

# RIEMANN ZEROS QUANTUM CHAOS FUNCTIONAL DETERMINANTS AND TRACE FORMULAE

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ABSTRACT: We study the relation between the Guzwiller Trace for a dynamical system and the Riemann-Weil trace formula for the Riemann zeros, using the Bohr-Sommerfeld quantization condition and the fractional calculus we obtain a method to define implicitly a potential  $f^{-1}(x)$ , we apply this method to define a Hamiltonian whose energies are the square of the Riemann zeros (imaginary part)  $E_n = \gamma_n^2$ , also we show that for big 'x' the potential is very close to an exponential function.

In this paper and for simplicity we use units so  $2m = 1 = \hbar$

- *Keywords:* = Riemann Hypothesis, WKB semiclassical approximation, Gutzwiller trace formula, Bohr-Sommerfeld quantization, exponential potential.

## 1. RIEMANN ZEROS AND TRACE FORMULAE

Given a Hamiltonian in one dimension plus boundary conditions on the real line  $[0, \infty)$

$$Hy(x) = E_n y(x) = -\frac{\hbar^2}{2m} \frac{d^2 y(x)}{dx^2} + f(x)y(x) \quad y(0) = 0 = y(\infty) \quad (1.1)$$

Can we recover the potential  $f(x)$  from spectral data ?, for example if we knew the Eigenvalue staircase of the problem (1.1)  $N(E) = \sum_{n=0}^{\infty} H(E - E_n)$ , then we could use the Bohr-Sommerfeld quantization condition, see [9] for our problem as

$$2\pi\hbar\left(n + \frac{1}{2}\right) = \iint_{R^2} dx dp H(E - H(qx, p)) = 2\sqrt{2m} \int_0^{a(E)} \sqrt{E_n - V(x)} dx = 2\sqrt{2m} \int_0^E \sqrt{E_n - x} \frac{df^{-1}}{dx} = \sqrt{2m} \sqrt{\pi} D_x^{-\frac{1}{2}} \quad (1.2)$$

The number 'a' is a turning point where the momentum is  $p = 0$  so  $f(a) = E$

The idea of the Borh-Sommerfeld quantization condition (1.2) is the following, we equate the smooth part of the spectral staircase to an integer plus 1/2

$$\langle N(E) \rangle = \frac{1}{2\pi\hbar} \iint_{R^2} dx dp H(E - H(qx, p)) = n + \frac{1}{2} \quad H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \quad (1.3)$$

Here , inside (1.2) we have used the definitions of the fractional derivative and integral of order 1/2. [10] (for fractional calculus)

$$\frac{d^{\frac{1}{2}} f(x)}{dx^{\frac{1}{2}}} = \frac{1}{\Gamma(1/2)} \frac{d}{dx} \int_0^x \frac{dt f(t)}{\sqrt{x-t}} \quad \frac{d^{-\frac{1}{2}} f(x)}{dx^{-\frac{1}{2}}} = \frac{1}{\Gamma(1/2)} \int_0^x dt \frac{f(t)}{\sqrt{x-t}} \quad (1.4)$$

Also for our Hamiltonian we have imposed boundary conditions on the half line  $[0, \infty)$  so the Eigenfunctions  $H y_n(x) = E_n y(x)$  satisfy the boundary conditions  $y_n(0) = 0 = y_n(\infty)$ .

From (1.4) we obtain that the inverse of the potencial can be described implicitly in terms of the half-derivative of smooth part of the Eigenvalue staircase as the

$$\text{function } f^{-1}(x) = \sqrt{\frac{2\pi\hbar^2}{m}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \langle N(x) \rangle.$$

This result (4) can be improved with the aid of the Gutzwiller's trace formula for the density of states [6] , formula (1.42) valid (it is assumed trough all the paper) in the limit  $\hbar \rightarrow 0$  for the Planck's constant.

$$\rho(E) = \langle \rho(E) \rangle + \frac{1}{\pi\hbar} \sum_{\gamma_p} \sum_{k=1}^{\infty} \frac{A_{\gamma_p} \cos\left(\frac{p}{\hbar} kl_{\gamma_p} - \frac{\pi}{2} k\mu_{\gamma_p}\right)}{\left|\det(M_{\gamma_p}^k - 1)\right|^{\frac{1}{2}}} + O(\hbar) \quad E = p^2 \quad (1.5)$$

With  $S_{\gamma}(E) = \int_C p dx = \sqrt{2mE} = \sqrt{2m} p l_{\gamma}$  being the action over the closed orbit for

the momentum,  $l_{\gamma}$  is the length of the closed orbit ,  $\mu_{\gamma_p}$  is a Maslov index and  $\det(M_{\gamma_p}^k - 1)$  is the determinant of the Monodromy Matrix,  $A_{\gamma_p}$  (see [6] for further references) are constants related to the orbits. Equation (1.5) is a better expression to evaluate the Eigenvalue staircase ( by integration) since  $\frac{dN(x)}{dx} = \rho(x)$  , also from expression (1.5) we can obtain a trace formula

$$\sum_{n=0}^{\infty} h(p_n) = \int_0^{\infty} dp \langle \rho(p) \rangle h(p) + \frac{1}{\hbar} \sum_{\gamma_p} \sum_{k=1}^{\infty} \frac{A_{\gamma_p} F_{\gamma_p}^k [h] \left(\frac{kl_{\gamma_p}}{\hbar}\right)}{\left|\det(M_{\gamma_p}^k - 1)\right|^{\frac{1}{2}}} + O(\hbar) \quad (1.6)$$

$$F_{\gamma_p}[h](u) = \frac{1}{\pi} \int_0^{\infty} dp h(p) \cos\left(pu - \frac{\pi}{2} \mu_{\gamma_p}\right) \quad (1.7)$$

Since the energy is related to the momentum of the particle by  $E = p^2$ , then we must choose an even function of the momentum  $h(p) = h(-p)$  so this test function may be also defined for negative 'p', in both cases the trace formulae (1.5) and (1.6) are real for real values of the Energy.

For the EXACT Eigenvalue staircase the half derivative can be evaluated

formally as  $f^{-1}(x) = \sqrt{\frac{2h^2}{m}} \sum_{n=0}^{\infty} \frac{H(E - p_n^2)}{\sqrt{E - p_n^2}}$  if we insert this function inside (1.6)

$$f^{-1}(x) = \sqrt{\frac{2h^2\pi}{m}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \langle N(x) \rangle + \sqrt{\frac{2\pi}{m}} \sum_{\gamma_p} \sum_{k=1}^{\infty} \frac{A_{\gamma_p} J_0\left(\frac{\sqrt{x}}{h} k l_{\gamma_p} - \frac{\pi}{2} k \mu_{\gamma_p}\right)}{\left|\det(M_{\gamma_p}^k - 1)\right|^{\frac{1}{2}}} + O(h) \quad (1.8)$$

Where we have used inside (1.8) the representation for the zeroeth order

Bessel function  $\frac{1}{\pi} \int_0^x \frac{dt \cos(ut)}{\sqrt{x^2 - t^2}} = \frac{J_0(ux)}{2}$ . In order to study the limit  $x \rightarrow \infty$  inside

(1.8) we can use the approximation for the Bessel function

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right) + O\left(\frac{1}{x}\right)$$

If we took the fractional derivative operator  $\sqrt{\frac{m}{2h^2\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}$  inside (1.8) we would obtain the trace formulae for the density of states (1.6)

Equation (1.8) defined the potential function for the Hamiltonian inside (1.8) which depends on the fractional derivative of the Smooth part of the Eigenvalue

staircase  $\langle N(E) \rangle = \frac{1}{2\pi h} \iint_{R^2} dx dp H(E - H(qx, p))$  plus a correction due to the closed orbits of the dynamical system, this correction will turn to be very important for the case of the potential and the Hamiltonian which yield to the Riemann zeros.

A good example of the Trace formula (1.5) is for the case of the Eigenvalue problem  $y(0) = 0 = y(\pi)$   $H = -\frac{d^2 y(x)}{dx^2} = E_n y(x)$ , in this case the density of states and the trace (1.5) is just the Poisson summation formula

$\sum_{m=-\infty}^{\infty} e^{2\pi i x m} = \sum_{m=-\infty}^{\infty} \delta(x-m)$  , the smooth part of the Eigenvalue staircase is given by  $\langle N(E) \rangle = \sqrt{E}$  , since the energies of the problems are  $E_n = n^2$  , and the correction to the inverse of the potential due to the length of the periodic orbits is (in terms of the Bessel function)  $\sum_{m=1}^{\infty} J_0(2\pi m \sqrt{x})$

- o *Riemann zeros and a potential:*

There is exist an analogue of the Gutzwiller's trace formula for the Riemann zeros, if we consider a dynamical system with the Maslov indices  $e^{\frac{\pi}{2} k \mu_{\gamma p}} = -1$  and length of the periodic orbits  $S_{\gamma}(E) = \log p_n$  (prime numbers), see [7]

$$\sum_{\gamma} h(\gamma) = 2h\left(\frac{i}{2}\right) - g(0) \ln \pi - 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} g(\ln n) + \frac{1}{2\pi} \int_{-\infty}^{\infty} ds h(s) \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{is}{2}\right) \quad (1.9)$$

Here,  $g(k) = \frac{1}{2\pi} \int_0^{\infty} dx h(x) \cos(kx) = g(-k)$   $h(x)$  and  $g(x)$  are test functions which

form a Fourier transform pair and  $\Lambda(n) = \begin{cases} \ln p & n = p^k \\ 0 & \text{otherwise} \end{cases}$  is the Mangoldt

function , formula (1.9) gives the relationship between a sum over the imaginary part of the Riemann zeros and a sum over the primes and prime powers.

By analogy with the Trace formula (1.5) the imaginary part of the zeros are not energies but rather the momenta of a certain Hamiltonian , the energies of the Hamiltonian will be the square of the imaginary part for the Riemann zeros  $E_n = \gamma_n^2$  , if we do the same reasoning we did for the Gutzwiller trace and set  $h = 2m = 1$  , then the potential which yields to the imaginary part of the Riemann Zeros is given by

$$f^{-1}(x) = 2 \sum_n \frac{H(x - \gamma_n^2)}{\sqrt{x - \gamma_n^2}} = \frac{4H\left(x + \frac{1}{4}\right)}{\sqrt{4x+1}} - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} J_0(\sqrt{x} \ln n) \quad x > 0 \quad (1.10)$$

$$\frac{1}{2\pi} \int_{-\sqrt{x}}^{\sqrt{x}} \frac{dr}{\sqrt{x-r^2}} \left( \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{ir}{2}\right) - \ln \pi \right)$$

We can see immediatly how the expressions (1.6) and (1.10) are connected, they both have a correction due to the length of the periodic orbits which includes the Bessel function term, in the case of the Riemann zeros , from the definition of Von Mangoldt function we have that the lenghts of the orbits are equal to the log of prime numbers (with repetition).

But what would happen for  $x < 0$  ?, due to the boundary condition  $y(0) = 0$  there is a infinite potential well at  $x=0$  so the potential would be

$$f(x) = \begin{cases} \text{defined implicitly by formula (1.10) for } x > 0 \\ \infty & \text{for } x \leq 0 \end{cases} \quad (1.11)$$

If we take the fractional derivative  $\frac{1}{2\sqrt{\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}$  inside (1.10) and use the identities

$$\sqrt{\pi} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} J_0(a\sqrt{x}) = \frac{\cos(a\sqrt{x})}{\sqrt{x}} \quad \delta(f(x)) = \sum_n \frac{\delta(x-x_n)}{|f'(x_n)|} \quad (1.12)$$

We obtain the distributional Riemann-Weil trace formula, so the density of states of our Hamiltonian , with the potential defined implicitly inside (14) is just the Riemann-Weil trace formula (on the momentum variable) [7]

$$\begin{aligned} \sum_{n=0}^{\infty} \delta(k-\gamma_n) + \sum_{n=0}^{\infty} \delta(k+\gamma_n) &= \frac{1}{2\pi} \frac{\zeta'}{\zeta} \left( \frac{1}{2} + ik \right) + \frac{1}{2\pi} \frac{\zeta'}{\zeta} \left( \frac{1}{2} - ik \right) - \frac{\ln \pi}{2\pi} \\ &+ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + i \frac{k}{2} \right) \frac{1}{4\pi} + \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} - i \frac{k}{2} \right) \frac{1}{4\pi} + \delta \left( k - \frac{i}{2} \right) + \delta \left( k + \frac{i}{2} \right) \end{aligned} \quad (1.13)$$

Where we have used the Shokhotsky's formula representation for the delta function  $-\frac{1}{\pi} \Im m \left( \frac{1}{x+i\epsilon-a} \right) = \delta(x-a)$  with  $a = \pm \frac{i}{2}$ .

In case  $x \gg 1$  , the smooth density of states can be well approximated by  $\langle N(x) \rangle \approx \frac{\sqrt{x}}{2\pi} \ln \left( \frac{\sqrt{x}}{2\pi e} \right)$  so in this case the trace formula inside (1.10) becomes

$$f^{-1}(x) \approx \frac{1}{\sqrt{\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \left( \sqrt{x} \ln \left( \frac{\sqrt{x}}{2\pi e} \right) \right) + \frac{2}{\sqrt{\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \arg \zeta \left( \frac{1}{2} + i\sqrt{x} \right) + O \left( \frac{1}{\sqrt{x}} \right) \quad (1.14)$$

We have used inside (1.14) the zeta regularization [ ] for the Dirichlet series

$$\frac{\zeta'}{\zeta} \left( \frac{1}{2} + is \right) =_{reg} - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\frac{1}{2}+is}} \quad \text{so in this case the term}$$

$$\frac{2}{\sqrt{\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \arg \zeta \left( \frac{1}{2} + i\sqrt{x} \right) = - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} J_0(\sqrt{x} \ln n) \quad (1.15)$$

Unfortunately the expressions for the inverse of the potential (1.10) and (1.14) can not be analytically invert ( we will study the asymptotic behaviour in the nex

section), however any function can be numerically inverted so the need only to reflect every point of  $f^{-1}(x)$  through the line  $y = x$  to get  $f(x)$

## 2. A TOY MODEL OF RIEMANN ZEROS WITH AN EXPONENTIAL POTENTIAL

For big energies the Eigenvalue staircase for a Hamiltonian whose energies are the square of the Riemann zeros is given by

$$N(x) = \frac{\sqrt{x}}{2\pi} \ln \left( \frac{\sqrt{x}}{2\pi e} \right) + \frac{7}{8} + O \left( \frac{1}{\sqrt{x}} \right) + \frac{1}{\pi} \arg \zeta \left( \frac{1}{2} + i\sqrt{x} \right) \quad (2.1)$$

Then the smooth part is given approximately by  $N_{smooth}(E) = \frac{\sqrt{E}}{2\pi} \ln \left( \frac{\sqrt{E}}{2\pi e} \right)$

To compute the half-derivative we use the representation for the logarithm

$$\ln(x) \approx \frac{x^\varepsilon - 1}{\varepsilon} \quad \varepsilon \rightarrow 0, \quad e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

in this case we get

$$f^{-1}(x) \approx \frac{(4\pi^2 e^2)^{-\varepsilon/2} A(\varepsilon) x^{\varepsilon/2} - B}{\sqrt{\pi} \varepsilon} \quad f(x) \approx 4\pi^2 e^2 \left( \frac{\varepsilon \sqrt{\pi} x + B}{A(\varepsilon)} \right)^{\frac{2}{\varepsilon}} \quad (2.2)$$

The constants are  $A(\varepsilon) = \frac{\Gamma\left(\frac{3+\varepsilon}{2}\right)}{\Gamma\left(1+\frac{\varepsilon}{2}\right)}$  and  $B = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ , and we have used

the property of the half-derivative of powers of 'x'  $\frac{d^{\frac{1}{2}} x^n}{dx^{\frac{1}{2}}} = \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} x^{n-\frac{1}{2}}$ ,

(Nishimoto [10])

The last expression inside (2.2) is equal to an exponential, so for the case of a Hamiltonian with boundary conditions  $y(0) = 0 = y(\infty)$  and that gives only the 'smooth' part of the staircase of the zeros via the WKB approximation

$$2 \int_0^{a=a(E)} \sqrt{E_n - \lambda e^{4x}} dx \approx N_{smooth}(E) = \frac{\sqrt{E}}{2\pi} \ln \left( \frac{\sqrt{E}}{2\pi e} \right) \text{ the potential is the following}$$

$$f_0(x) = \begin{cases} 4\pi^2 \exp\left(2 - \frac{2}{\sqrt{\pi}} \frac{\partial G(s)}{\partial s} \Big|_{s=0}\right) e^{4x} & x > 0 \\ \infty & x \leq 0 \end{cases} \quad \lambda = 4\pi^2 e^2 \exp\left(-\frac{2}{\sqrt{\pi}} \frac{\partial F(s)}{\partial s} \Big|_{s=0}\right) \quad (2.3)$$

With  $G(s) = \frac{\Gamma\left(\frac{3}{2} + s\right)}{\Gamma(1+s)}$ . So our toy model or approximate model for the

Riemann zeros is given by the Hamiltonian on the half line  $[0, \infty)$

$$E_n y(x) = -\frac{d^2 y(x)}{dx^2} + \lambda e^{4x} y(x) \quad y(0) = 0 = y(\infty) \quad E_n \approx \gamma_n^2 \quad \zeta\left(\frac{1}{2} + i\sqrt{E_n}\right) = 0 \quad (2.4)$$

And  $\lambda \approx 16\pi^2$  has been previously defined inside (2.3).

An advantage of this model is that is exactly solvable (Amore,[1]), if we impose boundary conditions on the half line  $[0, \infty)$  the quantization conditions for the energies are

$$0 = C_1 J_\mu\left(\frac{\sqrt{-\lambda}}{2}\right) + C_2 J_{-\mu}\left(\frac{\sqrt{-\lambda}}{2}\right) \quad \mu = \frac{i\sqrt{E_n}}{2} \quad C_1, C_2 \in \mathbb{C} \quad (2.5)$$

So the energies appear inside the index of a Bessel function, in general this problem may be generalized to arbitrary boundary conditions on the half line  $[u_0, \infty)$  for some real  $u_0$  so  $y(u_0) = 0 = y(\infty)$ , if we choose also that  $C_1 = -C_2$  then we may choose the  $u_0$  (if such  $u_0$  exists) so

$$1 = \frac{J_{\frac{ix}{2}}\left(\sqrt{\frac{-\lambda}{4}} e^{2u_0}\right)}{J_{-\frac{ix}{2}}\left(\sqrt{\frac{-\lambda}{4}} e^{2u_0}\right)} \approx \frac{\Gamma\left(\frac{1}{4} - \frac{ix}{2}\right) \zeta\left(\frac{1}{2} - ix\right)}{\Gamma\left(\frac{1}{4} + \frac{ix}{2}\right) \zeta\left(\frac{1}{2} + ix\right)} \pi^{ix} \quad (2.6)$$

Equation (2.6) is just the functional equation for the Riemann Zeta function on the critical line  $s = \frac{1}{2} + ix$ , this means that the quantization condition (2.5) may give the Riemann zeros and it is equivalent to the functional equation for the Riemann zeta function

### 3. ZETA REGULARIZATION FOR FUNCTIONAL DETERMINANTS AND THE RIEMANN XI- FUNCTION $\xi(s)$

Berry [5] has suggested the following Quantization condition for the Energies of a Quantum system

$$\Delta(E) = \det(E - H) = 0 \quad (3.1)$$

Here  $\Delta(E) = \prod_{n=0}^{\infty} (E - E_n)$  is the functional determinant of the system, for example for the Harmonic oscillator and the infinite potential well we have

$$\prod_{n=0}^{\infty} \left(1 - \frac{E}{n^2}\right) = \frac{\sin(\pi\sqrt{E})}{\pi\sqrt{E}} \quad \prod_{n=0}^{\infty} \left(1 - \frac{E}{n + \frac{1}{2}}\right) e^{\frac{E}{n+1}} = \frac{\cos(\pi E)}{\sqrt{\pi}} \frac{\Gamma\left(E + \frac{1}{2}\right)}{e^{-\gamma E}} \quad (3.2)$$

Here  $\gamma = 0.57721..$  is the Euler-Mascheroni constant.

In order to define a Functional determinant, one of the best method to use is the Zeta regularization [8] the zeta regularized determinant for an operator T having real eigenvalues  $\{\lambda_n\}$  is

$$\det(T) = \prod_{n=0}^{\infty} \lambda_n = \exp\left(-\frac{\partial Z(0)}{\partial s}\right) \quad Z(s) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n^s} \quad \partial_s Z(0) = -\sum_{n=0}^{\infty} \frac{\ln \lambda_n}{1} \quad (3.3)$$

Here Z(s) is the spectral zeta function associated to the operator T, in many cases we do not know this function so we need to use the representation

$$Z(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt \Theta(t) t^{s-1} \quad \Theta(t) = \sum_{n=0}^{\infty} e^{-tE_n} \quad (3.4)$$

This Theta function is defined only for  $t > 0$ , for our case with the potential

defined implicitly by the equation  $f^{-1}(x) = \sum_{n=0}^{\infty} \frac{H(x - \gamma_n^2)}{\sqrt{x - \gamma_n^2}} = 2\sqrt{\pi} \frac{d^{\frac{1}{2}} N(x)}{dx^{\frac{1}{2}}}$  we can

use the Semiclassical approximation for the Theta function

$$\Theta_{WKB}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_0^{\infty} dp e^{-tp^2 - tf(x)} = \frac{1}{\sqrt{\pi t}} \int_0^{\infty} dx e^{-tf(x)} = \sqrt{\frac{t}{\pi}} \int_0^{\infty} dr e^{-tr} \frac{f^{-1}(r)}{dr} \quad (3.5)$$

From the properties of the Laplace transform  $\int_0^{\infty} dt f(t) t^k = (-1)^k \frac{\partial F}{\partial s}$

$\int_0^{\infty} dt f(t) = F(s) \quad k = \frac{1}{2}$  and from the identity  $\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$  the last integral

inside (3.5) is equal to  $\sum_{n=0}^{\infty} e^{-t\gamma_n^2}$  with  $\zeta\left(\frac{1}{2} + i\gamma_n\right) = 0$ . If we take the Mellin



transform  $\frac{1}{\Gamma(s)} \int_0^\infty dt \Theta_{WKB}(t) t^{s-1} e^{-tE^2}$  inside (3.4) we obtain  $\sum_{n=0}^\infty \frac{1}{(E^2 + \gamma_n^2)^s} = Z(E, s)$ ,

and the sum is extended to the positive imaginary part of the Riemann zeros, from this last expression we can define the Riemann Xi-function on the critical line as the quotient of 2 functional determinants

$$\frac{\det(H - E)}{\det(H)} = \frac{\prod_{n=0}^\infty (\gamma_n^2 - E)}{\prod_{n=0}^\infty \gamma_n^2} = \prod_{n=0}^\infty \left(1 - \frac{E}{E_n}\right) = \frac{\xi\left(\frac{1}{2} + i\sqrt{E}\right)}{\xi(1/2)} = \exp\left(-\frac{d}{ds} Z(s, E)\Big|_{s=0} + \frac{d}{ds} Z(s, 0)\Big|_{s=0}\right)$$

(3.6)

So, from the expression (3.6) one observes that the functional determinant of a Hamiltonian  $H = p^2 + f(x)$  with a potential defined implicitly by

$$f^{-1}(x) = \sum_{n=0}^\infty \frac{H(x - \gamma_n^2)}{\sqrt{x - \gamma_n^2}} = 2\sqrt{\pi} \frac{d^{\frac{1}{2}} N(x)}{dx^{\frac{1}{2}}} = \frac{2}{\sqrt{\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \arg \zeta\left(\frac{1}{2} + i\sqrt{x}\right) \quad (3.7)$$

is exactly to the Riemann Xi-function on the critical line, hence Riemann Hypothesis must be true

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