f_7;\\
I_1 &= \partial_0f_1+\partial_1f_0+\partial_2f_3-\partial_3f_2+\partial_6f_5-\partial_5f_6-\partial_7f_4+\partial_4f_7;\\
I_2 &= \partial_0f_2+\partial_2f_0+\partial_3f_1-\partial_1f_3+\partial_4f_6-\partial_6f_4-\partial_7f_5+\partial_5f_7;\\
I_3 &= \partial_0f_3+\partial_3f_0+\partial_1f_2-\partial_2f_1+\partial_6f_7-\partial_7f_6+\partial_5f_4-\partial_4f_5;\\
I_4 &= \partial_0f_4+\partial_4f_0+\partial_3f_5-\partial_5f_3-\partial_2f_6+\partial_6f_2-\partial_1f_7+\partial_7f_1;\\
I_5 &= \partial_0f_5+\partial_5f_0+\partial_1f_6-\partial_6f_1+\partial_7f_2-\partial_2f_7-\partial_3f_4+\partial_4f_3;\\
I_6 &= \partial_0f_6+\partial_6f_0-\partial_1f_5+\partial_5f_1+\partial_2f_4-\partial_4f_2-\partial_3f_7+\partial_7f_3;\\
I_7 &= \partial_0f_7+\partial_7f_0+\partial_1f_4-\partial_4f_1+\partial_2f_5-\partial_5f_2-\partial_6f_3+\partial_3f_6.
\end{aligned} \tag{16}$$

The regularity condition (13) may now be considered as a homogeneous octonion wave equation for octonion variables without sources. On the other hand, equation (15) is considered as the inhomogeneous wave equation as

$$D\mathcal{F} = I. \tag{17}$$

where I is also an octonion. Similarly, we may also write the homogeneous as well as inhomogeneous octonion wave equations on using the right regularity condition (14). We may now interpret these octonion wave equations as the classical wave (field) equations of physical variables. Thus, one dimensional octonion representation is identical to eight dimensional spaces over the field of real numbers. It is isomorphic to four-dimensional space representation over the field of complex variables which is equivalent to two-dimensional space representation over quaternion field variables. Similarly, one dimensional quaternion space is isomorphic to four-dimensional space over the field of real numbers which is identical to two-dimensional space over the field of complex numbers.

4 Octonionic Superstring Theory

The octonionic representation of the super-string (SS) theory may be consider as the combination of four complex ($\mathbb{C}$) spaces namely associated with the gravitational (G-space), electromagnetic (EM-space), weak (W-space) and strong (S-space) interactions [46,47], i.e. unification of the four fundamental forces. So, we may write the octonionic superstring space as

$$\begin{aligned} \mathcal{O}_{SS} =& \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \\ &= (\mathcal{O}_g \, , \, \mathcal{O}_{em} \, , \, \mathcal{O}_w \, , \, \mathcal{O}_s) \Longrightarrow ((e_0, e_1) \, , \, (e_2, e_3) \, , \, (e_4, e_5) \, , \, (e_6, e_7)) \, , \end{aligned} \tag{18}$$

where $\mathcal{O}_g$, $\mathcal{O}_{em}$, $\mathcal{O}_w$, $\mathcal{O}_s$ are respectively known as gravitational, electromagnetic, weak and strong spaces in superstring theory related with the octonionic basis $(e_0, e_1), (e_2, e_3), (e_4, e_5), (e_6, e_7)$. Thus, the octonionic physical quantity $\mathbb{X} \in \mathcal{O}_{SS}$ is expressed as

$$\begin{aligned} \mathbb{X} =& X_g + X_{em} + X_w + X_s \\ =& (X_{g_0} e_0 + X_{g_1} e_1) + (X_{em_0} e_2 + X_{em_1} e_3) + (X_{w_0} e_4 + X_{w_1} e_5) + (X_{s_0} e_6 + X_{s_1} e_7). \end{aligned} \tag{19}$$

The octonionic differential operator in case of superstring theory (i.e. unification of four differential operator) may be written as

$$\begin{aligned} \boxplus_{SS} =& \boxdot_g + \boxdot_{em} + \boxdot_w + \boxdot_s \\ =& (\partial_{g_0} e_0 + \partial_{g_1} e_1) + (\partial_{em_0} e_2 + \partial_{em_1} e_3) + (\partial_{w_0} e_4 + \partial_{w_1} e_5) + (\partial_{s_0} e_6 + \partial_{s_1} e_7). \end{aligned} \tag{20}$$

Octonionic conjugate of equation (20) is described as

$$\overline{\boxplus}_{SS} = (\partial_{g_0} e_0 - \partial_{g_1} e_1) - (\partial_{em_0} e_2 + \partial_{em_1} e_3) - (\partial_{w_0} e_4 + \partial_{w_1} e_5) - (\partial_{s_0} e_6 + \partial_{s_1} e_7). \tag{21}$$

Thus, the octonionic superstring valued potential with the combination of four potentials may be expressed as

$$\begin{aligned}\mathbb{V}_{SS}=(V_g\,,\,V_{em}\,,\,V_w\,,\,V_s) &= ((V_0,V_1)\,,\,(V_2,V_3)\,,\,(V_4,V_5)\,,\,(V_6,V_7))\\ &= ((V_{g_0},V_{g_1})\,,\,(V_{em_0},V_{em_1})\,,\,(V_{w_0},V_{w_1})\,,\,(V_{s_0},V_{s_1}))\,,\end{aligned} \tag{22}$$

which is further simplified to

$$\mathbb{V}_{SS}=(V_{g_0}e_0+V_{g_1}e_1)+(V_{em_0}e_2+V_{em_1}e_3)+(V_{w_0}e_4+V_{w_1}e_5)+(V_{s_0}e_6+V_{s_1}e_7). \tag{23}$$

In order to obtain the octonionic potential wave equations of the superstring space, let us operate $\overline{\boxplus}_{SS}$ given by equation (21) to octonion superstring valued potential $\mathbb{V}_{SS}$ of equation (23) and we get

$$\begin{aligned}\overline{\boxplus}_{SS}\,\mathbb{V}_{SS} =& e_0\{\partial_{g_0}V_{g_0}+\partial_{g_1}V_{g_1}+\partial_{em_0}V_{em_0}+\partial_{em_1}V_{em_1}+\partial_{w_0}V_{w_0}+\partial_{w_1}V_{w_1}+\partial_{s_0}V_{s_0}+\partial_{s_1}V_{s_1}\}\\ &+e_1\{\partial_{g_0}V_{g_1}-\partial_{g_1}V_{g_0}-\partial_{em_0}V_{em_1}+\partial_{em_1}V_{em_0}-\partial_{w_0}V_{s_1}+\partial_{w_1}V_{s_0}-\partial_{s_0}V_{w_1}+\partial_{s_1}V_{w_0}\}\\ &+e_2\{\partial_{g_0}V_{em_0}+\partial_{g_1}V_{em_1}-\partial_{em_0}V_{g_0}-\partial_{em_1}V_{g_1}-\partial_{w_0}V_{s_0}-\partial_{w_1}V_{s_1}+\partial_{s_0}V_{w_0}+\partial_{s_1}V_{w_1}\}\\ &+e_3\{\partial_{g_0}V_{em_1}-\partial_{g_1}V_{em_0}+\partial_{em_0}V_{g_1}-\partial_{em_1}V_{g_0}+\partial_{w_0}V_{w_1}-\partial_{w_1}V_{w_0}-\partial_{s_0}V_{s_1}+\partial_{s_1}V_{s_0}\}\\ &+e_4\{\partial_{g_0}V_{w_0}+\partial_{g_1}V_{s_1}+\partial_{em_0}V_{s_0}-\partial_{em_1}V_{w_1}-\partial_{w_0}V_{g_0}+\partial_{w_1}V_{em_1}-\partial_{s_0}V_{em_0}-\partial_{s_1}V_{g_1}\}\\ &+e_5\{\partial_{g_0}V_{w_1}-\partial_{g_1}V_{s_0}+\partial_{em_0}V_{s_1}+\partial_{em_1}V_{w_0}-\partial_{w_0}V_{em_1}-\partial_{w_1}V_{g_0}+\partial_{s_0}V_{g_1}-\partial_{s_1}V_{em_0}\}\\ &+e_6\{\partial_{g_0}V_{s_0}+\partial_{g_1}V_{w_1}-\partial_{em_0}V_{w_0}+\partial_{em_1}V_{s_1}+\partial_{w_0}V_{em_0}-\partial_{w_1}V_{g_1}-\partial_{s_0}V_{g_0}-\partial_{s_1}V_{em_1}\}\\ &+e_7\{\partial_{g_0}V_{s_1}-\partial_{g_1}V_{w_0}-\partial_{em_0}V_{w_1}-\partial_{em_1}V_{s_0}+\partial_{w_0}V_{g_1}+\partial_{w_1}V_{em_0}+\partial_{s_0}V_{em_1}-\partial_{s_1}V_{g_0}\},\end{aligned} \tag{24}$$

which provides the following octonionic analogous of superstring theory as

$$\overline{\boxplus}_{SS}\,\mathbb{V}_{SS} \;=\; \mathbb{F}_{SS}=((F_0,F_1),\,(F_2,F_3),\,(F_4,F_5),\,(F_6,F_7))\,; \tag{25}$$

where $\mathbb{F}_{SS}(F_0,F_1,F_2,F_3,F_4,F_5,F_6,F_7)$ is an octonion which reproduces the superstring field strengths. So, equation (25) may further be expressed as

$$\begin{aligned}\mathbb{F}_{SS}=F_g+F_{em}+F_w+F_s &= ((F_{g_0},F_{g_1})\,,\,(F_{em_0},F_{em_1})\,,\,(F_{w_0},F_{w_1})\,,\,(F_{s_0},F_{s_1}))\\ &=(F_{g_0}e_0+F_{g_1}e_1)+(F_{em_0}e_2+F_{em_1}e_3)\\ &\quad+(F_{w_0}e_4+F_{w_1}e_5)+(F_{s_0}e_6+F_{s_1}e_7),\end{aligned} \tag{26}$$

where the first term ($F_g=F_{g_0},F_{g_1}$) is defined as the gravitational field strength in G-space, the second term ($F_{em}=F_{em_0},F_{em_1}$) is described as the electromagnetic field strength in EM-space, the third term ($F_w=F_{w_0},F_{w_1}$) provides the weak interaction field strength in W-space and the forth term ($F_s=F_{s_0},F_{s_1}$) is responsible for the strong field strength in S-space. Thus, the component of $\mathbb{F}_{SS}\{(F_{g_0}, F_{g_1}), (F_{em_0}, F_{em_1}), (F_{w_0}, F_{w_1}), (F_{s_0}, F_{s_1})\}$ are expressed as the following octonionic representation

$$
\begin{aligned}
F_{g_0}e_0 &= \{\partial_{g_0}V_{g_0}e_0 - \partial_{g_1}V_{g_0}e_1 - \partial_{em_0}V_{g_0}e_2 - \partial_{em_1}V_{g_0}e_3 - \partial_{w_0}V_{g_0}e_4 - \partial_{w_1}V_{g_0}e_5 - \partial_{s_0}V_{g_0}e_6 - \partial_{s_1}V_{g_0}e_7\} \\
F_{g_1}e_1 &= \{\partial_{g_1}V_{g_1}e_0 + \partial_{g_0}V_{g_1}e_1 - \partial_{em_1}V_{g_1}e_2 + \partial_{em_0}V_{g_1}e_3 - \partial_{s_1}V_{g_1}e_4 + \partial_{s_0}V_{g_1}e_5 - \partial_{w_1}V_{g_1}e_6 + \partial_{w_0}V_{g_1}e_7\} \\
F_{em_0}e_2 &= \{\partial_{em_0}V_{em_0}e_0 + \partial_{em_1}V_{em_0}e_1 + \partial_{g_0}V_{em_0}e_2 - \partial_{g_1}V_{em_0}e_3 - \partial_{s_0}V_{em_0}e_4 - \partial_{s_1}V_{em_0}e_5 + \partial_{w_0}V_{em_0}e_6 + \partial_{w_1}V_{em_0}e_7\} \\
F_{em_1}e_3 &= \{\partial_{em_1}V_{em_1}e_0 - \partial_{em_0}V_{em_1}e_1 + \partial_{g_1}V_{em_1}e_2 + \partial_{g_0}V_{em_1}e_3 + \partial_{w_1}V_{em_1}e_4 - \partial_{w_0}V_{em_1}e_5 - \partial_{s_1}V_{em_1}e_6 + \partial_{s_0}V_{em_1}e_7\} \\
F_{w_0}e_4 &= \{\partial_{w_0}V_{w_0}e_0 + \partial_{s_1}V_{w_0}e_1 + \partial_{s_0}V_{w_0}e_2 - \partial_{w_1}V_{w_0}e_3 + \partial_{g_0}V_{w_0}e_4 + \partial_{em_1}V_{w_0}e_5 - \partial_{em_0}V_{w_0}e_6 - \partial_{g_1}V_{w_0}e_7\} \\
F_{w_1}e_5 &= \{\partial_{w_1}V_{w_1}e_0 - \partial_{s_0}V_{w_1}e_1 + \partial_{s_1}V_{w_1}e_2 + \partial_{w_0}V_{w_1}e_3 - \partial_{em_1}V_{w_1}e_4 + \partial_{g_0}V_{w_1}e_5 + \partial_{g_1}V_{w_1}e_6 - \partial_{em_0}V_{w_1}e_7\} \\
F_{s_0}e_6 &= \{\partial_{s_0}V_{s_0}e_0 + \partial_{w_1}V_{s_0}e_1 - \partial_{w_0}V_{s_0}e_2 + \partial_{s_1}V_{s_0}e_3 + \partial_{em_0}V_{s_0}e_4 - \partial_{g_1}V_{s_0}e_5 + \partial_{g_0}V_{s_0}e_6 - \partial_{em_1}V_{s_0}e_7\} \\
F_{s_1}e_7 &= \{\partial_{s_1}V_{s_1}e_0 - \partial_{w_0}V_{s_1}e_1 - \partial_{w_1}V_{s_1}e_2 - \partial_{s_0}V_{s_1}e_3 + \partial_{g_1}V_{s_1}e_4 + \partial_{em_0}V_{s_1}e_5 + \partial_{em_1}V_{s_1}e_6 + \partial_{g_0}V_{s_1}e_7\}.
\end{aligned}
\tag{27}
$$

So, in order to obtain the octonionic superstring field equations, we apply the differential operator (20) to equation (26) as

$$
\begin{aligned}
\boxplus_{SS}\mathbb{F}_{SS} = \\
& -e_0\{\partial_{g_0}F_{g_0} + \partial_{g_1}F_{g_1} + \partial_{em_0}F_{em_0} + \partial_{em_1}F_{em_1} + \partial_{w_0}F_{w_0} + \partial_{w_1}F_{w_1} + \partial_{s_0}F_{s_0} + \partial_{s_1}F_{s_1}\} \\
& +e_1\{\partial_{g_0}F_{g_1} + \partial_{g_1}F_{g_0} + \partial_{em_0}F_{em_1} - \partial_{em_1}F_{em_0} + \partial_{w_0}F_{s_1} - \partial_{w_1}F_{s_0} + \partial_{s_0}F_{w_1} - \partial_{s_1}F_{w_0}\} \\
& +e_2\{\partial_{g_0}F_{em_0} - \partial_{g_1}F_{em_1} + \partial_{em_0}F_{g_0} + \partial_{em_1}F_{g_1} + \partial_{w_0}F_{s_0} + \partial_{w_1}F_{s_1} - \partial_{s_0}F_{w_0} - \partial_{s_1}F_{w_1}\} \\
& +e_3\{\partial_{g_0}F_{em_1} + \partial_{g_1}F_{em_0} - \partial_{em_0}F_{g_1} + \partial_{em_1}F_{g_0} - \partial_{w_0}F_{w_1} + \partial_{w_1}F_{w_0} + \partial_{s_0}F_{s_1} - \partial_{s_1}F_{s_0}\} \\
& +e_4\{\partial_{g_0}F_{w_0} - \partial_{g_1}F_{s_1} - \partial_{em_0}F_{s_0} + \partial_{em_1}F_{w_1} + \partial_{w_0}F_{g_0} - \partial_{w_1}F_{em_1} + \partial_{s_0}F_{em_0} + \partial_{s_1}F_{g_1}\} \\
& +e_5\{\partial_{g_0}F_{w_1} + \partial_{g_1}F_{s_0} - \partial_{em_0}F_{s_1} - \partial_{em_1}F_{w_0} + \partial_{w_0}F_{em_1} + \partial_{w_1}F_{g_0} - \partial_{s_0}F_{g_1} + \partial_{s_1}F_{em_0}\} \\
& +e_6\{\partial_{g_0}F_{s_0} - \partial_{g_1}F_{w_1} + \partial_{em_0}F_{w_0} - \partial_{em_1}F_{s_1} - \partial_{w_0}F_{em_0} + \partial_{w_1}F_{g_1} + \partial_{s_0}F_{g_0} + \partial_{s_1}F_{em_1}\} \\
& +e_7\{\partial_{g_0}F_{s_1} + \partial_{g_1}F_{w_0} + \partial_{em_0}F_{w_1} + \partial_{em_1}F_{s_0} - \partial_{w_0}F_{g_1} - \partial_{w_1}F_{em_0} - \partial_{s_0}F_{em_1} + \partial_{s_1}F_{g_0}\},
\end{aligned}
\tag{28}
$$

which can further be reduced and be written in following compact notation in terms of an octonionic superstring representation as

$$
\boxplus_{SS}\mathbb{F}_{SS} = \mathbb{J}_{SS} = \left(\left(J_0, J_1\right), \left(J_2, J_3\right), \left(J_4, J_5\right), \left(J_6, J_7\right)\right). \tag{29}
$$

Here $\mathbb{J}_{SS}(J_0, J_1, J_2, J_3, J_4, J_5, J_6, J_7)$ is an octonionic superstring current source, which may be expressed in the following matrix form,

$$\begin{pmatrix} J_{(g-g)} & J_{(g-g)} & J_{(g-em)} & J_{(g-em)} & J_{(g-w-s)} & J_{(g-w-s)} & J_{(g-s-w)} & J_{(g-s-w)} \\ J_{(em-em)} & J_{(em-em)} & J_{(em-g)} & J_{(em-g)} & J_{(em-s-w)} & J_{(em-s-w)} & J_{(em-w-s)} & J_{(em-w-s)} \\ J_{(w-w)} & J_{(w-s)} & J_{(w-s)} & J_{(w-w)} & J_{(w-g-em)} & J_{(w-em-g)} & J_{(w-em-g)} & J_{(w-g-em)} \\ J_{(s-s)} & J_{(s-w)} & J_{(s-s)} & J_{(s-w)} & J_{(s-em-g)} & J_{(s-g-em)} & J_{(s-g-em)} & J_{(s-em-g)} \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{pmatrix}, \Longrightarrow \mathbb{J}_{SS}. \tag{30}$$

The generalized Dirac-Maxwell's (GDM) is visualized in superstring theory by means of octonions and the generalized current has been discussed in equation (29). The generalized superstring current source described the various terms where $J_{(g-g)}$, $J_{(em-em)}, J_{(em-g)}, J_{(g-em)}$ defines as the octonionic representation of gravitational - gravitational, electromagnetic - electromagnetic, electromagnetic - gravitational, gravitational - electromagnetic current source and $J_{(w-w)}, J_{(s-s)}, J_{(s-w)}, J_{(w-s)}$ sectors of the octonionic current source respectively associated the weak-weak, strong-strong, strong-weak, weak strong interactions. Consequently, the remaining terms of equation (30) denotes as the parts of octonionic current sources for the combination of different three interactions. Thus, the present formulation has been provides the superstring theory in the terms of octonionic representations.

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