E8 Physics and Quasicrystals
Icosidodecahedron and Rhombic Triacontahedron

vixra 1301.0150 Frank Dodd (Tony) Smith Jr. - 2013

The E8 Physics Model (viXra 1108.0027) is based on the Lie Algebra E8.
240 E8 vertices = 112 D8 vertices + 128 D8 half-spinors where
D8 is the bivector Lie Algebra of the Real Clifford Algebra Cl(16) = Cl(8)xCl(8).
112 D8 vertices = (24 D4 + 24 D4) = 48 vertices from the D4xD4 subalgebra of D8
plus 64 = 8x8 vertices from the coset space D8 / D4xD4.
128 D8 half-spinor vertices = 64 ++half-half-spinors + 64 --half-half-spinors.
An 8-dim Octonionic Spacetime comes from the Cl(8) factors of Cl(16) and
a 4+4 = 8-dim Kaluza-Klein M4 x CP2 Spacetime emerges due to the freezing out of a preferred Quaternionic Subspace. Interpreting World-Lines as Strings leads to 26-dim Bosonic String Theory in which 10 dimensions reduce to 4-dim CP2 and a 6-dim Conformal Spacetime from which 4-dim M4 Physical Spacetime emerges.

Although the high-dimensional E8 structures are fundamental to the E8 Physics Model it may be useful to see the structures from the point of view of the familiar 3-dim Space where we live. To do that, start by looking the the E8 Root Vector lattice.

Algebraically, an E8 lattice corresponds to an Octonion Integral Domain.
There are 7 Independent E8 Lattice Octonion Integral Domains
corresponding to the 7 Octonion Imaginaries, as described by H. S. M. Coxeter
in "IntegralCayley Numbers" (Duke Math. J. 13 (1946) 561-578 and
in "Regular and Semi-Regular Polytopes III" (Math. Z. 200 (1988) 3-45).
Let { 1, i, j, k, e, ie, je, ke } be a basis of the Octonions.

The 112 D8 Root Vector vertices can be written as

(+/- 1 , +/- 1 , 0 , 0 , 0 , 0 , 0 , 0)

for all 4 possible +/- signs times all (8l2) = 28 permutations of pairs of basis elements.

The 128 D8 half-spinor vertices can be written in 7 different ways

(+/- (1 + i) +/- j +/- k +/- e +/- ie +/- je +/- ke) / 2
(+/- (1 + j) +/- i +/- k +/- e +/- ie +/- je +/- ke) / 2
(+/- (1 + k) +/- i +/- j +/- e +/- ie +/- je +/- ke) / 2
(+/- (1 + e) +/- i +/- j +/- k + +/- ie +/- je +/- ke) / 2
(+/- (1 + ie) +/- i +/- j +/- k +/- e +/- je +/- ke) / 2
(+/- (1 + je) +/- i +/- j +/- k +/- e +/- ie +/- ke) / 2
(+/- (1 + ke) +/- i +/- j +/- k +/- e +/- ie +/- je) / 2

in each of which
one Octonion Imaginary basis element is paired (same sign) with the Real basis element
to give 2^8-1) = 2^7 = 128 D8 half-spinor Root Vector vertices
so that
7 different E8 lattices, each with a 240-vertex Root Vector polytope around the origin,

can be constructed:

iE8 = (+/- 1 , +/- 1 , 0 , 0 , 0 , 0 , 0 , 0)
+ (+/- (1 + i) +/- j +/- k +/- e +/- ie +/- je +/- ke) / 2

jE8 = (+/- 1 , +/- 1 , 0 , 0 , 0 , 0 , 0 , 0)
+ (+/- (1 + j) +/- i +/- k +/- e +/- ie +/- je +/- ke) / 2

kE8 = (+/- 1 , +/- 1 , 0 , 0 , 0 , 0 , 0 , 0)
+ (+/- (1 + k) +/- i +/- j +/- e +/- ie +/- je +/- ke) / 2

eE8 = (+/- 1 , +/- 1 , 0 , 0 , 0 , 0 , 0 , 0)
+ (+/- (1 + e) +/- i +/- j +/- k + +/- ie +/- je +/- ke) / 2

ieE8 = (+/- 1 , +/- 1 , 0 , 0 , 0 , 0 , 0 , 0)
+ (+/- (1 + ie) +/- i +/- j +/- k +/- e +/- je +/- ke) / 2

jeE8 = (+/- 1 , +/- 1 , 0 , 0 , 0 , 0 , 0 , 0)
+ (+/- (1 + je) +/- i +/- j +/- k +/- e +/- ie +/- ke) / 2

keE8 = (+/- 1 , +/- 1 , 0 , 0 , 0 , 0 , 0 , 0)
+ (+/- (1 + ke) +/- i +/- j +/- k +/- e +/- ie +/- je) / 2

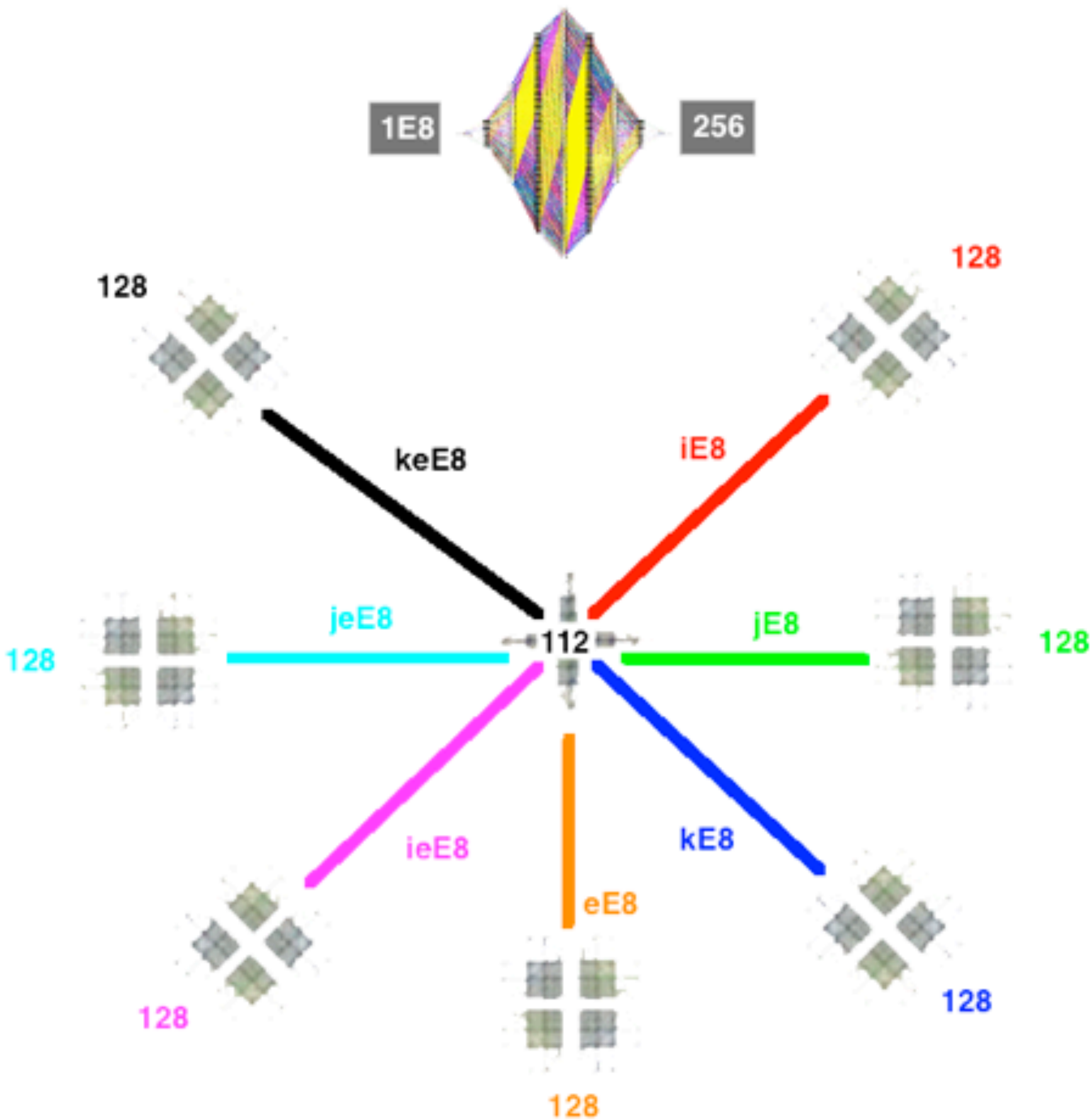
As Conway and Sloane say in "Sphere Packings, Lattices and Groups"
(Third Edition Springer
"... when n = 8 ... we can slide another copy of Dn in between the points of Dn ...
Formally, we define Dn+ = Dn u ([1] + Dn
When n = 8 ... the lattice D8+ ...[is]... known as E8 ...".

The D8 part of E8 contains the 112 D8 Root Vectors.
The 7 different E8 lattices correspond to 7 different ways to slide
the D8 half-spinor copy of D8 in between the points of the first D8
thus
producing 7 different E8 lattices each with a 112 + 128 = 240 Root Vector polytope.

For the same result from a different point of view, see Appendix A.

An 8th lattice 1E8 can be constructed as the Cl(8) Clifford Algebra Z8 cubic lattice
with 2^8 = 256 Root Vector vertices of the 8-dim Light Cone:
1E8 = (+/- 1 +/- i +/- j +/- k +/- e +/- ie +/- je +/- ke) / 2
1E8 can be useful in physics (such as in construction of 8-branes
as superpositions of E8 lattices in 26-dim Bosonic Theory of Strings as World-Lines).

All 8 lattice 1E8, ... , keE8 Root Vector vertices appear in the second layer out from the origin of any E8 lattice, which layer has 256 + 112 + 7(128+128) = 2160 vertices.
The 256 are the Cl(8) Light Cone vertices of 1E8.
The 112 are D8 Root Vectors that live in each of iE8, jE8, kE8, eE8, ieE8, jeE8, keE8.
The 7x128 are the D8 half-spinor vertices of iE8, jE8, kE8, eE8, ieE8, jeE8, keE8.
The other 7x128 correspond to the other D8 half-spinors that are not in E8 and so are not shown in the image below:



The 112 D8 Root Vector vertices in iE8, jE8, kE8, eE8, ieE8, jeE8, keE8

(+/- 1 , +/- 1 , 0 , 0 , 0 , 0 , 0 , 0)

for all 4 possible +/- signs times all (8l2) = 28 permutations of pairs of basis elements can be written in matrix form with each "4" representing possible signs and with the overall pattern of (1+2+3) + (4x4) + (3+2+1) representing the 28 permutations as

	1	i	j	k	e	ie	je	ke
1	-	4	4	4	4	4	4	4
i			4	4	4	4	4	4
j				4	4	4	4	4
k					4	4	4	4
e						4	4	4
ie							4	4
je								4
ke								-

The 4 x 6 = 24 in the (1,i,j,k) x (1,i,j,k) block corresponding to M4 Physical Spacetime are the Root Vectors of a D4 in D8 in E8 with a U(2,2) subgroup that contains the SU(2,2) = Spin(2,4) Conformal Group of Gravity.

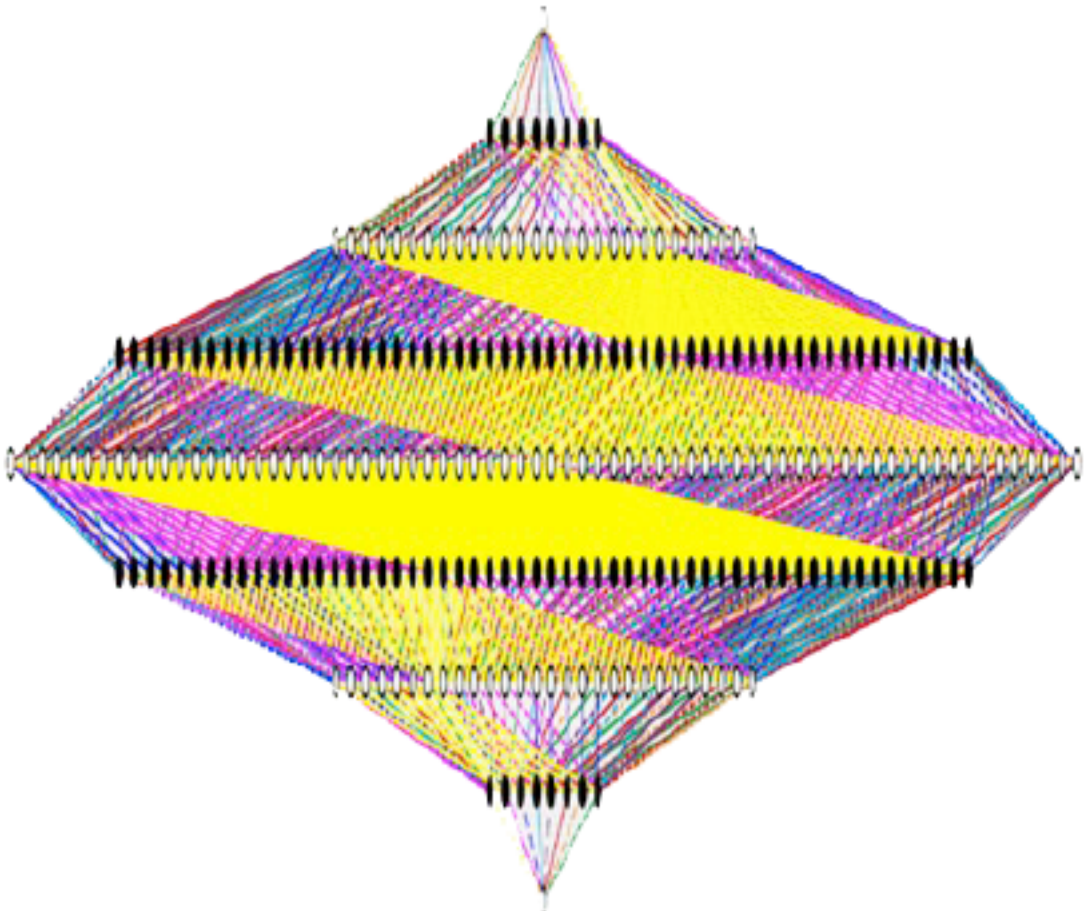
The 4 x 4x4 = 64 in the (1,i,j,k) x (e,ie,je,ke) block represents (4+4)-dim M4 x CP2 Kaluza-Klein Spacetime position and momentum.

The 4 x 6 = 24 in the (e,ie,je,ke) x (e,ie,je,ke) block corresponding to CP2 Internal Symmetry Space are the Root Vectors of another D4 in D8 in E8 with a U(4) subgroup that contains the SU(3) Color Force Group of the Standard Model.
The coset structure CP2 = SU(3) / U(1)xSU(2) gives the ElectroWeak U(1) and SU(2).

In each of the 7 E8 Root Vector sets for iE8, jE8, kE8, eE8, ieE8, jeE8, keE8
64 of the 128 D8 half-spinor vertices represent 8 components of 8 Fermion Particles
and
64 of the 128 D8 half-spinor vertices represent 8 components of 8 Fermion AntiParticles
where
the 8 fundamental Fermion Particle/AntiParticle types are:

neutrino, red down quark, green down quark, blue down quark;
blue up quark, green up quark, red up quark, electron.

The 256 vertices of **1E8** form an 8-cube with 1024 edges, 1792 square faces, 1792 cubic cells, 1120 tesseract 4-faces, 448 5-cube 5-faces, 112 6-cube 6-faces, and 16 7-cube 7-faces. This image in the format of African Adinkra for 256 Odu of IFA



shows Cl(8) graded structure 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 of 8-cube vertices. Physically they represent **Operators in H92 x Sl(8)** Generalized Heisenberg Algebra that is the **Maximal Contraction of E8**:

Odd-Grade Parts of Cl(8):

8+56 grades-1,3 = Fermion Particle 8-Component Creation (AntiParticle Annihilation)
56+8 grades-5,7 = Fermion AntiParticle 8-Component Creation (Particle Annihilation)

Even-Grade Subalgebra of Cl(8):

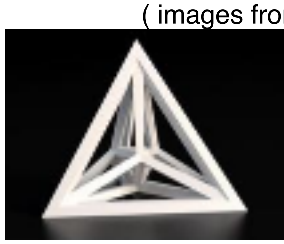
28 grade-2 = Gauge Boson Creation (16 for Gravity, 12 for Standard Model)
28 grade-6 = Gauge Boson Annihilation (16 for Gravity , 12 for Standard Model)
64 of grade-4 = 8-dim Position x Momentum
1+(3+3)+1 grades-0,4,8 = Primitive Idempotent:
(1+3) = Higgs Creation; (3+1) = Higgs Annihilation

The Fermionic **Odd-Grade parts of 1E8** correspond to 8+56+56+8 = **128** vertices of the **D8 half-spinors** of the 7 Root Vector Systems iE8, jE8, kE8, eE8, ieE8, jeE8, keE8.

The **Even-Grade parts of 1E8** correspond to (24+4)+64+(24+4) + 1+(3+3)+1 =
= 24+64+24 + (4+4) + 1+(3+3)+1 = (112 + (4+4)) + (1+3)+(3+1) =
= **112 vertices of D8 Root Vectors** + 8 of E8 Cartan Subalgebra + 8 Higgs Operators.

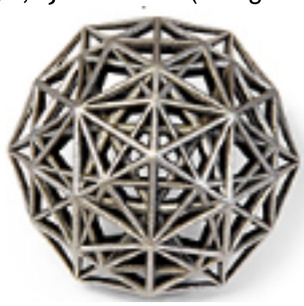
8 of E8 Cartan Subalgebra + 8 Higgs Operators = 2 copies of 4-dim 16-cell

(images from Bathsheba)

The 16-cell has 24 edges, midpoints of which are the 24 vertices of a 24-cell.
The 24-cell has 96 edges, Golden Ratio points of which when added to its 24 vertices, form the 96+24 = 120 vertices of a 600-cell.

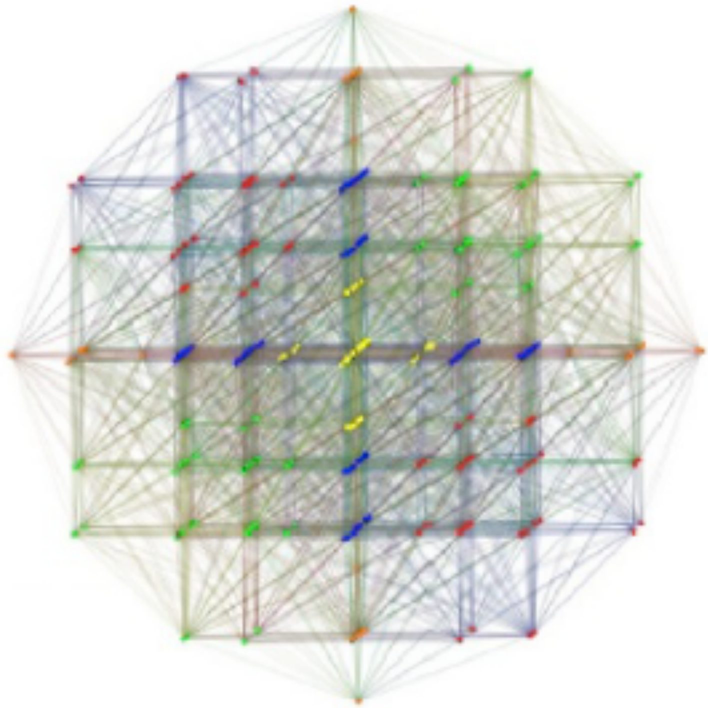
128 vertices of the D8 half-spinors + 112 vertices of D8 Root Vectors = 240 =
= 2 copies of 4-dim {3,3,5} 600-cell (images from Bathsheba)

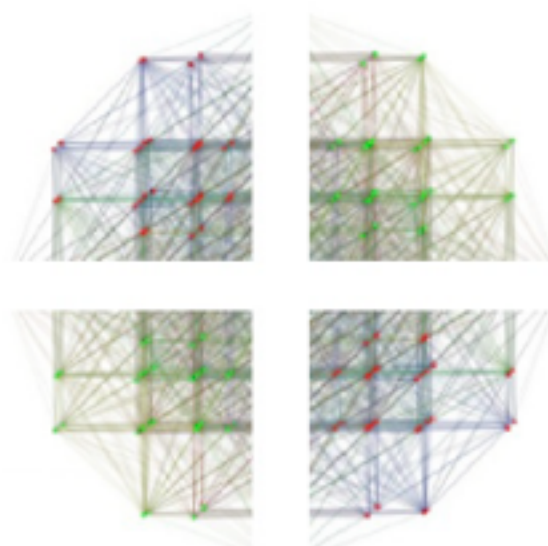
As described above, each 600-cell lives inside a 16-cell.

So,
the 256 vertices of **1E8**
(which represents Creation/Annihilation Operators in the Generalized Heisenberg Algebra H92 x Sl(8) that is the Maximal Contraction of E8)
contain
dual 16-cell structure of E8 Cartan Subalgebra + Cl(8) Primitive Idempotent Higgs
as well as
the dual 600-cell structure of the 240 E8 Root Vector vertices
of iE8, jE8, kE8, eE8, ieE8, jeE8, keE8.

The 240 vertices of iE8, jE8, kE8, eE8, ieE8, jeE8, keE8 and 240 of the 256 vertices of 1E8



can all be described as the **First Shell of an E8 Lattice**
which is made up of **112 D8 Root Vectors** plus **128 D8 half-spinor vertices**:



In "Regular and Semi-Regular Polytopes III" Coxeter describes that shell as

"... The eight-dimensional polytope4_21 ... in which the 240 vertices are distributed in 8 concentric tricontagons {30} ...

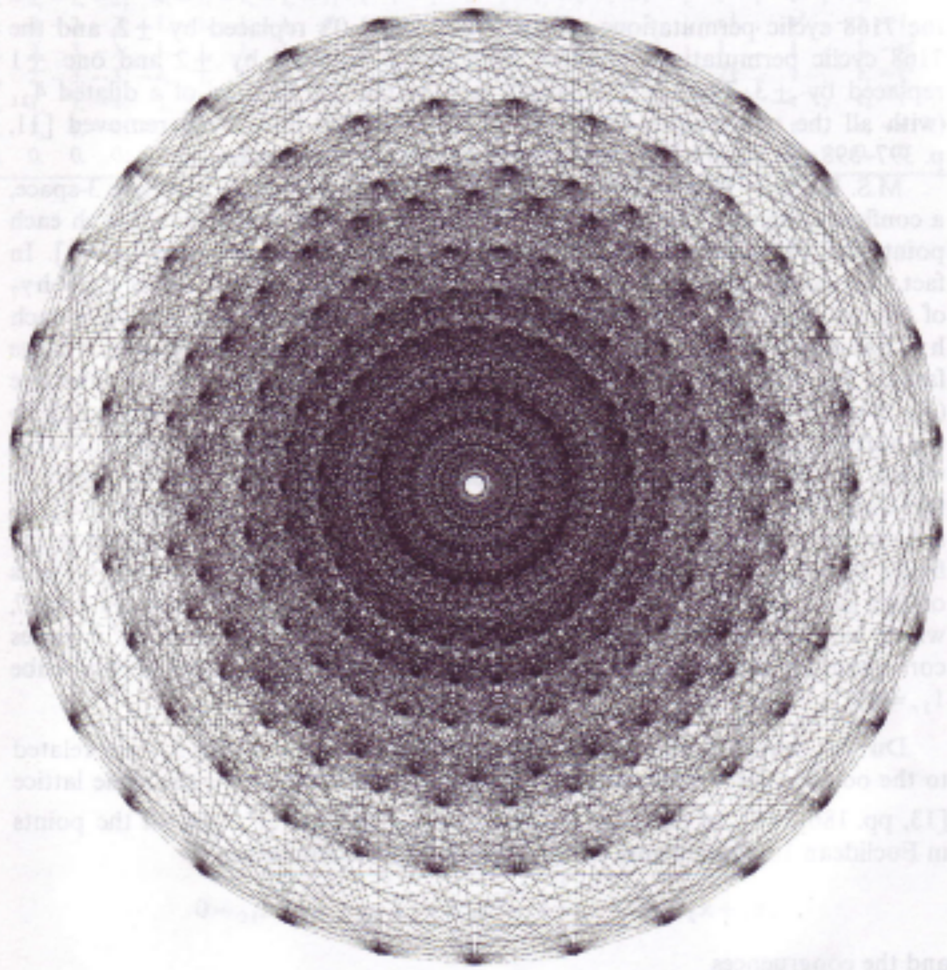
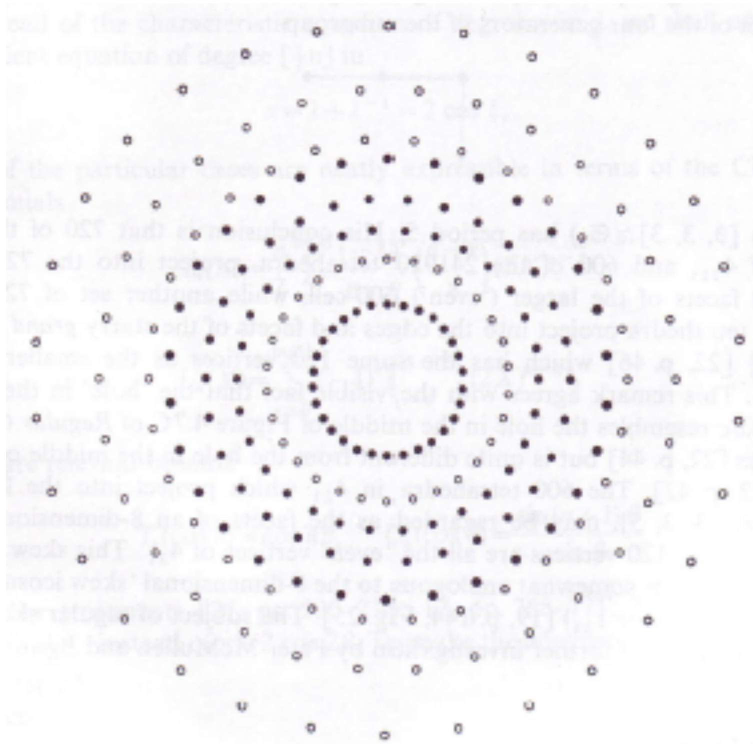


Fig. 3.8c. The eight-dimensional polytope 4_{21}

... The 120+120 vertices of the polytope 4_21 ...



...[are]... the 120+120 jvertices of two homothetic 600-cells {3,3,5}:

one having the coordinates ...[with T being the Golden Ratio]...

the even permutations of (+/- T , +/- 1 , +/- T^(-1) , 0) ,
the permutations of (+/- 2 , 0 , 0 , 0) ,
and (+/- 1 , +/- 1 , +/- 1 , +/- 1)
...[a total of 8x(1/2)x4! + 2x4 + 16x1 = 96+8+16 = 120]...

... while
the other has these same coordinates multiplied by T ...".

One 600-cell represents half of the 240 E8 Root Vector vertices:

56 of D8 vertices =
(12 of D4 + 12 of D4) = 24 vertices from D4xD4 subalgebra of D8
plus
32 = 8x4 vertices from the coset space D8 / D4xD4.

64 of the D8 half-spinor vertices = 32 ++half-half-spinors + 32 --half-half-spinors.

The 600-cell lives in a 3-dim sphere inside 4-dim Space. It is half of the E8 vertices.
With respect to the 3-sphere S3 the 120 vertices of the 600-cell look like:

1 - North Pole - Single Point (projected to center of Equatorial Icosidodecahedron)
12 - Arctic Circle - Icosahedron - half of 24 Root Vectors of one of the E8 D4
20 - North Temperate Zone - Dodecahedron part of Rhombic Triacontahedron
12 - Tropic of Cancer - Icosahedron part of Rhombic Triacontahedron
30 - Equator - Icosidodecahedron
12 - Tropic of Capricorn - Icosahedron part of Rhombic Triacontahedron
20 - South Temperate Zone - Dodecahedron part of Rhombic Triacontahedron
12 - Antarctic Circle - Icosahedron - half of 24 Root Vectors of another E8 D4
1 - South Pole - Single Point (projected to center of Equatorial Icosidodecahedron)

The colors represent E8 Physics Model physical interpretation:
Conformal Gravity Root Vector Gauge Bosons
Fermion Particles
Spacetime position and momentum
Fermion Antiparticles
Standard Model Gauge Bosons
Sadoc and Mosseri in their book "Geometric Frustration" (Cambridge 2006) Fig. A51
illustrate the shell structure of the 120 vertices of a 600-cell:

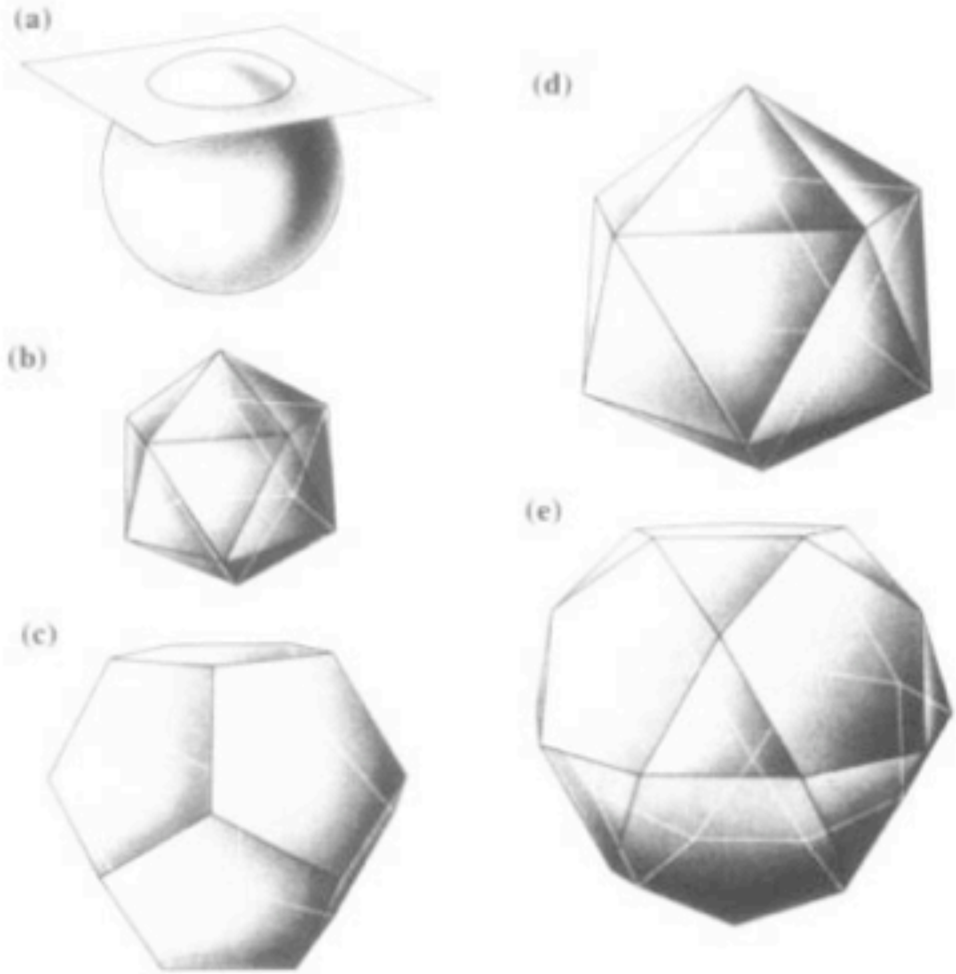
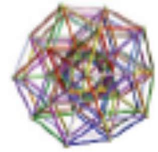


Fig. A5.1. The {3, 3, 5} polytope. Different flat sections in S^3 (with one site on top) give the following successive shells; (a) an icosahedral shell formed by the first 12 neighbours, (b) a dodecahedral shell, (c) a second and larger icosahedral shell, (d) an icosidodecahedral shell on the equatorial sphere. Then other shells are symmetrically disposed in the second 'south' hemi-hypersphere, relative to the equatorial sphere (e).

The 30-vertex Icosidodecahedron (e) cannot tile flat 3-dim space. Its dual, the 32-vertex Rhombic Triacontahedron, is a combination of the 12-vertex Icosahedron (d) and the 20-vertex Dodecahedron (c). It "forms the convex hull of ... orthographic



projection ... using the Golden ratio in the basis vectors ... of a 6-cube to 3 dimensions." (Wikipedia).

Physical Interpretation of

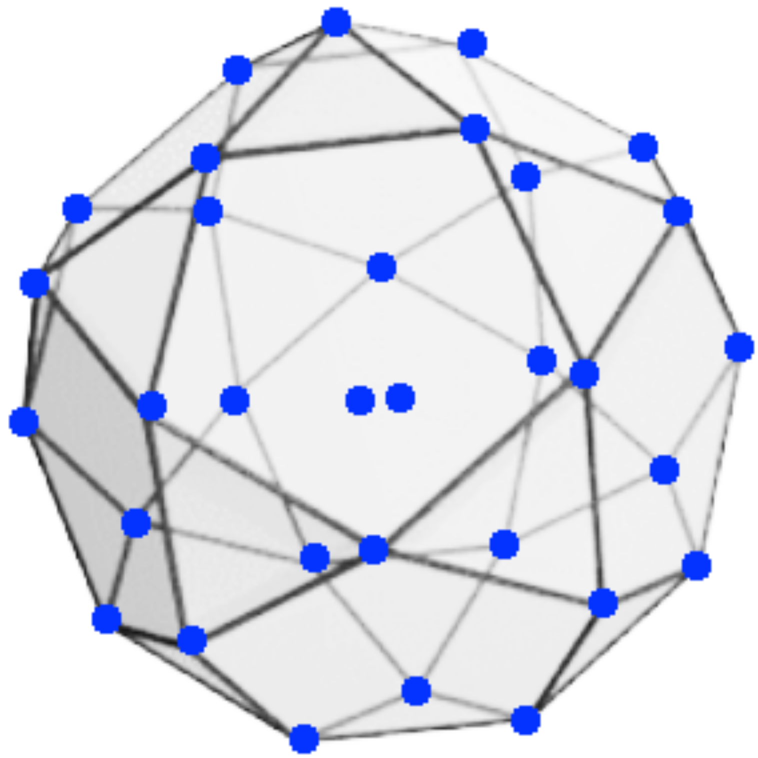
1 - North Pole - Single Point (projected to center of Equatorial Icosidodecahedron)
30 - Equator - Icosidodecahedron
1 - South Pole - Single Point (projected to center of Equatorial Icosidodecahedron)

is

8 components of 8-dim Kaluza-Klein M4xCP2 Spacetime Position

times

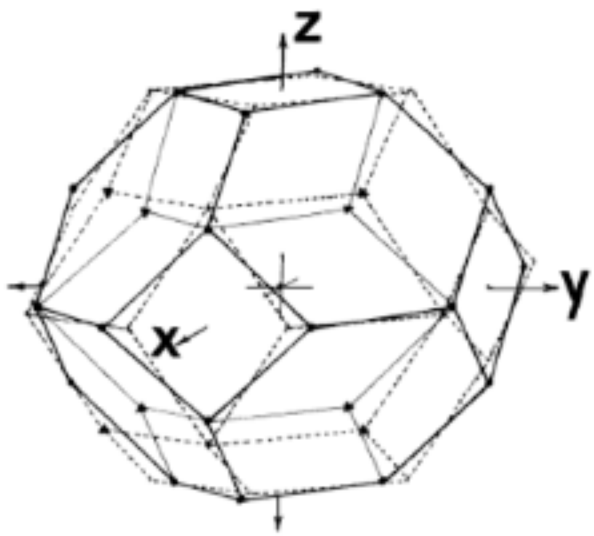
4 components of 4-dim M4 Physical Spacetime Momentum

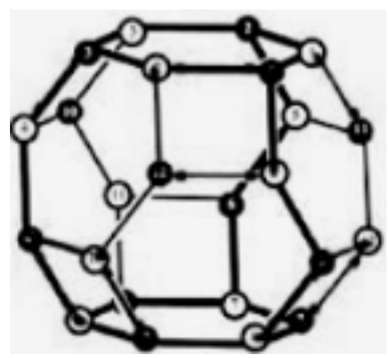


There are 64 - 32 = 32 of the 240 E8 in the half of E8 that did not go to the 600-cell. They correspond to 8 components of Position x 4 components of momentum in CP2. Since the CP2 Internal Symmetry Space is the small compactified part of M4xCP2 momentum in CP2 is substantially irrelevant to our 3-dim space M4 world.

The 30 Icosadodecahedron vertices are at pairwise intersections of 6 Great Circle Decagons. Let each Great Circle represent a generator of a spacetime translation. Then the Icosadodecahedron represents a 6-dim Spin(2,4) Conformal spacetime that acts conformally on 4-dim M4 Minkowski Physical Spacetime that lives inside 8-dim Kaluza-Klein M4xCP2 Spacetime (where CP2 = SU(3) / U(2)). Physically the 6 Great Circles of the Icosadodecahedron show that the 10-dim space of 26-dim String Theory of Strings as World-Lines reduces to 6-dim Conformal Physical Spacetime plus 4-dim CP2 Internal Symmetry Space.

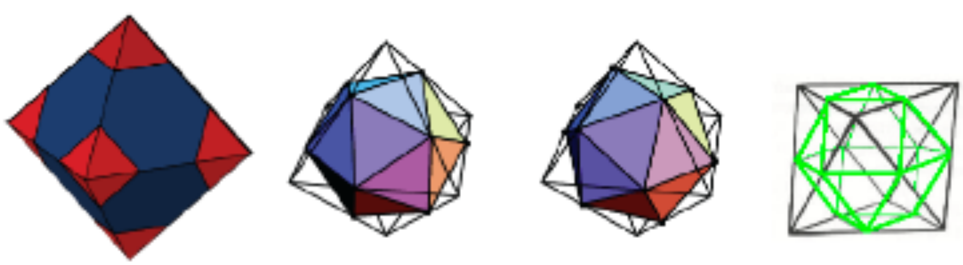
The 32-vertex Rhombic Triacontahedron does not itself tile 3-dim space
but it is important in 3-dim QuasiCrystal tiling.
Mackay (J. Mic. 146 (1987) 233-243) said "... the basic cluster, to be observed everywhere in the three-dimensional ... Penrose ... tiling ...[is]... a rhombic triacontahedron (RTH) ... The 3-D tiling can be regarded as an assembly of such RTH, party overlapping
...
a rhombic triacontahedron (RTH) ... can be deformed to ... a truncated octahedron ... [which is] the space-filling polyhedron for body-centered cubic close packing ...





By a similar process ... a cuboctahedr[on]... can be deformed to an icosahedron ...".

In the latter process, the Jitterbug, sets of points on the edges of an Octahedron correspond to the vertices of the Truncated Octahedron (1/3 and 2/3),
a pair of Icosahedra (Golden Ratio Points), and the Cuboctahedron (Mid-Point).



but the Rhombic Triacontahedron deformation process involves moving its vertices somewhat off the exact edges of the Octahedron and in adding to the 24 vertices of the Truncated Octahedron 8 more vertices corresponding to centers of its hexagonal faces and making 3 rhombohedral faces from each of its hexagonal faces.

Since the 3-dim space itself is due to the Icosidodecahedron Spacetime,
construction of a 3-dim space version of the E8 Physics Model does not require
tiling of 3-dim space by Icosidodecahedra (it would be redundant and inconsistent
to tile space with space) but
it is useful to consider tiling 3-dim space with the fermion particles and gauge bosons
that are actors on the stage of space, and they correspond to Rhombic Triacontahedra,
and
there are two ways to look at tiling 3-dim space by Rhombic Triacontahedra:

1 - Make a 3-dim QuasiCrystal of Rhombic Triacontaheda, partly overlapping,
as suggested by Mackay (J. Mic. 146 (1987) 233-243).

2 - Deform the Rhombic Triacontahedra to Truncated Octahedra and tile 3-space
with the Truncated Octahedra.

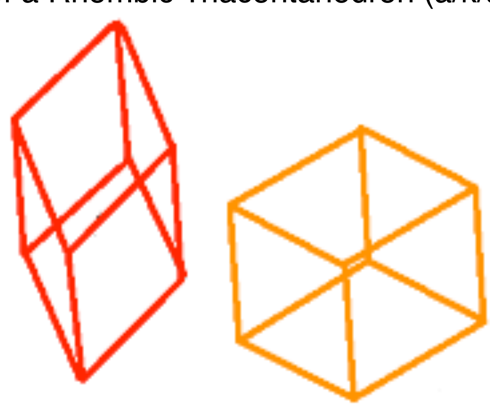
Whichever way is chosen, the first step is to describe the physical interpretation
of the Rhombic Triacontahedra, beginning with

20 - North Temperate Zone - Dodecahedron part of Rhombic Triacontahedron
12 - Tropic of Cancer - Icosahedron part of Rhombic Triacontahedron
and
12 - Tropic of Capricorn - Icosahedron part of Rhombic Triacontahedron
20 - South Temperate Zone - Dodecahedron part of Rhombic Triacontahedron

which are interpreted as fermion particles and fermion antiparticles, respectively.

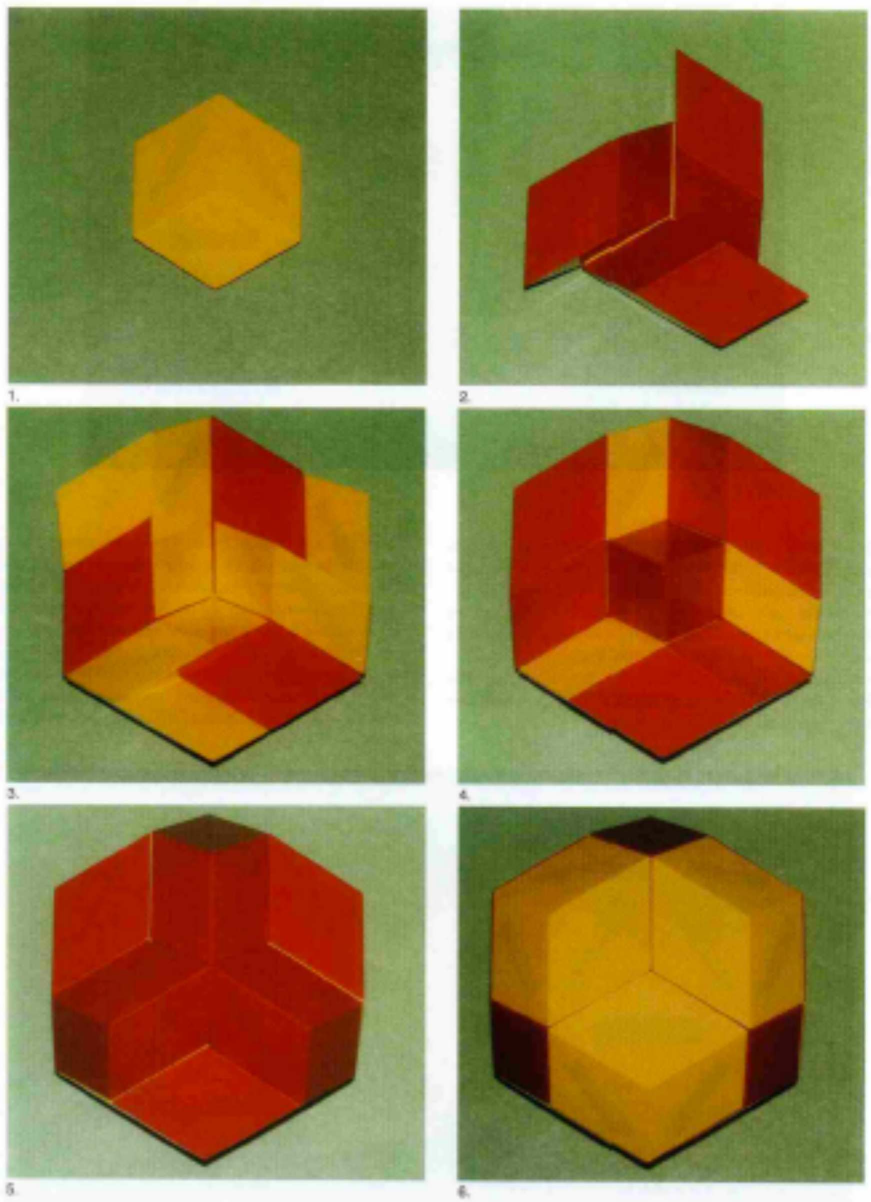
Since fermion particles are inherently Left-Handed
and fermion antiparticles are inherently Right-Handed,
the Rhombic Triacontahedra representing each should be constructed correspondingly
and
units of the 3-space tiling should contain a superposition of both Left and Right RTH,
as well as third RTH with no handedness to describe Gravity and the Standard Model.

The basic building blocks of a Rhombic Triacontahedron (a/k/a Kepler Ball) are



two golden rhombohedra (sharp "S" and flat "F") , using 10 of each.

Construction of Left-Handed and Right-Handed Rhombic Triacontahedra is described by Michael S. Longuet-Higgins in " Nested Triacontahedral Shells Or How to Grow a Quasi-crystal" (Mathematical Intelligencer 25 (Spring 2003) 25-43):
"... start with a flat rhombohedron,
placing on it three sharp rhombohedra in a left-handed symmetric way
and building up the rest of the ball maintaining always a three-fold axis of rotational symmetry ...



... (We could also start with right-handed symmetry, producing the mirror image.) ...".

Physical Interpretation of

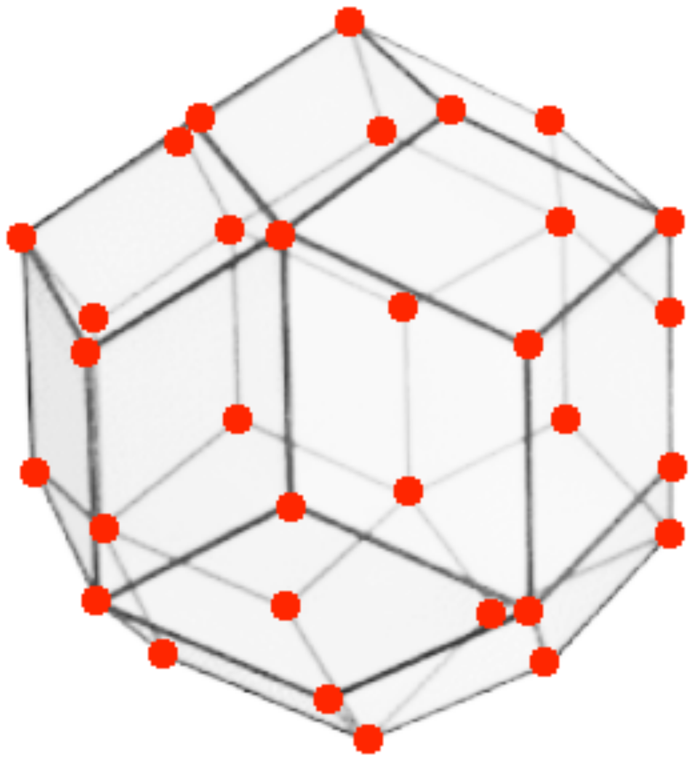
20 - North Temperate Zone - Dodecahedron part of Rhombic Triacontahedron
12 - Tropic of Cancer - Icosahedron part of Rhombic Triacontahedron

is

8 fundamental first-generation fermion particles

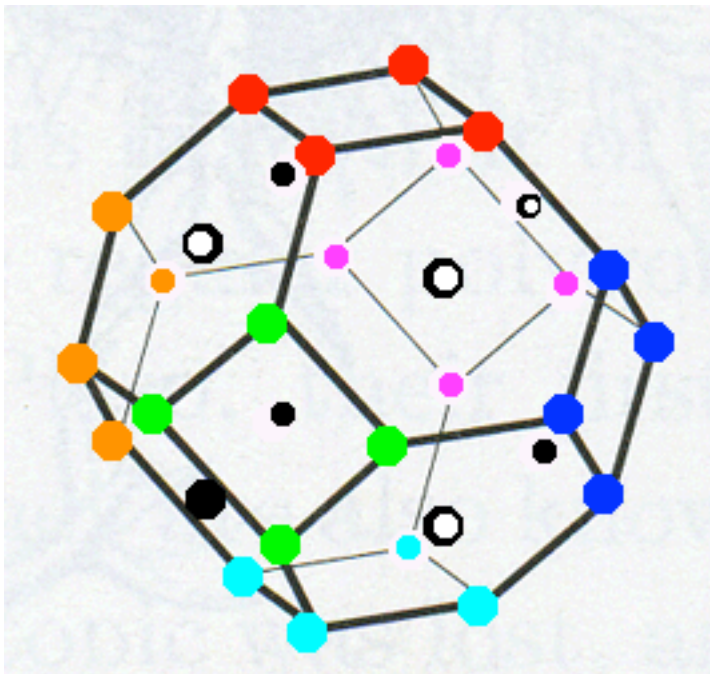
times

4 covariant components of 4-dim M4 Physical Spacetime Momentum



Left-Handed Rhombic Triacontahedron Kepler Ball.

As to which vertices correspond to which Fermion Particles or Antiparticles
the Truncated Octahedron point of view with 6 sets of 4 vertices for quarks
and 2 sets of 4 hexagon-centers for leptons, showing the 4 covariant components
with respect to M4 Physical Spacetime for each Fermion, is useful:



neutrino, red down quark, green down quark, blue down quark;

blue up quark, greem up quark , red up quark, electron

(orange, magenta, cyan, black are used for blue, green, red up quarks and electron)

Physical Interpretation of

12 - Tropic of Capricorn - Icosahedron part of Rhombic Triacontahedron

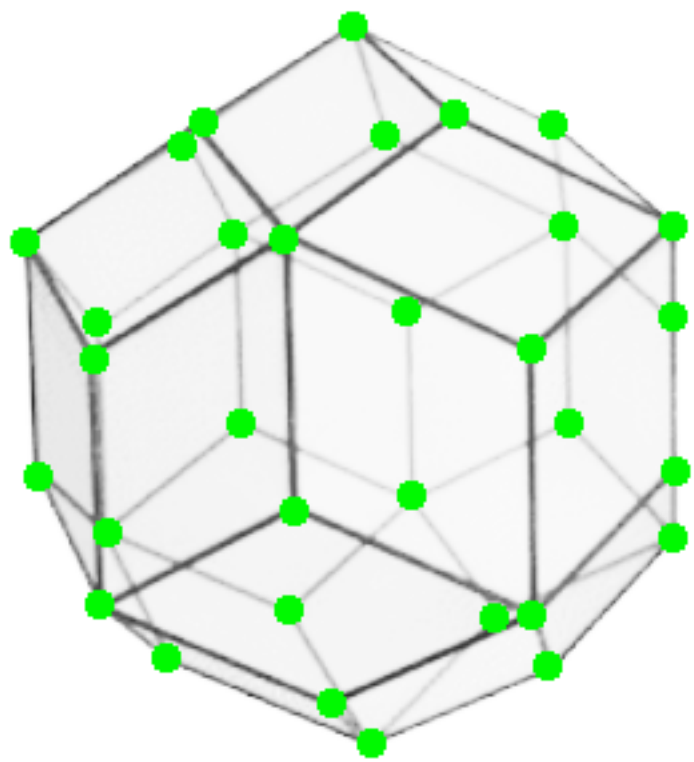
20 - South Temperate Zone - Dodecahedron part of Rhombic Triacontahedron

is

8 fundamental first-generation fermion antiparticles

times

4 covariant components of 4-dim M4 Physical Spacetime Momentum

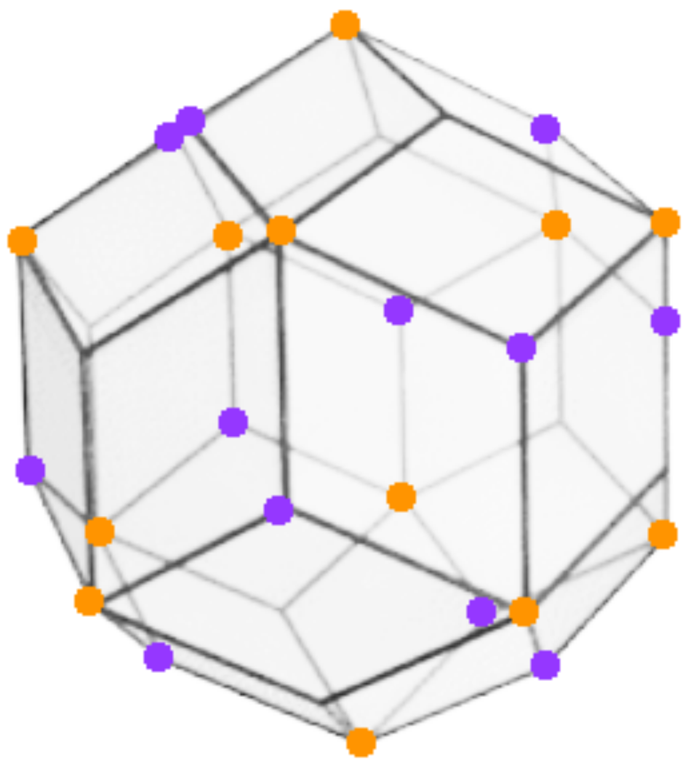


Right-Handed Rhombic Triacontahedron Kepler Ball.

Physical interpretation of the Rhombic Triacontahedra also includes

12 - Arctic Circle - Icosahedron - half of 24 Root Vectors of one of the E8 D4
and
12 - Antarctic Circle - Icosahedron - half of 24 Root Vectors of another E8 D4

which are interpreted as Gauge Bosons for Gravity and the Standard Model, respectively.

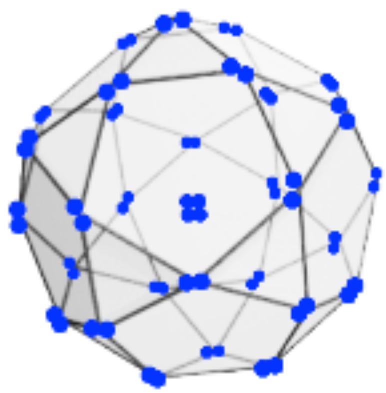


Rhombic Triacontahedron Kepler Ball with no handedness.

12 of the 20 3-edge vertices are 12 D4 Root Vectors for the Standard Model that combine with 4 of the 8 E8 Cartan SubAlgebra generators to form 12+4 = 16-dim U(4) that contains the Batakis Color Force SU(3) that gives the Standard Model through CP2 = SU(3) / U(1)xSU(2).
The 20-12 = 8 3-edge vertices that are not used correspond to the centers of the hexagonal faces of the Truncated Octahedron related to the Kepler Ball.

12 5-edge vertices are 12 D4 Root Vectors for Conformal Gravity
that combine with 4 of the 8 E8 Cartan SubAlgebra generators to
form 12+4 = 16-dim U(2,2) = U(1)xSU(2,2) where SU(2,2) = Spin(2,4)
The Conformal Lie Algebra SU(2,2) = Spin(2,4) has 15 dimensions,
and Longuet-Higgins (Mathematical Intelligencer 25 (Spring 2003) 25-43) says
"... a Kepler Ball may be thought of as a can of 15 worms
... with 3 worms passing through the centre of each rhombohedron
...
define a worm as a line drawn from the center of one face w1 of a Kepler Ball to the center of the opposite face w2 of the corresponding golden rhombohedron; then from w2 to the opposite face w3 of the adjacent rhomobohedron, and so on, ending at the face wn of the Kepler Ball opposite to w1. Thus a Kepler Ball may be thought of as a can of 15 worms, with 3 worms passing through the centre of each rhombohedron. The two ends of the worm lie on two opposite faces of the Ball. ... all of ... the worms ... will ... pass through two F's and two S's. ...".

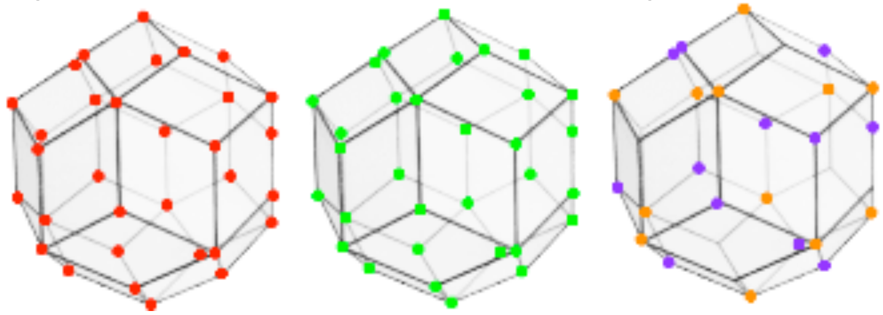
Compare the 15 worms based on faces of the Kepler Ball Rhombic Triacontahedron with the 30-vertex structure of its dual the Icosidodecahedron



whose physical interpretation is Spacetime. As the 6 Great Circle Decagons of the Icosidodecahedron represent 6-dim Conformal Physical Space and as the 15 worms represent the 15 antipodal pairs of the 30 Icosidodecahdron vertices and as each antipodal pair of vertices corresponds to a pair of Great Circle Decagons the 15 worms represent the 15 generators of the Conformal Group Spin(2,4) = SU(2,2).

3-space tiled by Deformation or QuasiCrystal

For tiling of 3-space the basic Rhombic Triacontahedra Kepler Ball should contain all 3:

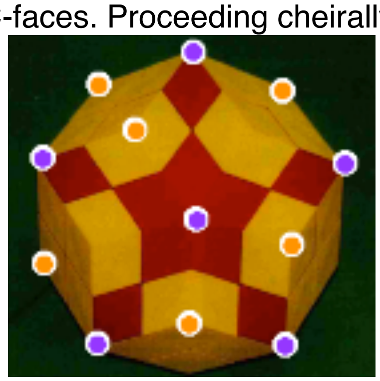


Left-Handed for Fermion Particles, Right-Handed for Fermion Antiparticles,
and no handedness for Gauge Bosons of Gravity and the Standard Model.

To construct such a 3-type Rhombic Triacontahedron Kepler Ball:

Start with a Left-Handed Kepler Ball for Fermion Particles and denote it by K(1).
Then, using K(1) as a nucleus, construct a K(2) Kepler Ball by adding to the K(1)

sharp "S"  and flat "F"  golden rhombohedra with dihedral angles
2 pi / 5 or 3 pi / 5 for S and pi / 5 or 4 pi / 5 for F as described in
" Nested Triacontahedral Shells Or How to Grow a Quasi-crystal"
by Michael S. Longuet-Higgins (Mathematical Intelligencer 25 (Spring 2003) 25-43):
"... To construct a K(2) ... label the thirty faces of a K(1) as follows: call
the five faces surrounding a given pentagonal vertex A's; the five adjoining faces B's;
the next ten adjoining faces (which are all parallell to the pentagonal axis) C's;
the next five D's; and the last five E's.
Taking the K(1), leave the A-faces bare and lay one F on each B-face.
Next lay an S on each of the C-faces. Proceeding cheirally ... we ... arrive ... at a K(2) ...



[I have added purple and orange indicators for K(2) vertices representing some of
the Root Vectors of U(2,2) for Conformal Gravity and of U(4) for the Standard Model.]
The view from the opposite end is similar ...
there is a second K(1), coaxial with the first, along the pentagonal axis ...".

K(1) is for Fermion Particles, second K(1) is for Fermion Antiparticles,
and K(2) is for Gauge Bosons of Gravity and the Standard Model.
K(2), containing Particle-Antiparticle Pairs, is the Basic Tiling Kepler Ball.

As remarked earlier,

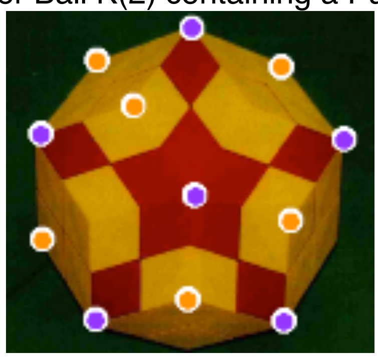
there are two ways to look at tiling 3-dim space by Rhombic Triacontahedra:

1 - **Make a 3-dim QuasiCrystal of Rhombic Triacontaheda**, partly overlapping, as suggested by Mackay (J. Mic. 146 (1987) 233-243).

2 - **Deform the Rhombic Triacontahedra to Truncated Octahedra** and tile 3-space with the Truncated Octahedra.

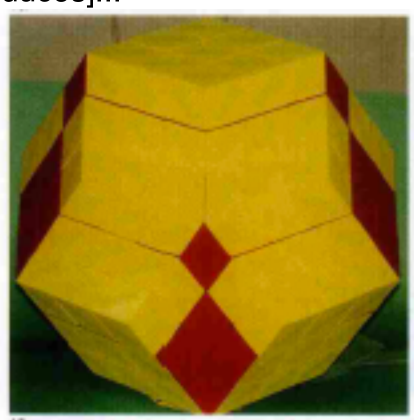
1 - Make a 3-dim QuasiCrystal of Rhombic Triacontaheda, partly overlapping, as suggested by Mackay (J. Mic. 146 (1987) 233-243).

Start with the Basic Tiling Kepler Ball K(2) containing a Particle-Antiparticle pair of K(1)s



Then adding to the K(2) sharp "S"  and flat "F"  golden rhombohedra construct a larger Rhombic Triacontahedron Kepler Ball K(3). Continue the process, adding to each K(n) sharp "S" and flat "F" golden rhombohedra to form K(n+1).
There are a number of ways to do that. One that I like is described in
" Nested Triacontahedral Shells Or How to Grow a Quasi-crystal"
by Michael S. Longuet-Higgins (Mathematical Intelligencer 25 (Spring 2003) 25-43):
"... in general ... it is possible ...to derive a Kepler Ball K(n+1) of side n+1 from a K(n) ...
Define a carpet of rhombohedra as an (n x n x 1) array of golden rhombohedra (of the same kind), covering an n x n rhombic face such as b(n), for example.
All the rhombohedra are oriented identically.
A fringe is an (n x 1 x 1) array, oriented similarly, adjoining the "edge" of two different arrays, and a tassel is a single cell, i.e., a (1 x 1 x 1) array at the join or extension of two or more fringes. ...
(1) Leave the a(n)-faces bare, and cover each of the b(n)-faces with a carpet of F's.
(2) Complete the a(n+1)'s with three fringes of F's and
lay a carpet of S's on each of the c(n)-faces.
(3) Turn the emodel over. Lay a carpet of S's on each of the d(n)-faces.
(4) Lay a carpet of F's cheirally on each e(n)-face and

a carpet of S's on each f(n)-face, with a cheiral fringe of S's.
(5) Lay a second carpet of F's, cheirally, on each of the carpets coveriing the e(n)-faces.
(6) Lay a carpet of F's on each of the f(n)-faces, and
fill in with fringes of F's and a tassel in the centre.
The latter will be the start of a coaxial [K*(1)].
(7) Cover the upper surface cheirally with a layer of F's,
leaving three zigzag canyons meeting at the centre.
(8) Fill in the canyons with F's and S's.
(9) Cover the F's with a layer of S's.
(10) Complete the [d(n)-face] with a carpet of F's. (This also completes the [e(n)-faces].)
(11) Add F's to complete the K(n+1).
... the outer shell is [not] cheiral
...
the whole K(n+1) is covered by a layer of rhombohedra no more than four deep
...
[such a] construction of K(3) from K(2) ...[produces]...



... in many respects the particular arrangements described here are not unique.
For example, in places where a triacontahedron occurs locally, ...[it]... may be replaced by a ... [triacontahedron of a different type] ...

the method of assembly ...
does not require the existence of such long-range forces
as would be needed to assemble an Ammann tiling

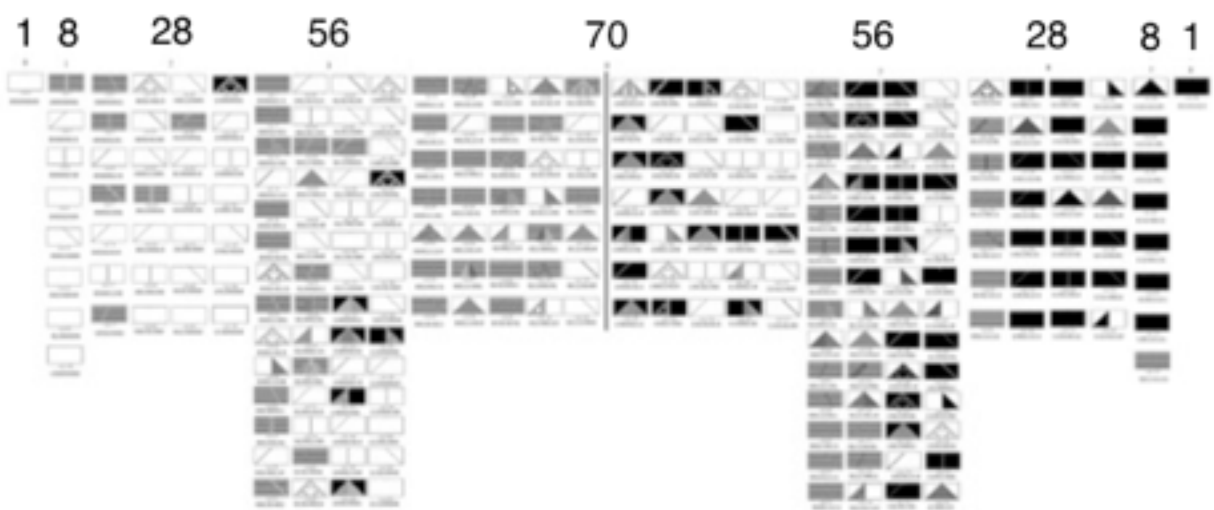
... ".

As Mackay (J. Mic. 146 (1987) 233-243) said "... the basic cluster, to be observed everywhere in the three-dimensional ... Penrose ... tiling ...[is]... a rhombic triacontahedron (RTH) ...
The 3-D tiling can be regarded as an assembly of such RTH, party overlapping ...".

2 - Deform the Rhombic Triacontahedra to Truncated Octahedra and tile 3-space with the Truncated Octahedra

Mackay (J. Mic. 146 (1987) 233-243) said "...a rhombic triacontahedron (RTH) ... can be deformed to ... a **truncated octahedron** ...
[which is] the **space-filling polyhedron for body-centered cubic close packing** ...".
Such a lattice of Truncated Octahedra (image from realwireless)

can form the basis for the spatial part of a 4-dim Feynman Checkerboard representation of the E8 Physics Model,
with the Feynman Checkerboard Rules being related to the 256 Cellular Automata corresponding to the 256 elements of the Cl(8) Clifford Algebra of the E8 Physics Model



Appendix A: Quaternionic View of E8 Lattices

1E8 has 256 first-shell vertices of the 8-dim uniform Octionionic form
(+/- 1 +/- i +/- j +/- k +/- e +/- ie +/- je +/- ke) / 2
that is consistent with the structure of the 256-dim Clifford Algebra Cl(8).

Introduction of Quaternionic substructure such as that related to (4+4)-dim M4 x CP2 Kaluza-Klein SpaceTime requires breaking the Octonionic 8-term form (+/- 1 +/- i +/- j +/- k +/- e +/- ie +/- je +/- ke) / 2 into Quaternionic 4-term forms like (+/- A +/- B +/- C +/- D) / 2.

To do that, consider that there are (8l4) = 70 ways to choose 4-term subsets of the 8 Octonionic basis element terms. Using all of them produces 224 4-term subsets in each of the 7 Octonion Imaginary E8 lattices iE8,jE8,kE8,eE8,ieE8,jeE8,keE8 each of which also has 16 1-term first-shell vertices.

56 of the 70 4-term subsets appear as 8 in each of the 7 Octonion Imaginary E8 lattices.

The other 70-56 = 14 4-term subsets occur in sets of 3 among 7x6 = 42 4-term subsets as indicated in the following detailed list of the 7 Octonion Imaginary E8 lattices:

eE8:

```
112 of D8 Root Vectors
16 appear in all 7 of iE8,jE8,kE8,eE8,ieE8,jeE8,keE8
±1, ±i, ±j, ±k, ±e, ±ie, ±je, ±ke
96 appear in 3 of iE8,jE8,kE8,eE8,ieE8,jeE8,keE8
(±1  ±ke ±e  ±k )/2     (±i  ±j  ±ie ±je)/2     kE8  , eE8  , keE8
(±1  ±je ±j  ±e )/2     (±ie ±ke ±k  ±i )/2     jE8  , eE8  , jeE8
(±1  ±e  ±ie ±i )/2     (±ke ±k  ±je ±j )/2     iE8  , eE8  , ieE8

128 of D8 half-spinors appear only in eE8
(±1  ±ie ±je ±ke)/2     (±e  ±i  ±j  ±k )/2
(±1  ±k  ±i  ±je)/2     (±j  ±ie ±ke ±e )/2
(±1  ±i  ±ke ±j )/2     (±k  ±je ±e  ±ie)/2
(±1  ±j  ±k  ±ie)/2     (±je ±e  ±i  ±ke)/2
```

iE8:

```
112 of D8 Root Vectors
16 appear in all 7 of iE8,jE8,kE8,eE8,ieE8,jeE8,keE8
±1, ±i, ±j, ±k, ±e, ±ie, ±je, ±ke
96 appear in 3 of iE8,jE8,kE8,eE8,ieE8,jeE8,keE8
(±1  ±ie ±i  ±e )/2     (±j  ±k  ±je ±ke)/2     iE8  , eE8  , ieE8
(±1  ±ke ±je ±i )/2     (±j  ±k  ±e  ±ie)/2     iE8  , jeE8 , keE8
(±1  ±i  ±k  ±j )/2     (±e  ±ie ±je ±ke)/2     iE8  , jE8  , kE8

128 of D8 half-spinors appear only in iE8
(±1  ±k  ±ke ±ie)/2     (±i  ±j  ±e  ±je)/2
(±1  ±e  ±j  ±ke)/2     (±i  ±k  ±ie ±je)/2
(±1  ±j  ±ie ±je)/2     (±i  ±k  ±e  ±ke)/2
(±1  ±je ±e  ±k )/2     (±i  ±j  ±ie ±ke)/2
```

jE8:

```
112 of D8 Root Vectors
16 appear in all 7 of iE8,jE8,kE8,eE8,ieE8,jeE8,keE8
±1, ±i, ±j, ±k, ±e, ±ie, ±je, ±ke
96 appear in 3 of iE8,jE8,kE8,eE8,ieE8,jeE8,keE8
(±1  ±k  ±j  ±i )/2     (±e  ±ie ±je ±ke)/2     iE8  , jE8  , kE8
(±1  ±ie ±ke ±j )/2     (±i  ±k  ±e  ±je)/2     jE8  , ieE8 , keE8
(±1  ±j  ±e  ±je)/2     (±i  ±k  ±ie ±ke)/2     jE8  , eE8  , jeE8

128 of D8 half-spinors appear only in jE8
(±1  ±e  ±ie ±k )/2     (±i  ±j  ±je ±ke)/2
(±1  ±i  ±je ±ie)/2     (±j  ±k  ±e  ±ke)/2
(±1  ±je ±k  ±ke)/2     (±i  ±j  ±e  ±ie)/2
(±1  ±ke ±i  ±e )/2     (±j  ±k  ±ie ±je)/2
```

kE8:

```
112 of D8 Root Vectors
16 appear in all 7 of iE8,jE8,kE8,eE8,ieE8,jeE8,keE8
±1, ±i, ±j, ±k, ±e, ±ie, ±je, ±ke
96 appear in 3 of iE8,jE8,kE8,eE8,ieE8,jeE8,keE8
(±1  ±je ±k  ±ie)/2     (±i  ±j  ±e  ±ke)/2     kE8  , ieE8 , jeE8
(±1  ±j  ±i  ±k )/2     (±e  ±ie ±je ±ke)/2     iE8  , jE8  , kE8
(±1  ±k  ±ke ±e )/2     (±i  ±j  ±ie ±je)/2     kE8  , eE8  , keE8

128 of D8 half-spinors appear only in kE8
(±1  ±ke ±j  ±je)/2     (±i  ±k  ±e  ±ie)/2
(±1  ±ie ±e  ±j )/2     (±i  ±k  ±je ±ke)/2
(±1  ±e  ±je ±i )/2     (±j  ±k  ±ie ±ke)/2
(±1  ±i  ±ie ±ke)/2     (±j  ±k  ±e  ±je)/2
```

ieE8:

```
112 of D8 Root Vectors
16 appear in all 7 of iE8,jE8,kE8,eE8,ieE8,jeE8,keE8
±1, ±i, ±j, ±k, ±e, ±ie, ±je, ±ke
96 appear in 3 of iE8,jE8,kE8,eE8,ieE8,jeE8,keE8
(±1  ±j  ±ie ±ke)/2     (±i  ±k  ±e  ±je)/2     jE8  , ieE8 , keE8
(±1  ±i  ±e  ±ie)/2     (±j  ±k  ±je ±ke)/2     iE8  , eE8  , ieE8
(±1  ±ie ±je ±k )/2     (±i  ±j  ±e  ±ke)/2     kE8  , ieE8 , jeE8

128 of D8 half-spinors appear only in ieE8
(±1  ±je ±i  ±j )/2     (±k  ±e  ±ie ±ke)/2
(±1  ±ke ±k  ±i )/2     (±j  ±e  ±ie ±je)/2
(±1  ±k  ±j  ±e )/2     (±i  ±ie ±je ±ke)/2
(±1  ±e  ±ke ±je)/2     (±i  ±j  ±k  ±ie)/2
```

jeE8:

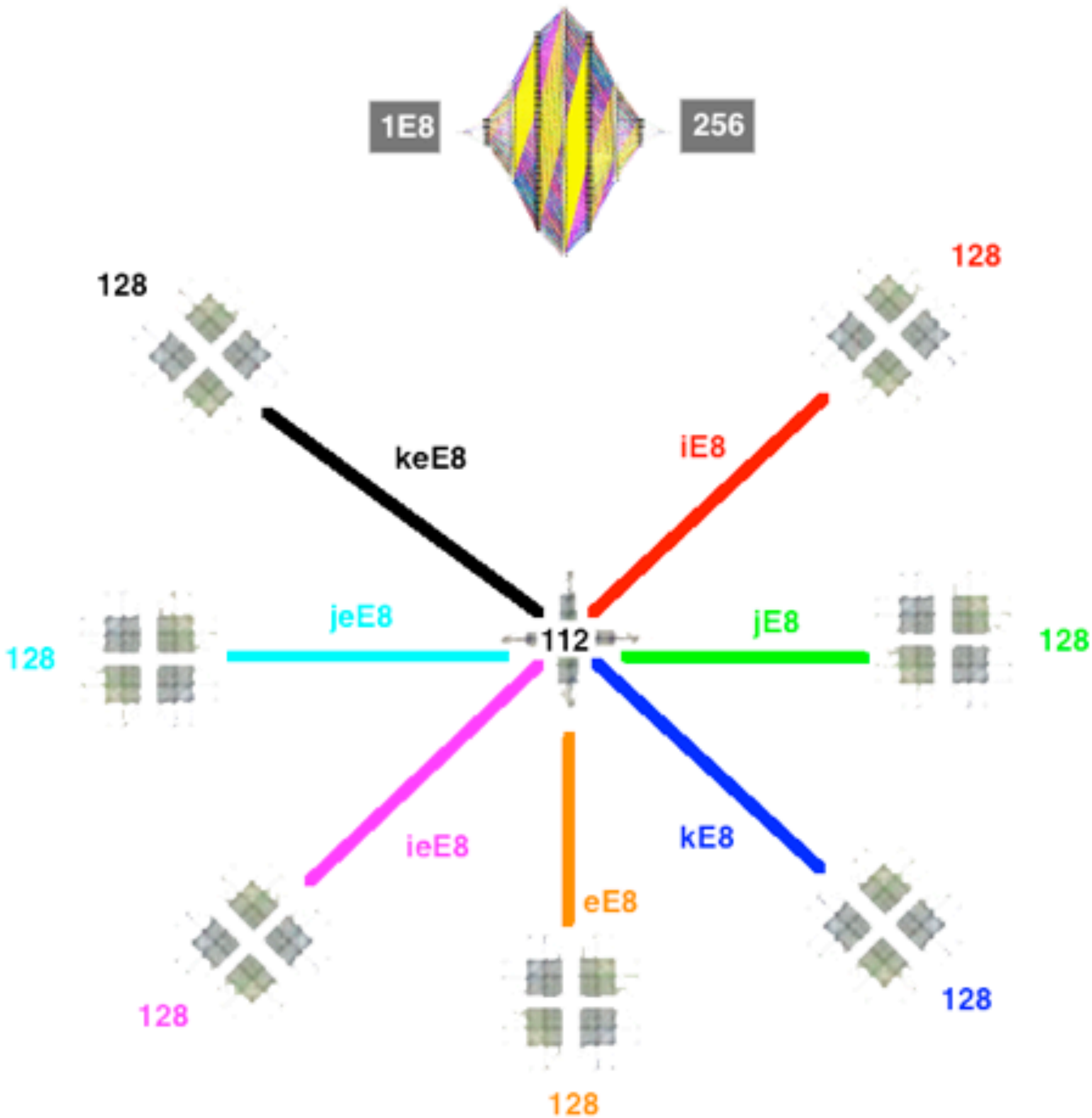
```
112 of D8 Root Vectors
16 appear in all 7 of iE8,jE8,kE8,eE8,ieE8,jeE8,keE8
±1, ±i, ±j, ±k, ±e, ±ie, ±je, ±ke
96 appear in 3 of iE8,jE8,kE8,eE8,ieE8,jeE8,keE8
(±1  ±e  ±je ±j )/2     (±i  ±k  ±ie ±ke)/2     jE8  , eE8  , jeE8
(±1  ±k  ±ie ±je)/2     (±i  ±j  ±e  ±ie)/2     kE8  , ieE8 , jeE8
(±1  ±je ±i  ±ke)/2     (±j  ±k  ±e  ±ie)/2     iE8  , jeE8 , keE8

128 of D8 half-spinors appear only in jeE8
(±1  ±i  ±k  ±e )/2     (±j  ±ie ±je ±ke)/2
(±1  ±j  ±ke ±k )/2     (±i  ±e  ±ie ±je)/2
(±1  ±ke ±e  ±ie)/2     (±i  ±j  ±k  ±je)/2
(±1  ±ie ±j  ±i )/2     (±k  ±e  ±je ±ke)/2
```

keE8:

```
112 of D8 Root Vectors
16 appear in all 7 of iE8,jE8,kE8,eE8,ieE8,jeE8,keE8
±1, ±i, ±j, ±k, ±e, ±ie, ±je, ±ke
96 appear in 3 of iE8,jE8,kE8,eE8,ieE8,jeE8,keE8
(±1  ±i  ±ke ±je)/2     (±j  ±k  ±e  ±ie)/2     iE8  , jeE8 , keE8
(±1  ±e  ±k  ±ke)/2     (±i  ±j  ±ie ±je)/2     kE8  , eE8  , keE8
(±1  ±ke ±j  ±ie)/2     (±i  ±k  ±e  ±je)/2     jE8  , ieE8 , keE8

128 of D8 half-spinors appear only in keE8
(±1  ±j  ±e  ±i )/2     (±k  ±ie ±je ±ke)/2
(±1  ±je ±ie ±e )/2     (±i  ±j  ±k  ±ke)/2
(±1  ±ie ±i  ±k )/2     (±j  ±e  ±je ±ke)/2
(±1  ±k  je  ±j )/2     (±i  ±e  ±ie ±ke)/2
```



1E8
256
128
keE8
128
iE8
jeE8
112
jE8
128
128
ieE8
kE8
eE8
128
128
128