Action-Reaction Paradox Resolution

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Abstract

According to Newton's Third Law, in a collision between two isolated particles 'action equals reaction'. However, in classical electrodynamics, this law is violated. In general, in a collision between two isolated charged particles, the momentum of the particles is not conserved. Typically, it is necessary to combine the field momentum with the particle momentum in order to 'balance the scales'. A paradox arises from the fact that, generally, particle momentum is conserved in the center of mass (cm) frame, but not in the lab frame. Here, we offer a resolution to this paradox in which the Third Law remains valid for collisions between charged particles, in all situations and in all frames, without the need to invoke the momentum of the field.

1 Introduction

We would like to evaluate the forces on two isolated 'point' charges q and q' with constant velocities $\mathbf{v}$ and $\mathbf{v}'$, respectively, for the non-relativistic, three-dimensional case (the complete relativistic, four-dimensional force equations, can be seen in the section "The Force Density Four-vector" at http://www.softcom.net/users/der555/newtransform.pdf). In the classical model, these forces are not always equal and opposite. We intend to show that this inequality of forces leads to a paradox, to which we offer a resolution.

This treatment does not take the momentum of the field into consideration, although it applies even when the field momentum is considered. For the complete conservation equations, which take into account the momentum of the field, please refer to the section "The Energy-Momentum Tensor" at http://www.softcom.net/users/der555/newtransform.pdf. We've used SI units, here, in contrast to the Gaussian units used in the paper, above.

2 The Paradox

The Lorentz force $\mathbf{F}$ on q due to q' is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the conventional electric and magnetic field three-vectors at the position of q due to q'. The Lorentz force $\mathbf{F}'$ on q' due to q is

$$\mathbf{F}' = q'(\mathbf{E}' + \mathbf{v}' \times \mathbf{B}') \tag{2}$$

where $\mathbf{E}'$ and $\mathbf{B}'$ are the conventional electric and magnetic field three-vectors at the position of q' due to q.

The electric forces $\mathbf{F}_e = q\mathbf{E}$ and $\mathbf{F}'_e = q'\mathbf{E}'$ on the particles are always equal in magnitude and opposite in direction, but the magnetic forces

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B} \tag{3}$$

and

$$\mathbf{F}'_m = q'\mathbf{v}' \times \mathbf{B}' \tag{4}$$

are *not*, in general, equal and opposite. In fact, for every case except the cases, one or both particles at rest, or particles on parallel paths, the magnetic forces are *not* equal and opposite. Therefore, the total forces on the particles are not, in general, equal and opposite.

However, in the center of mass (cm) frame, the total forces on the particles *are* equal and opposite at all times, due to symmetry. Therefore, we have the paradox that the forces on the particles are equal and opposite in the cm frame, but not in the lab frame.

3 The Resolution

I would like to offer a resolution to this paradox. Consider the additional force

$$\mathbf{F}_a = -q\mathbf{v}(\nabla \cdot \mathbf{A}) \tag{5}$$

on q due to q', where $\mathbf{A}$ is the vector potential, and the additional force

$$\mathbf{F}'_a = -q'\mathbf{v}'(\nabla \cdot \mathbf{A}') \tag{6}$$

on q' due to q.[1]

Therefore, our new law for the total force on q is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B} - \mathbf{v}(\nabla \cdot \mathbf{A})) \tag{7}$$

and for the total force on q'

$$\mathbf{F}' = q'(\mathbf{E}' + \mathbf{v}' \times \mathbf{B}' - \mathbf{v}'(\nabla \cdot \mathbf{A}')) \tag{8}$$

Considering, first, the forces on q, using (7), we note that the magnetic field, can also be written as

$$\mathbf{B} = \frac{1}{c^2}\mathbf{v}' \times \mathbf{E} \tag{9}$$

[1]These additional forces are not *ad hoc* additions. They are due to the time component of my electric field four-vector which is part of my force density equations at the URL above (these forces are not referred to as $\mathbf{F}_a$ and $\mathbf{F}'_a$ in my paper).

and, since $\mathbf{A} = \mathbf{v}'\phi/c^2$, we can write $\nabla \cdot \mathbf{A}$ as

$$\nabla \cdot \mathbf{A} = \nabla \cdot \left(\frac{\mathbf{v}'\phi}{c^2}\right) = \frac{1}{c^2}\,\mathbf{v}' \cdot (\nabla\phi) = -\frac{1}{c^2}\,\mathbf{v}' \cdot \mathbf{E} \tag{10}$$

where ϕ is the static electric potential. Inserting (9) and (10) into (7), we have

$$\mathbf{F} = q(\mathbf{E} + \frac{1}{c^2}\,(\mathbf{v} \times (\mathbf{v}' \times \mathbf{E}) + \mathbf{v}(\mathbf{v}' \cdot \mathbf{E})) \tag{11}$$

Using the vector identity

$$\mathbf{v} \times (\mathbf{v}' \times \mathbf{E}) = \mathbf{v}'(\mathbf{v} \cdot \mathbf{E}) - \mathbf{E}(\mathbf{v} \cdot \mathbf{v}') \tag{12}$$

we can write (11) as

$$\mathbf{F} = q(\mathbf{E} + \frac{1}{c^2}\,(\mathbf{v}'(\mathbf{v} \cdot \mathbf{E}) - \mathbf{E}(\mathbf{v} \cdot \mathbf{v}') + \mathbf{v}(\mathbf{v}' \cdot \mathbf{E})) \tag{13}$$

Since $q = q'$ and $\mathbf{E} = -\mathbf{E}'$, after rearranging terms, we can write (13) as

$$\mathbf{F} = -q'(\mathbf{E}' + \frac{1}{c^2}\,(\mathbf{v}(\mathbf{v}' \cdot \mathbf{E}') - \mathbf{E}'(\mathbf{v}' \cdot \mathbf{v}) + \mathbf{v}'(\mathbf{v} \cdot \mathbf{E}')) \tag{14}$$

From the vector identity

$$\mathbf{v}' \times (\mathbf{v} \times \mathbf{E}') = \mathbf{v}(\mathbf{v}' \cdot \mathbf{E}') - \mathbf{E}'(\mathbf{v}' \cdot \mathbf{v}) \tag{15}$$

we can write (14) as

$$\mathbf{F} = -q'(\mathbf{E}' + \frac{1}{c^2}\,(\mathbf{v}' \times (\mathbf{v} \times \mathbf{E}') + \mathbf{v}'(\mathbf{v} \cdot \mathbf{E}')) \tag{16}$$

or

$$\mathbf{F} = -q'(\mathbf{E}' + \mathbf{v}' \times \mathbf{B}' - \mathbf{v}'(\nabla \cdot \mathbf{A}')) \tag{17}$$

From (8), we can now write (17) as

$$\mathbf{F} = -\mathbf{F}' \tag{18}$$

We see from (18) that the total force $\mathbf{F}$ on q and the total force $\mathbf{F}'$ on q' are equal and opposite, thus action equals reaction in every case for two isolated particles.

We have considered only the spatial components of the force above, nevertheless, it is interesting to note that the time components of the force F_t and F_t' are equal, but *not* opposite. We start by considering the time component of the force F_t on q

$$F_t = q(\frac{1}{c}\,\mathbf{v} \cdot \mathbf{E} + c\,\nabla \cdot \mathbf{A}) \tag{19}$$

or, substituting (10) into (19), we get

$$F_t = q(\frac{1}{c}\,\mathbf{v} \cdot \mathbf{E} - \frac{1}{c}\,\mathbf{v}' \cdot \mathbf{E}) \tag{20}$$

The time component of the force F_t' on q' is

$$F_t' = q'(\frac{1}{c}\,\mathbf{v}' \cdot \mathbf{E}' - \frac{1}{c}\,\mathbf{v} \cdot \mathbf{E}') \tag{21}$$

By substituting $q = q'$ and $\mathbf{E} = -\mathbf{E}'$ into (20), we can write

$$F_t = q'(\frac{1}{c}\mathbf{v}' \cdot \mathbf{E}' - \frac{1}{c}\mathbf{v} \cdot \mathbf{E}') \tag{22}$$

Note that the right-hand side of (22) is the same as the right-hand side of (21), so

$$F_t = F_t' \tag{23}$$

Thus, the time component of the force F_t on q and the time component of the force F_t' on q' are equal, but not opposite, for two isolated particles.

It must be noted that there are additional terms on the right-hand sides of the force equations at http://www.softcom.net/users/der555/newtransform.pdf that have been omitted in this discussion. Nevertheless, the inclusion of these terms, here, would leave $\mathbf{F} = -\mathbf{F}'$ and $F_t = F_t'$.