# Exact solution of Helmholtz equation for the case of non-paraxial Gaussian beams.

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**Keywords**: Helmholtz equation, paraxial approximation, Gaussian beam.

A new type of exact solution of the full 3 dimensional *spatial* Helmholtz equation for the case of non-paraxial Gaussian beams is presented here.
We consider appropriate representation of the solution for Gaussian beams *in a spherical coordinate system*, then implement it in the full 3 dimensional Helmholtz Eq.
Analyzing the structure of the final equation, we obtain one of the simple exact solutions which is proved to satisfy to such an equation for Gaussian beams.
Also the proper examples of implementing of the paraxial approximation for Gaussian beam could easily be obtained for a new type of exact solution of Helmholtz equation.

## 1. Introduction.

The full 3-dimensional *spatial* Helmholtz equation provides solutions that describe the propagation of waves over space (*e.g.*, *electromagnetic waves*); it should be presented in a spherical coordinate system $R, \theta, \varphi$ as below [1-2]:

$$\Delta A + k^2 A = 0 , \tag{1.1}$$

- where $\Delta$ - is the Laplacian, $k$ is the wavenumber, and $A$ is the amplitude.

Besides, in spherical coordinate system [3]:

$$\Delta A = \frac{\partial^2 A}{\partial R^2} + \frac{2}{R}\frac{\partial A}{\partial R} + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 A}{\partial \varphi^2} + \frac{1}{R^2}\frac{\partial^2 A}{\partial \theta^2} + \frac{1}{R^2} ctg\,\theta \frac{\partial A}{\partial \theta} .$$

Let us search for solutions of Eq. (1.1) in a form of Gaussian beams, which could be presented in Cartesian coordinate system as below [4]:

$$A = a \cdot \frac{w_0}{w(z)} \exp\left[ -\frac{x^2+y^2}{w^2(z)} - ikz - ik\frac{x^2+y^2}{2R(z)} + i\zeta(z) \right]$$

- where $w$ (z), $R$ (z), $\zeta$ (z) – some functions, describing the appropriate parameters of a beam; the last expression could be also represented as below

$$\exp\left[ i\left( \zeta(z) - kz + i\cdot \ln w(z) + \left( \frac{i}{w^2(z)} - \frac{k}{2R(z)} \right)\cdot (x^2+y^2) \right) \right] = \exp\left[ i\left( p(z) + \frac{x^2+y^2}{2q(z)} \right) \right]$$

- here $p$ (z) - is the complex phase-shift of the waves during their propagation along the $z$ axis; $q$ (z) - is the proper complex parameter of a beam, which is determining the Gaussian profile of a wave in the transverse plane at position $z$.

If the direction of axis *z* coincides with the main direction of propagation of a beam, the last expression could be transformed in a spherical coordinate system as below:

$$A = a \cdot \exp\left[ i\left( p(R,\varphi) + \frac{R^2 \cdot \sin^2\theta}{2q(R,\varphi)} \right) \right] .$$

Then having substituted the last expression (for Gaussian beam) into Eq. (1.1), we obtain ($\theta \neq 0$):

$$\frac{\partial^2 p(R,\varphi)}{\partial R^2} + \frac{\partial^2 \left( \frac{R^2}{q(R,\varphi)} \right)}{\partial R^2} \frac{\sin^2\theta}{2} + i\left( \frac{\partial p(R,\varphi)}{\partial R} + \frac{\partial \left( \frac{R^2}{q(R,\varphi)} \right)}{\partial R} \frac{\sin^2\theta}{2} \right)^2 + \frac{2}{R}\left( \frac{\partial p(R,\varphi)}{\partial R} + \frac{\partial \left( \frac{R^2}{q(R,\varphi)} \right)}{\partial R} \frac{\sin^2\theta}{2} \right) +$$

$$+ \frac{1}{R^2 \sin^2\theta} \cdot \left\{ \frac{\partial^2 p(R,\varphi)}{\partial \varphi^2} + \frac{\partial^2 \left( \frac{1}{q(R,\varphi)} \right)}{\partial \varphi^2} \cdot \frac{R^2 \sin^2\theta}{2} + i\left( \frac{\partial p(R,\varphi)}{\partial \varphi} + \frac{\partial \left( \frac{1}{q(R,\varphi)} \right)}{\partial \varphi} \cdot \frac{R^2 \sin^2\theta}{2} \right)^2 \right\} +$$

$$+ \left( \frac{1}{q(R,\varphi)} \right)\left\{ 2\cos^2\theta - \sin^2\theta + i\left( \frac{R^2 \cdot \sin^2\theta \cdot \cos^2\theta}{q(R,\varphi)} \right) \right\} - ik^2 = 0 \tag{1.2}$$

## 2. Exact solution.

Let us assume: $C \cdot q\,(R,\ \varphi) = R^2$, $C = const$. For such a case, Eq. (1.2) could be reduced as below ($\theta \neq 0$):

$$\frac{\partial^2 p(R,\varphi)}{\partial R^2} + i\left(\frac{\partial p(R,\varphi)}{\partial R}\right)^2 + \frac{2}{R}\left(\frac{\partial p(R,\varphi)}{\partial R}\right) + \frac{1}{R^2 \sin^2\theta}\cdot\left\{ \frac{\partial^2 p(R,\varphi)}{\partial \varphi^2} + i\left(\frac{\partial p(R,\varphi)}{\partial \varphi}\right)^2 \right\} +$$

$$+ \frac{C}{R^2}\cdot(2\cos^2\theta - \sin^2\theta + i\,C\cdot\sin^2\theta\cdot\cos^2\theta) - ik^2 = 0\ , \qquad (1.3)$$

- besides, Eq. (1.3) could be presented as a system of two equations:

$$\begin{cases} \dfrac{\partial^2 p(R,\varphi)}{\partial R^2} + i\left(\dfrac{\partial p(R,\varphi)}{\partial R}\right)^2 + \dfrac{2}{R}\left(\dfrac{\partial p(R,\varphi)}{\partial R}\right) - ik^2 = 0\ , \\ \dfrac{1}{R^2 \sin^2\theta}\cdot\left\{ \dfrac{\partial^2 p(R,\varphi)}{\partial \varphi^2} + i\left(\dfrac{\partial p(R,\varphi)}{\partial \varphi}\right)^2 \right\} + \\ + \dfrac{C}{R^2}\cdot(2\cos^2\theta - \sin^2\theta + i\,C\cdot\sin^2\theta\cdot\cos^2\theta) = 0\ , \end{cases}$$

- or

$$\begin{cases} \dfrac{\partial^2 p(R,\varphi)}{\partial R^2} + i\left(\dfrac{\partial p(R,\varphi)}{\partial R}\right)^2 + \dfrac{2}{R}\left(\dfrac{\partial p(R,\varphi)}{\partial R}\right) - ik^2 = 0\ , \\ \dfrac{\partial^2 p(R,\varphi)}{\partial \varphi^2} + i\left(\dfrac{\partial p(R,\varphi)}{\partial \varphi}\right)^2 + C\cdot\sin^2\theta\cdot\{2 + \sin^2\theta(-3 + i\,C\cdot\cos^2\theta)\} = 0\ . \end{cases}$$

Let us denote as below

$$i\cdot C\cdot sin^2\theta\cdot\{2 + sin^2\theta(-3 + i\,C\cdot cos^2\theta)\} = f^2(\theta),$$

- where we could choose $C = m{\cdot}i$ in the expression above; besides, for such a case we *should* choose $m \le -1$ *or* $m > 0$.

Then one of the simplest solution of 2-nd equation of the last system is obviously presented below

$$\begin{cases} \dfrac{d^2 p(R)}{d R^2} + i\left(\dfrac{d\, p(R)}{d R}\right)^2 + \dfrac{2}{R}\left(\dfrac{d\, p(R)}{d R}\right) - i k^2 = 0\,, \\ \left(\dfrac{\partial\, p(R,\varphi)}{\partial \varphi}\right)^2 = f^2(\theta)\,, \quad \Rightarrow \quad p(R,\varphi) = f(\theta)\cdot\varphi + p(R)\,, \end{cases}$$

- where $p(R) = p_1(R) + i \cdot p_2(R)$, so we obtain from the 1-st equation of system above:

$$\begin{cases} \dfrac{d^2 p_1(R)}{d R^2} + 2\left[\dfrac{1}{R} - \left(\dfrac{d\, p_2(R)}{d R}\right)\right]\cdot\dfrac{d\, p_1(R)}{d R} = 0\,, \\ i\cdot\left\{\dfrac{d^2 p_2(R)}{d R^2} + \left(\dfrac{d\, p_1(R)}{d R}\right)^2 - \left(\dfrac{d\, p_2(R)}{d R}\right)^2 - k^2\right\} = 0\,. \end{cases}$$

Besides, we obviously conclude that one of the simplest solution of 1-st equation of system above is $p_1(R) = const = C_1$, but function $p_2(R)$ should be defined from the 2-nd equation of system above:

$$\frac{d^2 p_2(R)}{d R^2} - \left(\frac{d\, p_2(R)}{d R}\right)^2 - k^2 = 0 \;\Rightarrow\; \frac{\dfrac{d^2 p_2(R)}{d R^2}}{\left(\dfrac{d\, p_2(R)}{d R}\right)^2 + k^2} = 1\,,$$

$$\Rightarrow\; \frac{1}{k} arctg\left(\frac{1}{k}\frac{d\, p_2(R)}{d R}\right) = R \;\Rightarrow\; \frac{d\, p_2(R)}{d R} = k\, tg(k R)\,,$$

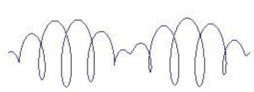
$$\Rightarrow\; p_2(R) = k\int tg(k R)\, d R = -\ln\cos(k R)\,.$$

Finally, one of the simplest exact solution of Eq. (1.2) is presented below:

$$A = a\cdot\exp\left[\, i\left( f(\theta)\cdot\varphi + C_1 - i\,\ln\cos(k R) + \frac{1}{2} C\cdot\sin^2\theta\right)\right],$$

$$A = A_0\cdot\cos(k R)\cdot\exp\left[\, i\left( f(\theta)\cdot\varphi + \frac{C\cdot\sin^2\theta}{2}\right)\right].$$

Let us also imagine the solution above in a spherical coordinate system *R, θ, φ*:



As for the proper example of paraxial approximation for such an exact solution of Helmholtz equation, it could be easily obtained in the case $\theta \to 0$ (see the last expression above).

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