Exact solution of Helmholtz equation for the case of non-paraxial Gaussian beams.

Sergey V. Ershkov
Institute for Time Nature Explorations,
M.V. Lomonosov's Moscow State University,
Leninskie gory, 1-12, Moscow 119991, Russia.

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A new type of exact solutions of the full 3 dimensional *spatial* Helmholtz equation for the case of non-paraxial Gaussian beams is presented here.
We consider appropriate representation of the solution for Gaussian beams *in a spherical coordinate system* by substituting it to the full 3 dimensional spatial Helmholtz Equation.
Analyzing the structure of the final equation, we obtain that governing equations for the components of our solution are represented by the proper *Riccati* equations of complex value, which has no analytical solution in general case.
But we find one of the possible exact solution which is proved to satisfy to such an equations for Gaussian beams.

1. Introduction.

The full 3-dimensional *spatial* Helmholtz equation provides solutions that describe the propagation of waves over space (*e.g.*, *electromagnetic waves*); it should be presented in a spherical coordinate system R, θ, φ as below [1-2]:

$$\Delta A + k^2 A = 0\,, \qquad (1.1)$$

- where Δ - is the Laplacian, k is the wavenumber, and A is the amplitude.

Besides, in spherical coordinate system [3]:

$$\Delta A = \frac{\partial^2 A}{\partial R^2} + \frac{2}{R}\frac{\partial A}{\partial R} + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 A}{\partial \varphi^2} + \frac{1}{R^2}\frac{\partial^2 A}{\partial \theta^2} + \frac{1}{R^2} ctg\,\theta \frac{\partial A}{\partial \theta} \quad .$$

Let us search for solutions of Eq. (1.1) in a *classical* form of Gaussian beams [4-8], which could be presented in Cartesian coordinate system as below [9]:

$$A = a \cdot \frac{w_0}{w(z)} \exp\left[-\frac{x^2+y^2}{w^2(z)} - ikz - ik\frac{x^2+y^2}{2R(z)} + i\zeta(z) \right]$$

- where $w(z)$, $R(z)$, $\zeta(z)$ – are some functions, describing the appropriate parameters of a beam; the last expression could be also represented as below

$$\exp\left[i\left(\zeta(z) - kz + i\cdot \ln w(z) + \left(\frac{i}{w^2(z)} - \frac{k}{2R(z)} \right)\cdot (x^2+y^2) \right) \right] = \exp\left[i\left(p(z) + \frac{x^2+y^2}{2q(z)} \right) \right]$$

- here p(z) is the complex phase-shift of the waves during their propagation along the z axis; q(z) is the proper complex parameter of a beam, which is determining the Gaussian profile of a wave in the transverse plane at position z.

The last expression could be transformed to the form below in a *spherical* coordinate system:

$$A = a \cdot \exp\left[i \left(p(R,\theta) + \frac{R^2 \cdot \sin^2\theta}{2\,q(R,\theta)} \right) \right] \qquad (*)$$

Then having substituted the expression (*) into Eq. (1.1), we should obtain ($\theta \neq 0$):

$$\frac{\partial^2 p(R,\theta)}{\partial R^2} + \frac{\partial^2 \left(\frac{R^2}{q(R,\theta)} \right)}{\partial R^2} \frac{\sin^2\theta}{2} + i \cdot \left(\frac{\partial p(R,\theta)}{\partial R} + \frac{\partial \left(\frac{R^2}{q(R,\theta)} \right)}{\partial R} \frac{\sin^2\theta}{2} \right)^2 + \frac{2}{R} \cdot \left(\frac{\partial p(R,\theta)}{\partial R} + \frac{\partial \left(\frac{R^2}{q(R,\theta)} \right)}{\partial R} \frac{\sin^2\theta}{2} \right) +$$

$$+ \frac{1}{R^2} \cdot \frac{\partial^2 p(R,\theta)}{\partial \theta^2} + \frac{1}{2} \frac{\partial^2 \left(\frac{\sin^2\theta}{q(R,\theta)} \right)}{\partial \theta^2} + \frac{i}{R^2} \cdot \left(\frac{\partial p(R,\theta)}{\partial \theta} + \frac{\partial \left(\frac{\sin^2\theta}{q(R,\theta)} \right)}{\partial \theta} \frac{R^2}{2} \right)^2 + \frac{ctg\,\theta}{R^2} \cdot \left(\frac{\partial p(R,\theta)}{\partial \theta} + \frac{\partial \left(\frac{\sin^2\theta}{q(R,\theta)} \right)}{\partial \theta} \frac{R^2}{2} \right) = \qquad (1.2)$$

$$= i \cdot k^2 \, .$$

2. Exact solutions.

Let us re-designate appropriate term in (*) as below:

$$f\,(R,\theta) = p(R,\theta) + \frac{R^2 \cdot \sin^2\theta}{2\,q(R,\theta)} \quad .$$

In such a case, Eq. (1.2) could be transformed as below ($\theta \neq 0$):

$$\frac{\partial^2 f(R,\theta)}{\partial R^2} + i\cdot\left(\frac{\partial f(R,\theta)}{\partial R}\right)^2 + \frac{2}{R}\cdot\left(\frac{\partial f(R,\theta)}{\partial R}\right) +$$

$$+\frac{1}{R^2}\cdot\left(\frac{\partial^2 f(R,\theta)}{\partial \theta^2} + i\cdot\left(\frac{\partial f(R,\theta)}{\partial \theta}\right)^2 + ctg\,\theta\cdot\left(\frac{\partial f(R,\theta)}{\partial \theta}\right)\right) - i\cdot k^2 = 0 \tag{2.1}$$

Thus, all possible solutions for representing of Gaussian beams in a form (*) are described by the Equation (2.1).

Let us assume as below:

$$\frac{\partial^2 f(R,\theta)}{\partial \theta^2} + i\cdot\left(\frac{\partial f(R,\theta)}{\partial \theta}\right)^2 + ctg\,\theta\cdot\left(\frac{\partial f(R,\theta)}{\partial \theta}\right) = C \tag{2.2}$$

- here C – is a constant of *complex* value. For such a case, Eq. (2.1) could be reduced as below ($\theta \neq 0$):

$$\frac{\partial^2 f(R,\theta)}{\partial R^2} + i\cdot\left(\frac{\partial f(R,\theta)}{\partial R}\right)^2 + \frac{2}{R}\cdot\left(\frac{\partial f(R,\theta)}{\partial R}\right) + \frac{C}{R^2} - i\cdot k^2 = 0 \tag{2.3}$$

Besides, one of the obvious solutions of PDE-equations (2.2)-(2.3):

$$f(R,\ \theta) = f_1(R) + f_2(\theta) \tag{**}$$

- where $f_1(R)$, $f_2(\theta)$ – are the functions of *complex* value.

3. Presentation of exact solution.

In such a case, Eq. (2.2) could be represented as below:

$$\left(\frac{d\, f_2}{d\,\theta}\right) = y(\theta) \quad \Rightarrow \quad y'(\theta) = -\, i\cdot y^2 - ctg\,\theta\cdot y + C\,, \tag{3.1}$$

$$y(\theta) = \sin^{-1}\theta\cdot u(\theta) \quad \Rightarrow \quad u'(\theta) = -\,(i\cdot\sin^{-1}\theta)\cdot u^2 + C\cdot\sin\theta\,,$$

- where the last equation is known to be the *Riccati* ODE [3], which has no solution in general case. But if $C = 0$, Eq. (3.1) has a proper solution (C_0 = const):

$$u'(\theta) = -\,(i\cdot\sin^{-1}\theta)\cdot u^2\,, \qquad u(\theta) = \frac{1}{\left(C_0 + i\cdot\int\sin^{-1}\theta\, d\,\theta\right)} \quad \Rightarrow \tag{3.2}$$

$$\frac{d\, f_2}{d\,\theta} = \frac{\sin^{-1}\theta}{\left(C_0 + i\cdot\int\sin^{-1}\theta\, d\,\theta\right)} \quad (C_0 = 0) \;\Rightarrow\; f_2 = -i\cdot\ln\left(\int\sin^{-1}\theta\, d\,\theta\right)$$

Besides, Eq. (2.3) could be presented as below ($C = 0$):

$$\left(\frac{d\, f_1}{d\, R}\right) = y_1(R) \quad \Rightarrow \quad y_1{}'(R) = -\, i\cdot y_1^2 - \frac{2}{R}\, y_1 - \left(\frac{C}{R^2} - i\cdot k^2\right), \tag{3.3}$$

$$f_1(R) = \int y_1(R)\, d\, R\;.$$

- where the last *Riccati* ODE (3.3) has a proper solution below in case of $C = 0$ (see [3], the case 1.104).

Indeed, let us assume ($k \neq 0$; R_0 = const):

$$y_1 = u_1 + \frac{i}{R}, \quad y_1'(R) = -i \cdot y_1^2 - \frac{2}{R} y_1 + i \cdot k^2 \;\Rightarrow$$

$$\Rightarrow u_1'(R) = -i \cdot u_1^2 + i \cdot k^2 \;\Rightarrow\; \int \frac{d\,u_1}{k^2 - u_1^2} = i \cdot (R + R_0)$$

$$\Rightarrow \begin{cases} u_1 = k \cdot th\left(i \cdot k \cdot (R + R_0)\right), \; \left| i \cdot tg\,(k \cdot (R + R_0)) \right| < 1, \\ u_1 = k \cdot cth\left(i \cdot k \cdot (R + R_0)\right), \; \left| i \cdot tg\,(k \cdot (R + R_0)) \right| > 1, \end{cases}$$

- then, we obtain (R_0 = 0):

$$\begin{cases} f_1 = -i \cdot \ln\, ch\left(i \cdot k \cdot R\right) + i \cdot \ln R, \; \left| k \cdot R \right| < \pi/4\,, \\ f_1 = -i \cdot \ln\, sh\left(i \cdot k \cdot R\right) + i \cdot \ln R, \; \left| k \cdot R \right| > \pi/4\,. \end{cases} \tag{3.4}$$

Taking into consideration the expression (**) for the solution as well as (3.2)-(3.4), let us finally present a new type of *non-paraxial* Gaussian beams, which is proved to satisfy to the Helmholtz equation (1.1), as below:

$$\begin{cases} A = a \cdot \left(\int \sin^{-1}\theta\, d\,\theta \right) \cdot \dfrac{ch\left(i \cdot k \cdot R\right)}{R}, \; \left| k \cdot R \right| < \pi/4, \\ A = a \cdot \left(\int \sin^{-1}\theta\, d\,\theta \right) \cdot \dfrac{sh\left(i \cdot k \cdot R\right)}{R}, \; \left| k \cdot R \right| > \pi/4\,, \end{cases}$$

- or

$$\begin{cases} A = a \cdot \ln\left(tg\dfrac{\theta}{2}\right) \cdot \dfrac{\cos(k \cdot R)}{R}, \quad |k \cdot R| < \pi/4 \ , \\[2ex] A = a \cdot \ln\left(tg\dfrac{\theta}{2}\right) \cdot \dfrac{i \cdot \sin(k \cdot R)}{R}, \quad |k \cdot R| > \pi/4 \ . \end{cases} \qquad (3.5)$$

4. Discussions & conclusion.

A new type of exact solutions of the full 3 dimensional *spatial* Helmholtz equation for the case of non-paraxial Gaussian beams is presented here.

We consider appropriate representation of the solution for Gaussian beams *in a spherical coordinate system* by substituting it to the full 3 dimensional spatial Helmholtz Equation.

Analyzing the structure of the final equation, we obtain that governing equations for the components of our solution are represented by the proper *Riccati* equations of complex value, which has no analytical solution in general case.

But we find one of the possible exact solution (3.5) which is proved to satisfy to such an equations for Gaussian beams (*).

Indeed, since the functions $g(R) = \sin(k \cdot R)/R$ or $g(R) = \cos(k \cdot R)/R$ in (3.5) are itself an exact solutions of the full Helmholtz equation (1.1), the formula for the Laplacian in spherical coordinates gives for $A = h(\theta) \cdot g(R)$, $h(\theta) = \ln tg(\theta/2)$:

$$\frac{1}{R^2}\frac{\partial^2 A}{\partial \theta^2} + \frac{1}{R^2} ctg\,\theta \frac{\partial A}{\partial \theta} = 0 \quad \rightarrow \quad \frac{d\left(\dfrac{1}{\sin\theta}\right)}{d\,\theta} + ctg\,\theta \cdot \left(\frac{1}{\sin\theta}\right) = 0 \, ,$$

- which is obviously valid.

As for the appropriate example of *paraxial* approximation for such a *non-paraxial* exact solution (3.5) of the full Helmholtz equation (1.1), it could be easily obtained in the case $\theta \rightarrow 0$ (see the expression above).

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