Octonion Dark Matter

B. C. Chanyal, P. S. Bisht and O. P. S. Negi

Department of Physics
Kumaun University
S. S. J. Campus
Almora -263601 (U.K.) India

Email:- bcchanyal@gmail.com
ps_bisht123@rediffmail.com
ops_negi@yahoo.co.in

Abstract

In this paper, we have made an attempt to discuss the role of octonions in gravity and dark matter where, we have described the octonion space as the combination of two quaternionic spaces namely gravitational G-space and electromagnetic EM-space. It is shown that octonionic hot dark matter contains the photon and graviton (i.e. massless particles) while the octonionic cold dark matter is associated with the $W^{\pm}, Z^{0}$ (massive) bosons.

1 Introduction

The Standard Model (SM) [1]-[5] of particle physics summarizes all [6]-[11] we know about the fundamental forces of electromagnetism, as well as the weak and strong interactions [12] (without gravity). The Standard Model consists of elementary particles grouped into two classes [12]: bosons (particles that transmit forces) and fermions (particles that make up matter). The bosons have particle spin that is either 0, 1 or 2. The fermions have spin 1/2. On the other hand, particle physics strives to identify the building blocks of matter and describe the interactions that bind them: the set of instructions needed to create a universe. Our most succinct and (we believe) accurate set of instructions is encapsulated in a quantum field theory [1, 3, 4] called the Standard Model, which describes a universe [13] made up of six types of quarks and six types of leptons, bound together by three fundamental forces: strong, weak, and electromagnetic. The standard model is a relativistic quantum field theory [1, 2, 3] that incorporates the basic principles of quantum mechanics and special relativity. Like quantum electrodynamics (QED) the standard model is a gauge theory [14]. However, with the non-Abelian gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ instead of the simple Abelian $U(1)_{em}$ gauge group of QED. The gauge bosons are the photons mediating the electromagnetic interactions, the $W^{\pm}$ and Z^{0} bosons mediating the weak interactions [12], as well as the gluons mediating the strong interactions [2, 3, 12]. Gauge theories can exist in several phases: in the Coulomb phase with massless gauge bosons (like in QED), in the Higgs-phase with spontaneously broken gauge symmetry [14] and with massive gauge bosons (e.g. the $W^{\pm}$ and Z^0 bosons), and in the confinement phase, in which the gauge bosons do not appear in the spectrum (like the gluons in quantum chromodynamics (QCD)). On the other hand, The Standard Model was formulated in the 1970s and tentatively established by experiments in the early 1980s. Nearly three decades of exacting experiments have tested and verified the theory in meticulous detail, confirming all of its predictions. Thus, the Standard Model of particle physics is the most successful theory of nature in history, but increasingly there are signs that it must be extended by adding new particles that play roles in high-energy reactions [15, 16].

Despite being the most successful theory of particle physics to date, the Standard Model is not perfect [17, 18]. The deficiencies of the Standard Model on the bans of experimental observations which are not yet explain, are described as

➤ The standard model does not provide an explanation of gravity [19]. Moreover it is incompatible with the most successful theory of gravity to date, general relativity.

➤ Cosmological observations tell us that the standard model is able to explain only about 4% of the energy present in the universe. Of the missing 96%, about 24% should be dark matter [20], i.e. matter that behaves just like the other matter we know, but which interacts only weakly with the standard model fields. The rest should be dark energy, a constant energy density for the vacuum. Attempts to explain the dark energy in terms of vacuum energy of the standard model lead to a mismatch of 120 orders of magnitude.

➤ According to the standard model the neutrinos are massless particles [21]. However, neutrino oscillation experiments have shown that neutrinos do have mass. Mass terms for the neutrinos can be added to the standard model by hand, but these lead to new theoretical problems [21]. (For example, the mass terms need to be extraordinarily small).

➤ The universe is made out of mostly matter. However, the standard model predicts that matter and anti-matter [22] should have been created in (almost) equal amounts, which would have annihilated each other as the universe cooled.

The standard model is also incomplete with respect to theoretical problems associated with

➤ **Hierarchy problem** – the standard model introduces particle masses through a process known as spontaneous symmetry breaking caused by the Higgs field. Within the standard model, the mass of the Higgs gets some very large quantum corrections due to the presence of virtual particles (mostly virtual top quarks) [23]. These corrections are much larger than the actual mass of the Higgs. This means that the bare mass parameter of the Higgs in the standard model must be fine tuned in such a way that almost completely cancels the quantum corrections. This level of fine tuning is deemed unnatural by many theorists.

➤ **Strong CP problem** – theoretically it can be argued that the standard model should contain a term that breaks CP symmetry [24] —relating matter to antimatter— in the strong interaction sector. Experimentally, however, no such violation has been found, implying that the coefficient of this term is very close to zero. This fine tuning is also considered unnatural.

➤ **Number of parameters** – the standard model depends on 19 numerical parameters. Their values are known from experiment, but the origin of the values is unknown. Some theorists have tried to find relations between different parameters, for example, between the masses of particles in different generations.

On the other hand, the two fundamental mathematical structures (division algebras) a physicist uses in his everyday life are the real ($\mathbb{R}$) and the complex ($\mathbb{C}$) numbers. Complex numbers are described as pairs of real numbers with a specific multiplication laws. One can however go even further and build two other sets of numbers, known in mathematics as quaternions ($\mathbb{H}$) [25] and octonions ($\mathbb{O}$) [26]. The quaternions, formed as pairs of complex numbers are non-commutative whereas the octonions, formed as pairs of quaternion numbers are both non-commutative and non-associative. The four sets of numbers are mathematically known as division algebras. The octonions are the last division algebra, no further generalization being consistent with the laws of mathematics. So, there exists four normed division algebras [27]: the real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) [25, 28], and octonions ($\mathbb{O}$) [26, 29]. Thus octonions are regarded as a super-set of quaternions in the same way that quaternions are a super-set of complex numbers, i.e.

- ➤ Scalars are represented by 1 number.
- ➤ Complex numbers are represented by 2 numbers (1 real and 1 imaginary).
- ➤ Quaternions are represented by 4 numbers (1 real and 3 imaginary).
- ➤ Octonions are represented by 8 numbers (1 real and 7 imaginary).

2 Octonion Definition

An octonion x is expressed [30, 31] as a set of eight real numbers

$$\begin{aligned} x=(x_0,\, x_1, ..., x_7) &= x_0e_0+x_1e_1+x_2e_2+x_3e_3+x_4e_4+x_5e_5+x_6e_6+x_7e_7 \\ &= x_0e_0+\sum_{A=1}^{7} x_A e_A \qquad (A=1,2,.....,7) \end{aligned} \tag{1}$$

where $e_A(A=1,2,.....,7)$ are imaginary octonion units and e_0 is the multiplicative unit element. The octet $(e_0,e_1,e_2,e_3,e_4,e_5,e_6,e_7)$ is known as the octonion basis and its elements satisfy the following multiplication rules

$$e_0=1, \qquad e_0e_A=e_Ae_0=e_A \qquad e_Ae_B=-\delta_{AB}e_0+f_{ABC}\,e_C. \quad (A,B,C=1,2,......7) \tag{2}$$

The structure constants f_{ABC} are completely antisymmetric and take the value 1 i.e. $f_{ABC}=+1=$ (123), (471), (257), (165), (624), (543), (736). Here the octonion algebra $\mathcal{O}$ is described over the algebra of rational numbers having the vector space of dimension 8. Octonion algebra is non associative and multiplication rules for its basis elements given by equations (2,3) are then generalized in the following table:

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_7	$-e_6$	e_5	$-e_4$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	$-e_5$	e_4	e_7	$-e_6$
e_4	$-e_7$	$-e_6$	e_5	-1	$-e_3$	e_2	e_1
e_5	e_6	$-e_7$	$-e_4$	e_3	-1	$-e_1$	e_2
e_6	$-e_5$	e_4	$-e_7$	$-e_2$	e_1	-1	e_3
e_7	e_4	e_5	e_6	$-e_1$	$-e_2$	$-e_3$	-1

Table1- Octonion Multiplication table

Hence, we get the following relations among octonion basis elements i.e.

$$[e_A,\ e_B] \;=\; 2f_{ABC}e_C; \quad \{e_A,\ e_B\} = -\delta_{AB}e_0; \quad e_A(e_Be_C) \neq (e_Ae_B)e_C; \tag{3}$$

where brackets [ ] and { } are used respectively for commutation and the anti commutation relations while δ_{AB} is the usual Kronecker delta-Dirac symbol.Octonion conjugate is thus defined as,

$$\begin{aligned}\bar{x} &= x_0e_0 - x_1e_1 - x_2e_2 - x_3e_3 - x_4e_4 - x_5e_5 - x_6e_6 - x_7e_7 \\ &= x_0e_0 \;-\sum_{A=1}^{7} x_Ae_A \qquad (A = 1, 2, ..., 7).\end{aligned} \tag{4}$$

An Octonion can be decomposed in terms of its scalar ($Sc(x)$) and vector ($Vec(x)$) parts as

$$Sc(x) \;=\; \frac{1}{2}(x + \bar{x}) = x_0; \quad Vec(x) = \frac{1}{2}(x - \bar{x}) = \sum_{A=1}^{7} x_Ae_A \tag{5}$$

Conjugates of product of two octonions and its own are described as

$$(\overline{xy}) \;=\; \bar{y}\,\bar{x}\;; \quad \overline{(\bar{x})} \;= x \tag{6}$$

while the scalar product of two octonions is defined as

$$\langle x\,,\,y\,\rangle \;= \textstyle\sum_{\alpha=0}^{7} \;\; x_\alpha y_\alpha = \displaystyle\frac{1}{2}(x\,\bar{y} + y\,\bar{x}) = \frac{1}{2}(\bar{x}\,y + \bar{y}\,x) \tag{7}$$

which can be written in terms of octonion units as

$$\langle e_A\,,\,e_B\,\rangle \;=\; \frac{1}{2}(e_A\overline{e_B} + e_B\overline{e_A}) = \frac{1}{2}(\overline{e_A}e_B + \overline{e_B}e_A) = \delta_{AB}. \tag{8}$$

The norm of the octonion $N(x)$ is defined as

$$N(x)=\bar{x}x \quad = x\,\bar{x} \quad = \sum_{\alpha=0}^{7} x_{\alpha}^{2} e_0 \tag{9}$$

which is zero if $x=0$, and is always positive otherwise. It also satisfies the following property of normed algebra

$$N(xy) \quad = N(x)N(y) \quad = N(y)N(x). \tag{10}$$

As such, for a nonzero octonion x , we define its inverse as

$$x^{-1} \quad = \quad \frac{\bar{x}}{N(x)} \tag{11}$$

which shows that

$$x^{-1}x \quad = xx^{-1} \quad = 1.e_0; \quad (xy)^{-1} = y^{-1}x^{-1}. \tag{12}$$

3 Octonion Gravitational and Electromagnetic interactions

Let us identify the octonion space (eight dimensional) as the combination of two quaternionic spaces namely associated with the gravitational interaction (G-space) and electromagnetic interaction (EM-space) [32, 33]. So, we may write the octonionic (gravitational-electromagnetic) space as

$$\mathcal{O}=\left(\mathcal{O}_{g-space}\,,\;\mathcal{O}_{em-space}\right)\Longrightarrow\left(\left(e_0,e_1,e_2,e_3\right),\;\left(e_4,e_5,e_6,e_7\right)\right), \tag{13}$$

where $(\mathcal{O}_{g-space})$ is octonionic gravitational space consists e_0, e_1, e_2, e_3 octonion basis and $(\mathcal{O}_{em-space})$ is octonionic electromagnetic space consists e_4, e_5, e_6, e_7. So

$$\mathcal{O} =(e_0,e_1,e_2,e_3,e_4,e_5,e_6,e_7) = (\mathcal{O}_g+\mathcal{O}_{em}). \tag{14}$$

Any physical quantity $X \in \mathcal{O}$ may be written as

$$\begin{aligned} X = X_g + X_{em} &=(X_{g_0}e_0+X_{g_1}e_1+X_{g_2}e_2+X_{g_3}e_3)+(X_{em_0}e_4+X_{em_1}e_5+X_{em_2}e_6+X_{em_3}e_7) \\ &=\sum_{j=0}^{3} X_{g_j}e_j + e_7\sum_{j=0}^{3} X_{em_j}e_j. \end{aligned} \tag{15}$$

Accordingly, the octonion differential operator $\boxdot$ [34, 35, 36] also may be written as the combination of the two quaternionic space (G-space & EM-space) [32] in the terms of eight dimensional space as

$$\boxdot=\boxdot_{g}+\boxdot_{em}=(\partial_{g_0}e_0+\partial_{g_1}e_1+\partial_{g_2}e_2+\partial_{g_3}e_3)+(\partial_{em_0}e_4+\partial_{em_1}e_5+\partial_{em_2}e_6+\partial_{em_3}e_7)$$
$$=\sum_{j=0}^{3}\partial_{g_j}e_j+e_7\sum_{j=0}^{3}\partial_{em_j}e_j. \tag{16}$$

Thus, the octonion conjugate of equation (16) may then be written as

$$\overline{\boxdot}=\overline{\boxdot}_{g}+\overline{\boxdot}_{em}=(\partial_{g_0}e_0-\partial_{g_1}e_1-\partial_{g_2}e_2-\partial_{g_3}e_3)+(-\partial_{em_0}e_4-\partial_{em_1}e_5-\partial_{em_2}e_6-\partial_{em_3}e_7)$$
$$=\partial_{g_0}e_0-\sum_{j=1}^{3}\partial_{g_j}e_j-e_7\sum_{j=0}^{3}\partial_{em_j}e_j. \tag{17}$$

Accordingly, the octonion valued potential, in eight dimensional formalism may also be written as the combinations of two four dimensional quaternionic spaces (i.e. G-space and EM-space) as

$$\begin{aligned}\mathbb{V}=(V_g\,,\ V_{em})&=((V_0,V_1,V_2,V_3)\,,\ (V_4,V_5,V_6,V_7))\\&=((V_{g_0},V_{g_1},V_{g_2},V_{g_3})\,,\ (V_{em_0},V_{em_1},V_{em_2},V_{em_3}))\,,\end{aligned} \tag{18}$$

which can further be reduced to

$$\begin{aligned}\mathbb{V}=&(V_{g_0}e_0+V_{g_1}e_1+V_{g_2}e_2+V_{g_3}e_3)+(V_{em_0}e_4+V_{em_1}e_5+V_{em_2}e_6+V_{em_3}e_7)\\=&\sum_{j=0}^{3}V_{g_j}e_j+e_7\sum_{j=0}^{3}V_{em_j}e_j.\end{aligned} \tag{19}$$

As such, we may obtain the octonion potential wave equation for gravitational-electromagnetic space by operating $\overline{\boxdot}$ given by equation (17) to octonion potential $\mathbb{V}$ (19) in the following manner,

$$\begin{aligned}\overline{\boxdot}\,\mathbb{V}=&e_0\{(\partial_{g_0}V_{g_0}+\partial_{g_1}V_{g_1}+\partial_{g_2}V_{g_2}+\partial_{g_3}V_{g_3})+(\partial_{em_0}V_{em_0}+\partial_{em_1}V_{em_1}+\partial_{em_2}V_{em_2}+\partial_{em_3}V_{em_3})\}\\&+e_1\{(\partial_{g_0}V_{g_1}-\partial_{g_1}V_{g_0}-\partial_{g_2}V_{g_3}+\partial_{g_3}V_{g_2})+(-\partial_{em_0}V_{em_3}+\partial_{em_1}V_{em_2}-\partial_{em_2}V_{em_1}+\partial_{em_3}V_{em_0})\}\\&+e_2\{(\partial_{g_0}V_{g_2}-\partial_{g_2}V_{g_0}+\partial_{g_1}V_{g_3}-\partial_{g_3}V_{g_1})+(-\partial_{em_0}V_{em_2}-\partial_{em_1}V_{em_3}+\partial_{em_2}V_{em_0}+\partial_{em_3}V_{em_1})\}\\&+e_3\{(\partial_{g_0}V_{g_3}-\partial_{g_3}V_{g_0}-\partial_{g_1}V_{g_2}+\partial_{g_2}V_{g_1})+(-\partial_{em_1}V_{em_0}+\partial_{em_0}V_{em_1}-\partial_{em_2}V_{em_3}+\partial_{em_3}V_{em_2})\}\\&+e_4\{(\partial_{g_0}V_{em_0}+\partial_{g_1}V_{em_3}+\partial_{g_2}V_{em_2}-\partial_{g_3}V_{em_1})+(-\partial_{em_0}V_{g_0}+\partial_{em_1}V_{g_3}-\partial_{em_2}V_{g_2}-\partial_{em_3}V_{g_1})\}\\&+e_5\{(\partial_{g_0}V_{em_1}-\partial_{g_1}V_{em_2}+\partial_{g_2}V_{em_3}+\partial_{g_3}V_{em_0})+(-\partial_{em_1}V_{g_0}-\partial_{em_0}V_{g_3}+\partial_{em_2}V_{g_1}-\partial_{em_3}V_{g_2})\}\\&+e_6\{(\partial_{g_0}V_{em_2}+\partial_{g_1}V_{em_1}-\partial_{g_2}V_{em_0}+\partial_{g_3}V_{em_3})+(-\partial_{em_2}V_{g_0}+\partial_{em_0}V_{g_2}-\partial_{em_1}V_{g_1}-\partial_{em_3}V_{g_3})\}\\&+e_7\{(\partial_{g_0}V_{em_3}-\partial_{g_1}V_{em_0}-\partial_{g_2}V_{em_1}-\partial_{g_3}V_{em_2})+(-\partial_{em_3}V_{g_0}+\partial_{em_0}V_{g_1}+\partial_{em_1}V_{g_2}+\partial_{em_2}V_{g_3})\}.\end{aligned} \tag{20}$$

which can further be reduced to

$$\overline{\boxdot}\,\mathbb{V}\ \ =\ \ \mathbb{F}=((F_0,F_1,F_2,F_3)\,,(F_4,F_5,F_6,F_7))\,, \tag{21}$$

where $\mathbb{F}(F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7)$ is also an octonion reproduces the field strength of generalized gravitational-electromagnetic fields of dyons. Thus, we may be express $\mathbb{F}$ as

$$\begin{aligned}\mathbb{F} = F_g + F_{em} =& \left(\left(F_{g_0}, F_{g_1}, F_{g_2}, F_{g_3}\right), \left(F_{em_0}, F_{em_1}, F_{em_2}, F_{em_3}\right)\right) \\ =& (F_{g_0}e_0 + F_{g_1}e_1 + F_{g_2}e_2 + F_{g_3}e_3) + (F_{em_0}e_4 + F_{em_1}e_5 + F_{em_2}e_6 + F_{em_3}e_7), \end{aligned} \tag{22}$$

where the component of $\mathbb{F}\,(F_{g_0}, F_{g_1}, F_{g_2}, F_{g_3}, F_{em_0}, F_{em_1}, F_{em_2}, F_{em_3})$ are expressed as

$$\begin{aligned}
F_{g_0} =& \{(\partial_{g_0}V_{g_0} + \partial_{g_1}V_{g_1} + \partial_{g_2}V_{g_2} + \partial_{g_3}V_{g_3}) + e_7(-\partial_{em_3}V_{g_0} + \partial_{em_0}V_{g_1} + \partial_{em_1}V_{g_2} + \partial_{em_2}V_{g_3})\} \\
F_{g_1} =& \{(\partial_{g_0}V_{g_1} - \partial_{g_1}V_{g_0} - \partial_{g_2}V_{g_3} + \partial_{g_3}V_{g_2}) + e_7(-\partial_{em_0}V_{g_0} + \partial_{em_1}V_{g_3} - \partial_{em_2}V_{g_2} - \partial_{em_3}V_{g_1})\} \\
F_{g_2} =& \{(\partial_{g_0}V_{g_2} - \partial_{g_2}V_{g_0} + \partial_{g_1}V_{g_3} - \partial_{g_3}V_{g_1}) + e_7(-\partial_{em_1}V_{g_0} - \partial_{em_0}V_{g_3} + \partial_{em_2}V_{g_1} - \partial_{em_3}V_{g_2})\} \\
F_{g_3} =& \{(\partial_{g_0}V_{g_3} - \partial_{g_3}V_{g_0} - \partial_{g_1}V_{g_2} + \partial_{g_2}V_{g_1}) + e_7(-\partial_{em_2}V_{g_0} + \partial_{em_0}V_{g_2} - \partial_{em_1}V_{g_1} - \partial_{em_3}V_{g_3})\} \\
F_{em_0} =& \{(\partial_{g_0}V_{em_0} + \partial_{g_1}V_{em_3} + \partial_{g_2}V_{em_2} - \partial_{g_3}V_{em_1}) + e_7(-\partial_{em_0}V_{em_3} + \partial_{em_1}V_{em_2} - \partial_{em_2}V_{em_1} + \partial_{em_3}V_{em_0})\} \\
F_{em_1} =& \{(\partial_{g_0}V_{em_1} - \partial_{g_1}V_{em_2} + \partial_{g_2}V_{em_3} + \partial_{g_3}V_{em_0}) + e_7(-\partial_{em_0}V_{em_2} - \partial_{em_1}V_{em_3} + \partial_{em_2}V_{em_0} + \partial_{em_3}V_{em_1})\} \\
F_{em_2} =& \{(\partial_{g_0}V_{em_2} + \partial_{g_1}V_{em_1} - \partial_{g_2}V_{em_0} + \partial_{g_3}V_{em_3}) + e_7(-\partial_{em_1}V_{em_0} + \partial_{em_0}V_{em_1} - \partial_{em_2}V_{em_3} + \partial_{em_3}V_{em_2})\} \\
F_{em_3} =& \{(\partial_{g_0}V_{em_3} - \partial_{g_1}V_{em_0} - \partial_{g_2}V_{em_1} - \partial_{g_3}V_{em_2}) + e_7(\partial_{em_0}V_{em_0} + \partial_{em_1}V_{em_1} + \partial_{em_2}V_{em_2} + \partial_{em_3}V_{em_3})\}
\end{aligned} \tag{23}$$

using the Lorentz Gauge conditions in the equation (23), i.e. $F_{g_0} = F_{em_3} = 0$. Thus, equation (22) may be written as

$$\mathbb{F} = F_g + F_{em} = (F_{g_1}e_1 + F_{g_2}e_2 + F_{g_3}e_3) + (F_{em_0}e_4 + F_{em_1}e_5 + F_{em_2}e_6). \tag{24}$$

Here, the first term $(F_g = F_{g_1}, F_{g_2}, F_{g_3})$ is defined as the field strength of the gravitational interaction in G-space while the second term $(F_{em} = F_{em_0}, F_{em_1}, F_{em_2})$ is associated with the field strength of the electromagnetic interaction in EM-space. Hence, we may obtain the octonionic field equation in gravitational-electromagnetic space on applying the differential operator (16) to equation (24) as

$$\begin{aligned}
\boxdot\,\mathbb{F} =& -e_0\{(\partial_{g_1}F_{g_1} + \partial_{g_2}F_{g_2} + \partial_{g_3}F_{g_3}) + (\partial_{em_0}F_{em_0} + \partial_{em_1}F_{em_1} + \partial_{em_2}F_{em_2})\} \\
&+e_1\{(\partial_{g_0}F_{g_1} + \partial_{g_2}F_{g_3} + \partial_{g_3}F_{g_2}) + (-\partial_{em_1}F_{em_2} + \partial_{em_2}F_{em_1} - \partial_{em_3}F_{em_0})\} \\
&+e_2\{(\partial_{g_0}F_{g_2} - \partial_{g_1}F_{g_3} + \partial_{g_3}F_{g_1}) + (-\partial_{em_2}F_{em_0} + \partial_{em_0}F_{em_2} - \partial_{em_3}F_{em_1})\} \\
&+e_3\{(\partial_{g_0}F_{g_3} + \partial_{g_1}F_{g_2} - \partial_{g_2}F_{g_1}) + (-\partial_{em_0}F_{em_1} + \partial_{em_1}F_{em_0} - \partial_{em_3}F_{em_2})\} \\
&+e_4\{(\partial_{g_0}F_{em_0} - \partial_{g_2}F_{em_2} + \partial_{g_3}F_{em_1}) + (-\partial_{em_1}F_{g_3} + \partial_{em_2}F_{g_2} + \partial_{em_3}F_{g_1})\} \\
&+e_5\{(\partial_{g_0}F_{em_1} + \partial_{g_1}F_{em_2} - \partial_{g_3}F_{em_0}) + (-\partial_{em_2}F_{g_1} + \partial_{em_0}F_{g_3} + \partial_{em_3}F_{g_2})\} \\
&+e_6\{(\partial_{g_0}F_{em_2} - \partial_{g_1}F_{em_1} + \partial_{g_2}F_{em_0}) + (-\partial_{em_0}F_{g_2} + \partial_{em_1}F_{g_1} + \partial_{em_3}F_{g_3})\} \\
&+e_7\{(\partial_{g_1}F_{em_0} + \partial_{g_2}F_{em_1} + \partial_{g_3}F_{em_2}) + (-\partial_{em_0}F_{g_1} - \partial_{em_1}F_{g_2} - \partial_{em_2}F_{g_3})\}.
\end{aligned} \tag{25}$$

which is further reduced to the compact notation in terms of an octonionic gravitational-electromagnetic space as

$$\boxdot\,\mathbb{F} \;=\; \mathbb{J} = \left(\left(J_0, J_1, J_2, J_3\right), \left(J_4, J_5, J_6, J_7\right)\right), \tag{26}$$

where $\mathbb{J}(J_0, J_1, J_2, J_3, J_4, J_5, J_6, J_7)$ is also an octonion reproduces the field current source of dyons. So, it may be expressed as

$$\begin{aligned}\mathbb{J} = & (J_0, J_1, J_2, J_3, J_4, J_5, J_6, J_7)\\ = & (J_{(g-g)_0} + J_{(em-em)_0})e_0\\ & +(J_{(g-g)_1} + J_{(em-em)_1})e_1\\ & +(J_{(g-g)_2} + J_{(em-em)_2})e_2\\ & +(J_{(g-g)_3} + J_{(em-em)_3})e_3\\ & +\{J_{(em-g)_0} + J_{(g-em)_0})e_4\\ & +(J_{(em-g)_1} + J_{(g-em)_1})e_5\\ & +(J_{(em-g)_2} + J_{(g-em)_2})e_6\\ & +(J_{(em-g)_3} + J_{(g-em)_3})e_7. \end{aligned} \tag{27}$$

Here $J_{(g-g)}, J_{(em-em)}, J_{(em-g)}, J_{(g-em)}$ are defined for the octonionic current source respectively for gravitational-gravitational, electromagnetic-electromagnetic, electromagnetic-gravitational, gravitational-electromagnetic interaction [32, 33, 34, 35, 36]. As such, the components of octonionic current source $\mathbb{J}$ are described as

$$\begin{aligned}
J_{(g-g)_0} =&(\partial_{g_1}F_{g_1} + \partial_{g_2}F_{g_2} + \partial_{g_3}F_{g_3}), \quad J_{(em-em)_0} = (\partial_{em_0}F_{em_0} + \partial_{em_1}F_{em_1} + \partial_{em_2}F_{em_2});\\
J_{(g-g)_1} =&(\partial_{g_0}F_{g_1} + \partial_{g_2}F_{g_3} + \partial_{g_3}F_{g_2}), \quad J_{(em-em)_1} = (-\partial_{em_1}F_{em_2} + \partial_{em_2}F_{em_1} - \partial_{em_3}F_{em_0});\\
J_{(g-g)_2} =&(\partial_{g_0}F_{g_2} - \partial_{g_1}F_{g_3} + \partial_{g_3}F_{g_1}), \quad J_{(em-em)_2} = (-\partial_{em_2}F_{em_0} + \partial_{em_0}F_{em_2} - \partial_{em_3}F_{em_1});\\
J_{(g-g)_3} =&(\partial_{g_0}F_{g_3} + \partial_{g_1}F_{g_2} - \partial_{g_2}F_{g_1}), \quad J_{(em-em)_3} = (-\partial_{em_0}F_{em_1} + \partial_{em_1}F_{em_0} - \partial_{em_3}F_{em_2});\\
J_{(em-g)_0} =&(\partial_{g_0}F_{em_0} - \partial_{g_2}F_{em_2} + \partial_{g_3}F_{em_1}), \quad J_{(g-em)_0} = (-\partial_{em_1}F_{g_3} + \partial_{em_2}F_{g_2} + \partial_{em_3}F_{g_1});\\
J_{(em-g)_1} =&(\partial_{g_0}F_{em_1} + \partial_{g_1}F_{em_2} - \partial_{g_3}F_{em_0}), \quad J_{(g-em)_1} = (-\partial_{em_2}F_{g_1} + \partial_{em_0}F_{g_3} + \partial_{em_3}F_{g_2});\\
J_{(em-g)_2} =&(\partial_{g_0}F_{em_2} - \partial_{g_1}F_{em_1} + \partial_{g_2}F_{em_0}), \quad J_{(g-em)_2} = (-\partial_{em_0}F_{g_2} + \partial_{em_1}F_{g_1} + \partial_{em_3}F_{g_3});\\
J_{(em-g)_3} =&(\partial_{g_1}F_{em_0} + \partial_{g_2}F_{em_1} + \partial_{g_3}F_{em_2}), \quad J_{(g-em)_3} = (-\partial_{em_0}F_{g_1} - \partial_{em_1}F_{g_2} - \partial_{em_2}F_{g_3});
\end{aligned} \tag{28}$$

which are analogous to the generalized Dirac-Maxwell's (GDM) equations in presence of gravitational-gravitational (G-G), electromagnetic-electromagnetic (EM-EM), electromagnetic-gravitational (EM-G), gravitational-electromagnetic (G-EM) interaction.
Consequently, the octonionic radius vector ($\mathbb{R} = R_0, R_1, R_2, R_3, R_4, R_5, R_6, R_7$) is defined the combination of two quaternionic space in the following manner,

$$\begin{aligned}\mathbb{R} =&(R_0, R_1, R_2, R_3)\ ,\ (R_4, R_5, R_6, R_7)\\ =&(R_0e_0 + R_1e_1 + R_2e_2 + R_3e_3) + (R_4e_4 + R_5e_5 + R_6e_6 + R_7e_7). \end{aligned} \tag{29}$$

which yields the velocity ($\boldsymbol{v}$) in the octonionic (gravitational-electromagnetic) representation as

$$\begin{aligned}\boldsymbol{v} =&\frac{\partial\mathbb{R}}{\partial t} = \frac{\partial}{\partial t}\{(R_0e_0 + R_1e_1 + R_2e_2 + R_3e_3) + (R_4e_4 + R_5e_5 + R_6e_6 + R_7e_7)\}\\ =&(v_0e_0 + v_1e_1 + v_2e_2 + v_3e_3) + (v_4e_4 + v_5e_5 + v_6e_6 + v_7e_7), \end{aligned} \tag{30}$$

we may also described the charge and mass [33] of the particle in octonionic space as

$$\begin{aligned}
J_{(g-g)_0} &\cong Q_{(g-g)_0} v_0, & J_{(em-em)_0} &\cong Q_{(em-em)_0} v_0; \\
J_{(g-g)_1} &\cong Q_{(g-g)_1} v_1, & J_{(em-em)_1} &\cong Q_{(em-em)_1} v_1; \\
J_{(g-g)_2} &\cong Q_{(g-g)_2} v_2, & J_{(em-em)_2} &\cong Q_{(em-em)_2} v_2; \\
J_{(g-g)_3} &\cong Q_{(g-g)_3} v_3, & J_{(em-em)_3} &\cong Q_{(em-em)_3} v_3; \\
J_{(em-g)_0} &\cong Q_{(em-g)_0} v_4, & J_{(g-em)_0} &\cong Q_{(g-em)_0} v_4; \\
J_{(em-g)_1} &\cong Q_{(em-g)_1} v_5, & J_{(g-em)_1} &\cong Q_{(g-em)_1} v_5; \\
J_{(em-g)_2} &\cong Q_{(em-g)_2} v_6, & J_{(g-em)_2} &\cong Q_{(g-em)_2} v_6; \\
J_{(em-g)_3} &\cong Q_{(em-g)_3} v_7, & J_{(g-em)_3} &\cong Q_{(g-em)_3} v_7;
\end{aligned} \tag{31}$$

where $Q_{(g-g)}, Q_{(em-g)}, Q_{(g-em)}$ are respectively denoted the "Mass" of the gravitational - gravitational (G-G), electromagnetic - gravitational (EM-G), gravitational - electromagnetic (G-EM) interactions while $Q_{(em-em)}$ represent the "Charge" of the electromagnetic - electromagnetic (EM-EM) interaction. So, the $Q_{(g-g)}$,$Q_{(em-g)}, Q_{(g-em)}$ and $Q_{(em-em)}$ respectively describe the "Generalized mass" and "Generalized charge" [33]. From the equations (27), (28) and (31), we may obtain the four-type of subfields in the octonionic electromagnetic-gravitational fields [32, 33] as

- ➤ Gravitational-Gravitational (G-G) subfield.
- ➤ Electromagnetic-Gravitational (EM-G) subfield.
- ➤ Electromagnetic-Electromagnetic (EM-EM) subfield.
- ➤ Gravitational-Electromagnetic (G-EM) subfield.

Thus, from above four subfields, we have describe the octonion dark matter in the following section.

4 Octonion Dark Matter

The *Dark Matter* [20, 22] is a type of matter hypothesized to account for a large part of the total mass in the universe. Dark matter cannot be seen directly with telescopes which is neither emits nor absorbs light or other electromagnetic radiation at any significant level. Instead, its existence and properties are inferred from its gravitational effects on visible matter, radiation and the large scale structure of the universe. The majority of dark matter in the universe cannot be baryons, and thus does not form atoms. It also cannot interact with ordinary matter as electromagnetic forces, i.e. the dark matter particles do not carry any electric charge. The nonbaryonic dark matter may include the photon, graviton, intermediate bosons and neutrinos, or supersymmetric particles. Unlike baryonic matter, nonbaryonic dark matter does not contribute to the formulation of the elements in the universe as its presence is revealed only via its gravitational attraction. Thus, the nonbaryonic dark matter [20, 21, 22] is evident through its gravitational effect only. There are two type of nonbaryonic dark matter respectively defined as hot dark matter and cold dark matter. Here, we have made an attempt to express the nonbaryonic dark matter in terms of octonion representation in the following subsections.

4.1 Octonion Hot Dark Matter (OHDM):

Octonions hot dark matter assumed to compose of particles that have zero or near-zero mass. The special theory of relativity requires that massless particles move at the speed of light while near-zero mass particles move at nearly the speed of light. Thus, the octonionic hot dark matter may be associated with the gravitational-gravitational (G-G) and electromagnetic-electromagnetic (EM-EM) subfields. Thus, the octonionic hot dark matter (OHDM) includes the photon and graviton. As such, we may write the quantum equation for octonionic hot dark matter in terms of potential, field and current equations. So, the potential wave equations from (15) and (16), may be written in the quaternionic (G-G) space as

$$\begin{aligned}\boxdot_g X_g =&(\partial_{g_0}e_0+\partial_{g_1}e_1+\partial_{g_2}e_2+\partial_{g_3}e_3)\cdot(X_{g_0}e_0+X_{g_1}e_1+X_{g_2}e_2+X_{g_3}e_3)\\ =&V_{(g-g)_0}e_0+V_{(g-g)_1}e_1+V_{(g-g)_2}e_2+V_{(g-g)_3}e_3,\quad \text{(for G-G space)}\end{aligned}\tag{32}$$

which may further be written for EM-EM sector as

$$\begin{aligned}\boxdot_{em} X_{em} =&(\partial_{em_0}e_4+\partial_{em_1}e_5+\partial_{em_2}e_6+\partial_{em_3}e_7)\cdot(X_{em_0}e_4+X_{em_1}e_5+X_{em_2}e_6+X_{em_3}e_7)\\ =&V_{(em-em)_0}e_0+V_{(em-em)_1}e_1+V_{(em-em)_2}e_2+V_{(em-em)_3}e_3,\text{(for EM-EM space)}\end{aligned}\tag{33}$$

Thus, equations (17) and (19) reduces to

$$\begin{aligned}\overline{\boxdot}_g V_g =&(\partial_{g_0}e_0-\partial_{g_1}e_1-\partial_{g_2}e_2-\partial_{g_3}e_3)\cdot(V_{g_0}e_0+V_{g_1}e_1+V_{g_2}e_2+V_{g_3}e_3)\\ =&F_{(g-g)_0}e_0+F_{(g-g)_1}e_1+F_{(g-g)_2}e_2+F_{(g-g)_3}e_3,\quad \text{(for G-G space)}\end{aligned}\tag{34}$$

and

$$\begin{aligned}\overline{\boxdot}_{em} V_{em} =&(-\partial_{em_0}e_4-\partial_{em_1}e_5-\partial_{em_2}e_6-\partial_{em_3}e_7)\cdot(V_{em_0}e_4+V_{em_1}e_5+V_{em_2}e_6+V_{em_3}e_7)\\ =&F_{(em-em)_0}e_0+F_{(em-em)_1}e_1+F_{(em-em)_2}e_2+F_{(em-em)_3}e_3,\text{(for EM-EM space)}\end{aligned}\tag{35}$$

Accordingly, the field source equations from (15) and (24), are respectively described as

$$\begin{aligned}\boxdot_g F_g =&(\partial_{g_0}e_0+\partial_{g_1}e_1+\partial_{g_2}e_2+\partial_{g_3}e_3)\cdot(F_{g_0}e_0+F_{g_1}e_1+F_{g_2}e_2+F_{g_3}e_3)\\ =&J_{(g-g)_0}e_0+J_{(g-g)_1}e_1+J_{(g-g)_2}e_2+J_{(g-g)_3}e_3,\quad \text{(for G-G space)}\end{aligned}\tag{36}$$

and

$$\begin{aligned}\boxdot_{em} F_{em} =&(\partial_{em_0}e_4+\partial_{em_1}e_5+\partial_{em_2}e_6+\partial_{em_3}e_7)\cdot(F_{em_0}e_4+F_{em_1}e_5+F_{em_2}e_6+F_{em_3}e_7)\\ =&J_{(em-em)_0}e_0+J_{(em-em)_1}e_1+J_{(em-em)_2}e_2+J_{(em-em)_3}e_3.\text{(for EM-EM space)}\end{aligned}\tag{37}$$

These two equations (36), (37) describe the generalized Dirac-Maxwell's equations of dyons in terms of octonionic hot dark matter comparizing gravitational-gravitational (G-G) and electromagnetic-electromagnetic (EM-EM) interactions. Hence, we may conclude that the quantum equations for octonionic hot dark matter (i.e. photon and graviton) are expressed in the terms of quaternionic representations of octonions.

4.2 Octonion Cold Dark Matter (OCDM):

Like wise, the octonions cold dark matter may be described as the composition of the massive objects moving at sub-relativistic velocities. So, the difference between the octonions cold dark matter (OCDM) and the octonions hot dark matter (OHDM) is significant in the formulation of structure, because the velocities of octonions hot dark matter cause it to wipe out structure on small scales.

Thus, the octonions cold dark matter is associated with the electromagnetic-gravitational (EM-G) and gravitational-electromagnetic (G-EM) subfields. Hence, the octonions cold dark matter (OCDM) is assumed to include intermediate particles (i.e. $W^{\pm}, Z^{o}$ particles). So, we may write the quantum equations for octonions cold dark matter in terms of potential, field and current equations. The potential wave equations from (15) and (16) may then be written respectively as

$$\begin{aligned}\boxdot_{em} X_g =& (\partial_{em_0} e_4 + \partial_{em_1} e_5 + \partial_{em_2} e_6 + \partial_{em_3} e_7) \cdot (X_{g_0} e_0 + X_{g_1} e_1 + X_{g_2} e_2 + X_{g_3} e_3) \\ =& V_{(em-g)_0} e_4 + V_{(em-g)_1} e_5 + V_{(em-g)_2} e_6 + V_{(em-g)_3} e_7, \quad \text{(for EM-G space)}\end{aligned} \tag{38}$$

and

$$\begin{aligned}\boxdot_{g} X_{em} =& (\partial_{g_0} e_0 + \partial_{g_1} e_1 + \partial_{g_2} e_2 + \partial_{g_3} e_3) \cdot (X_{em_0} e_4 + X_{em_1} e_5 + X_{em_2} e_6 + X_{em_3} e_7) \\ =& V_{(g-em)_0} e_0 + V_{(g-em)_1} e_1 + V_{(g-em)_2} e_2 + V_{(g-em)_3} e_3 + V_{(g-em)_4} e_4 + V_{(g-em)_5} e_5 \\ &+ V_{(g-em)_6} e_6 + V_{(g-em)_7} e_7. \quad \text{(for G-EM space)}\end{aligned} \tag{39}$$

Accordingly, the field equations from (17) and (19) are respectively described as

$$\begin{aligned}\overline{\boxdot}_{em} V_g =& (\partial_{em_0} e_4 - \partial_{em_1} e_5 - \partial_{em_2} e_6 - \partial_{em_3} e_7) \cdot (V_{g_0} e_0 + V_{g_1} e_1 + V_{g_2} e_2 + V_{g_3} e_3) \\ =& F_{(em-g)_0} e_4 + F_{(em-g)_1} e_5 + F_{(em-g)_2} e_6 + F_{(em-g)_3} e_7, \quad \text{(for EM-G space)}\end{aligned} \tag{40}$$

and

$$\begin{aligned}\overline{\boxdot}_{g} V_{em} =& (\partial_{g_0} e_0 - \partial_{g_1} e_1 - \partial_{g_2} e_2 - \partial_{g_3} e_3) \cdot (V_{em_0} e_4 + V_{em_1} e_5 + V_{em_2} e_6 + V_{em_3} e_7) \\ =& F_{(g-em)_0} e_0 + F_{(g-em)_1} e_1 + F_{(g-em)_2} e_2 + F_{(g-em)_3} e_3 + F_{(g-em)_4} e_4 + F_{(g-em)_5} e_5 \\ &+ F_{(g-em)_6} e_6 + F_{(g-em)_7} e_7. \quad \text{(for G-EM space)}\end{aligned} \tag{41}$$

On the other hand the field source equations (15) and (24) are expressed as

$$\begin{aligned}\boxdot_{em} F_g =& (\partial_{em_0} e_4 + \partial_{em_1} e_5 + \partial_{em_2} e_6 + \partial_{em_3} e_7) \cdot (F_{g_0} e_0 + F_{g_1} e_1 + F_{g_2} e_2 + F_{g_3} e_3) \\ =& J_{(em-g)_0} e_4 + J_{(em-g)_1} e_5 + J_{(em-g)_2} e_6 + J_{(em-g)_3} e_7, \quad \text{(for EM-G space)}\end{aligned} \tag{42}$$

and

$$\begin{aligned}\boxdot_{g} F_{em} =& (\partial_{g_0} e_4 + \partial_{g_1} e_5 + \partial_{g_2} e_6 + \partial_{g_3} e_7) \cdot (F_{g_0} e_0 + F_{g_1} e_1 + F_{g_2} e_2 + F_{g_3} e_3) \\ =& J_{(g-em)_0} e_0 + J_{(g-em)_1} e_1 + J_{(g-em)_2} e_2 + J_{(g-em)_3} e_3 + J_{(g-em)_4} e_4 + J_{(g-em)_5} e_5 \\ &+ J_{(g-em)_6} e_6 + J_{(g-em)_7} e_7. \quad \text{(for G-EM space)}\end{aligned} \tag{43}$$

These equation on simplification, describe the generalized Dirac-Maxwell's equations of dyons for octonionic cold dark matter in the presence of electromagnetic-gravitational (EM-G) and gravitational-electromagnetic (G-EM) interactions. So, the quantum equations for octonionic cold dark matter (i.e. $W^{\pm}, Z^{o}$ particles) may easily be expressed in the terms of simpler and compact notation of octonions representations.

Thus, the nonbaryonic dark matter is evident through its gravitational effect only. Octonions hot dark matter is composed of particles that have zero or near-zero mass. So, the octonionic hot dark matter (OHDM) includes the photon and graviton. As such, we have established the various quantum equation for octonionic hot dark matter in terms of potential, field and current equations given by equations (32)-(37). It is concluded that the quantum equations for octonionic hot dark matter (i.e. photon and graviton) are expressed in the terms of quaternionic representations of octonions. Accordingly, the octonions cold dark matter has been described as the composition of the massive objects moving at sub-relativistic velocities. Hence, the octonions cold dark matter (OCDM) includes the intermediate particles (i.e. $W^{\pm}, Z^{o}$ particles) may easily be expressed in the terms of octonions representations. So, we have established the quantum equations for octonions cold dark matter in terms of potential, field and current equations given by equations (38)-(43).

References

[1] M. E. Peskin and D. V. Schroeder, "**An introduction to quantum field theory**", Book, Harper-Collins, (1995).

[2] T. P. Cheng and L. F. Li, "**Gauge theory of elementary particle physics**", Book, Oxford University Press, (1982).

[3] S. Weinberg, "**The quantum theory of fields**", Book, Cambridge University Press, **2** (1996).

[4] A. Zee, "**Quantum Field Theory in a Nutshell**", Book, Princeton University Press, Second Edition (2010).

[5] S. F. Novaes, "**Standard Model: An Introduction**", arXiv: hep-ph/0001283, (2000).

[6] S. L. Glashow, "**Partial-symmetries of weak interactions**", Nuclear Physics **22** (1961), 579.

[7] S. Weinberg, "**A Model of Leptons**", Physical Review Letters, **19** (1967), 1264.

[8] A. Salam, "**Elementary Particle Physics: Relativistic Groups and Analyticity**", Eighth Nobel Symposium, Stockholm: Almquvist and Wiksell, (1968) 367.

[9] F. Englert and R. Brout, "**Broken Symmetry and the Mass of Gauge Vector Mesons**", Physical Review Letters **13** (1964), 321.

[10] P. W. Higgs, "**Broken Symmetries and the Masses of Gauge Bosons**", Physical Review Letters, **13** (1964), 508.

[11] G. S. Guralnik, C. R. Hagen and T. W. B. Kibble, "**Global Conservation Laws and Massless Particles**", Physical Review Letters, **13** (1964), 585.

[12] D. J. Griffiths, "**Introduction to Elementary Particles**", John Wiley & Sons (1987).

[13] R. Harnik, G. D. Kribs, and G. Perez, "**A universe without weak interactions**", Physical Review D, **74** (2006), 035006.

[14] C. Becchi, **"Introduction to Gauge Theories"**, arXiv: hep-ph/9705211, (1997).

[15] K. Nakamura, "**Review of Particle Physics**", Journal of Physics G: Nuclear and Particle Physics, **37** (2010), 075021.

[16] A. Boyle, "**Milestone in Higgs quest: Scientists find new particle**", MSNBC, Retrieved July 5, (2012)

[17] C. Arzt, M. B. Einhorn and J. Wudka, "**Patterns of deviation from the standardmodel**", Nuclear Physics B, **433** (1995), 41.

[18] A. G. Cohen, D. B. Kaplan and A. E. Nelson, "**The more Minimal Supersymmetric StandardModel**", Physics Letters B, **388** (1996), 588.

[19] V. A. Kostelecký, "**Gravity, Lorentz violation, and the standard model**", Phys. Rev. D, **69** (2004), 105009.

[20] G. Bertone, D. Hooper and J. Silk, "**Particle darkmatter: evidence, candidates and constraints**", Physics Reports, Vol. **405** (2005), 279.

[21] G. Altarelli and F. Feruglio, "**Neutrino masses and mixings: a theoretical perspective**", Physics Reports, **320** (1999), 295.

[22] M. Dine, "**Origin of the matter-antimatter asymmetry**", Rev. Mod. Phys., **76** (2003), 1.

[23] H. C. Cheng and I. Low, "**TeV symmetry and the little hierarchy problem**", JHEP, **09** (2003), 051.

[24] H. Y. Cheng, "**The strong CP problem revisited**", Physics Reports, **158** (1988), 1.

[25] W. R. Hamilton, **"Elements of Quaternions"** , Vol. I & II, Chelsea Publishing, New York (1969) 1185.

[26] A. Cayley, "**On certain results relating to quaternions**", Phil. Mag., **26** (1845), 210.

[27] R. Schafer, **"Introduction to Non-Associative Algebras"**, Dover, New York, (1995).

[28] P. G. Tait, **"An elementary treatise on quaternions"**, Oxford, (1867); 2^{nd} Ed.; Cambridge, (1873); 3^{rd}, (1890).

[29] J. Graves, **"On a connection between the general theory of normal couples and the theory of complete quadratic functions of two variables"**, Phil. Mag., **26** (1845), 320.

[30] J. C. Baez, **"The Octonions"**, arXiv: math. RA/0105155v4, (2002).

[31] G. M. Dixon, **"Division Algebras:: Octonions Quaternions Complex Numbers and the Algebraic Design of Physics"**, book, Springer (1994).

[32] Z. Weng, "**Octonionic Quantum Interplays of Dark Matter and Ordinary Matter**", arXiv: physics.gen-ph/0702019v4, (2008).

[33] Z. Weng, "**Octonionic electromagnetic and gravitational interactions and dark matter**", arXiv: physics.class-ph/0612102v8, (2009).

[34] B. C. Chanyal, P. S. Bisht and O. P. S. Negi , **"Generalized Octonion Electrodynamics"**, Int. J. Theor. Phys., **49** (2010), 1333.

[35] B. C. Chanyal, P. S. Bisht and O. P. S. Negi, **"Generalized Split-Octonion Electrodynamics"**, Int. J. Theor. Phys., **50** (2011), 1919.

[36] B. C. Chanyal, P. S. Bisht, Tianjun Li and O. P. S. Negi, **"Octonion Quantum Chromodynamics",** Int. J. Theor. Phys., **51** (2012), 3410.