The Gravitational Energy of the Universe

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Abstract

The gravitational energy, momentum, and stress are calculated for the Robertson-Walker metric. The principle of energy conservation is applied, in conjunction with the Friedmann equations. Together, they show that the cosmological constant Λ is non-zero, the curvature index $k = 0$, and the acceleration $\ddot{R}$ is positive. It is shown that the gravitational field accounts for two-thirds of the energy in the Universe.

1. Introduction

In the standard treatment of cosmology, no attempt is made to directly calculate the energy, momentum, and stress of the gravitational field. The following calculation derives from the scalar, three-vector theory of gravitation.[1,2] In this theory, displacements in time and space are expressed in the form

$$c\,dt = e_0(x)dx^0 \qquad\qquad d\mathbf{r} = \mathbf{e}_i(x)dx^i \tag{1}$$

where $e_\mu = (e_0,\, \mathbf{e}_i)$ is a scalar, 3-vector basis. The fundamental interval[1] is given by

$$\begin{aligned} ds^2 &= c^2dt^2 - d\mathbf{r}^2 = (e_0 dx^0)^2 - \mathbf{e}_i \cdot \mathbf{e}_j\, dx^i dx^j \\ &= g_{\mu\nu}dx^\mu dx^\nu \end{aligned} \tag{2}$$

where

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij} & \\ 0 & & & \end{pmatrix} \tag{3}$$

is the scalar, 3-vector metric.

The basis $e_\mu(x)$ varies from point to point according to the formula

$$\nabla_\nu e_\mu = e_\lambda Q^\lambda_{\mu\nu} \tag{4}$$

This separates into scalar and 3-vector parts

$$\nabla_\nu e_0 = e_0 Q^0_{0\nu} \tag{5}$$

$$\nabla_\nu \mathbf{e}_i = \mathbf{e}_k Q^k_{i\nu} \tag{6}$$

By definition $Q^k_{0\nu} = Q^0_{i\nu} \equiv 0$. The $Q^\mu_{\nu\lambda}$ are related to the Christofel coefficients as follows:

[1] This interval is invariant under a Lorentz transformation. At any point P, the vector $d\mathbf{r}$ may be projected onto an orthonormal 3-frame: $\mathbf{i}\cdot d\mathbf{r}$, $\mathbf{j}\cdot d\mathbf{r}$, $\mathbf{k}\cdot d\mathbf{r}$. These projections, together with the time interval dt, are then transformed into new values, which are observed in a relatively moving 3-frame. No coordinates are involved with this procedure.

$$Q^0_{0\lambda} = \Gamma^0_{0\lambda} = \frac{1}{2} g^{00} \partial_\lambda g_{00} \tag{7}$$

$$Q^i_{j0} = \Gamma^i_{j0} = \frac{1}{2} g^{in} \partial_0 g_{nj} \tag{8}$$

$$Q^i_{jk} = \Gamma^i_{jk} = \frac{1}{2} g^{in} \left(\partial_k g_{jn} + \partial_j g_{nk} - \partial_n g_{jk}\right) \tag{9}$$

They comprise the formula

$$Q^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} + g^{\mu\rho} g_{\lambda\eta} Q^\eta_{[\nu\rho]} \tag{10}$$

where

$$Q^\mu_{[\nu\lambda]} \equiv Q^\mu_{\nu\lambda} - Q^\mu_{\lambda\nu} \tag{11}$$

An observer is free to introduce new coordinates $\{x^{\mu'}\}$. The new coordinates must be *at rest* with respect to the old. Displacements (1) will then be invariant, so that the scalar, 3-vector character is preserved. The coordinate transformations are of the form $x^{0'} = x^{0'}(x^0)$ and $x^{i'} = x^{i'}(x^j)$. In particular, $g_{\mu\nu}$ (3) transforms as a tensor

$$g_{0'0'} = \frac{\partial x^0}{\partial x^{0'}} \frac{\partial x^0}{\partial x^{0'}} g_{00} \qquad\qquad g_{i'j'} = \frac{\partial x^m}{\partial x^{i'}} \frac{\partial x^n}{\partial x^{j'}} g_{mn} \tag{12}$$

The non-zero components of $Q^\mu_{[\nu\lambda]}$ (11) are

$$Q^0_{[0i]} = Q^0_{0i} = \frac{1}{2} g^{00} \partial_i g_{00} \qquad\qquad Q^i_{[j0]} = Q^i_{j0} = \frac{1}{2} g^{in} \partial_0 g_{nj} \tag{13}$$

They transform as tensor components

$$Q^{0'}_{0'i'} = \frac{\partial x^n}{\partial x^{i'}} Q^0_{0n} \qquad\qquad Q^{i'}_{j'0'} = \frac{\partial x^{i'}}{\partial x^m} \frac{\partial x^n}{\partial x^{j'}} \frac{\partial x^0}{\partial x^{0'}} Q^m_{n0} \tag{14}$$

This field strength tensor plays a central role in dynamics. It serves to define the gravitational energy tensor

$$T^{(g)}_{\mu\nu} = \frac{c^4}{8\pi G} \left\{ Q^\rho_{[\lambda\mu]} Q^\lambda_{[\rho\nu]} + Q_\mu Q_\nu - \frac{1}{2} g_{\mu\nu} g^{\eta\tau} (Q^\rho_{[\lambda\eta]} Q^\lambda_{[\rho\tau]} + Q_\eta Q_\tau) \right\} \tag{15}$$

where $Q_\mu = Q^\rho_{[\rho\mu]}$. The coefficient is chosen such that $T^{(g)}_{\mu\nu}$ reduces to the Newtonian stress-energy tensor

$$
\begin{aligned}
T_{00}^{(g)} &= \frac{1}{8\pi G}(\nabla\psi)^2 && (16)\\
T_{0i}^{(g)} &= 0 && (17)\\
T_{ij}^{(g)} &= \frac{1}{4\pi G}\Big\{\partial_i\psi\,\partial_j\psi - \frac{1}{2}\delta_{ij}(\nabla\psi)^2\Big\} && (18)
\end{aligned}
$$

when

$$
g_{00} = 1 + \frac{2}{c^2}\psi \tag{19}
$$

2. Gravitational energy, momentum, and stress

The Robertson-Walker metric is given by [3]

$$
ds^2 = (dx^0)^2 - \frac{R^2(t)}{r_0^2(1+kr^2/4r_0^2)^2}\Big(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\Big) \tag{20}
$$

where $k = -1, 0$, or $+1$. Since $g_{00} = 1$, all $Q^0_{0i} = 0$. This leaves only energy and stress components in (15)

$$
\begin{aligned}
T_{00}^{(g)} &= \frac{c^4}{16\pi G}\left(Q^m_{n0}Q^n_{m0} + Q_0Q_0\right) && (21)\\
T_{0i}^{(g)} &= 0 && (22)\\
T_{ij}^{(g)} &= -\frac{c^4}{16\pi G}\, g_{ij}\left(Q^m_{n0}Q^n_{m0} + Q_0Q_0\right) = -g_{ij}T_{00}^{(g)} && (23)
\end{aligned}
$$

A straightforward calculation yields

$$
Q^m_{n0}Q^n_{m0} = 3\,\frac{\dot R^2}{R^2} \qquad\qquad Q_0Q_0 = 9\,\frac{\dot R^2}{R^2} \tag{24}
$$

where $\dot R$ is the derivative with respect to $x^0 = ct$. Therefore, the non-zero components of the mixed energy tensor are

$$
T_0^{(g)0} = \frac{3c^4}{4\pi G}\frac{\dot R^2}{R^2} \qquad\qquad T_i^{(g)j} = -\delta_i^{\,j}\,\frac{3c^4}{4\pi G}\frac{\dot R^2}{R^2} \tag{25}
$$

The stresses are compressive and correspond to an equation of state

$$p_g = \rho_g c^2 \tag{26}$$

According to (25), the gravitational energy and pressure are functions of time alone.

3. Field equations

The gravitational field equations are derived by variation of the action

$$\delta \int \Big\{ \frac{c^4}{16\pi G}\Big(g^{\mu\nu}R_{\mu\nu} - 2\Lambda\Big) + L^{(m)} \Big\}\sqrt{-g}\, d^4x = 0 \tag{27}$$

There are seven field equations, corresponding to the seven variations $\delta g^{\mu\nu} = (\delta g^{00}, \delta g^{ij})$

$$R_\mu{}^\nu - \frac{1}{2}\delta_\mu{}^\nu R + \Lambda \delta_\mu{}^\nu = -\frac{8\pi G}{c^4} T_\mu^{(m)\nu} \tag{28}$$

Components $R_0{}^i$ and $T_0^{(m)i}$ do not appear. Substitute the Robertson-Walker metric (20) and the material energy tensor

$$T_\mu^{(m)\nu} = \begin{pmatrix} \rho_m c^2 & & & \\ & -p_m & & 0 \\ 0 & & -p_m & \\ & & & -p_m \end{pmatrix} \tag{29}$$

in order to obtain the Friedmann equations [3]

$$\begin{aligned} \frac{3\dot{R}^2}{R^2} + \frac{3k}{R^2} - \Lambda &= \frac{8\pi G}{c^4}\rho_m c^2 & (30) \\ \frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} - \Lambda &= -\frac{8\pi G}{c^4} p_m & (31) \end{aligned}$$

At the present time, the material pressure $p_m \ll \rho_m c^2$. It will be ignored for the remainder of the paper ($p_m = 0$). Equation (31) then becomes

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} - \Lambda = 0 \tag{32}$$

Eliminate k and $\dot{R}^2$ from (30) and (32) to find

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G}{3c^2}\,\rho_m \tag{33}$$

It follows that a positive cosmological constant is required, if $\ddot{R} > 0$.

4. Energy conservation

The differential law of energy and momentum conservation is (appendix)

$$\mathrm{div}\, T_\mu{}^\nu = T_\mu{}^\nu{}_{;\nu} + Q^\beta_{[\alpha\mu]} T_\beta{}^\alpha = 0 \tag{34}$$

where $T_\mu{}^\nu{}_{;\nu}$ is the covariant derivative. The total density of energy, momentum, and stress is given by

$$T_\mu{}^\nu = T^{(g)\nu}_\mu + T^{(m)\nu}_\mu + T^{(\Lambda)\nu}_\mu \tag{35}$$

The final term is implied by the cosmological constant in the field equations. It is yet to be determined, but it must have the form $T^{(\Lambda)\nu}_\mu = C\delta_\mu{}^\nu$ where C is a constant. The material equations of motion give

$$T^{(m)\nu}_{\mu\ ;\nu} = 0 \tag{36}$$

so that energy conservation is expressed by

$$\begin{aligned} \mathrm{div}\, T_0{}^\nu &= T^{(g)\nu}_{0\ ;\nu} + T^{(\Lambda)\nu}_{0\ ;\nu} + Q^j_{[i0]}(T^{(g)} + T^{(m)} + T^{(\Lambda)})^i_j \\ &= \partial_0 T^{(g)0}_0 + \Gamma^n_{n0} T^{(g)0}_0 + \Gamma^n_{n0} T^{(\Lambda)0}_0 = 0 \end{aligned} \tag{37}$$

Substitute (25) to find

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{4\pi G}{c^4} T^{(\Lambda)0}_0 = 0 \tag{38}$$

Comparison with (32) shows that $k = 0$ and

$$T^{(\Lambda)\nu}_\mu = -\frac{c^4}{4\pi G}\,\Lambda\delta_\mu{}^\nu \tag{39}$$

This energy tensor corresponds to an equation of state

$$p_\Lambda = -\rho_\Lambda c^2 \tag{40}$$

In equation (30), set $k = 0$ and rearrange to find

$$\Lambda = \frac{3\dot{R}^2}{R^2} - \frac{8\pi G}{c^2}\,\rho_m \tag{41}$$

Substitution into (35) gives

$$\begin{aligned} T_0^{\ 0} &= \frac{3c^4}{4\pi G}\frac{\dot{R}^2}{R^2} + \rho_m c^2 - \frac{c^4}{4\pi G}\,\Lambda \\ &= 3\rho_m c^2 \end{aligned} \tag{42}$$

Therefore, the gravitational field (with the cosmological term) accounts for two-thirds of the energy in the Universe.

5. Concluding remarks

Formula (41) makes possible an evaluation of the constant Λ, in terms of the mass density and the Hubble ratio

$$\frac{H}{c} = \frac{\dot{R}}{R} \tag{43}$$

The experimental value of the Hubble constant is stated to be

$$H_0 = 71\,\frac{\text{km-s}^{-1}}{\text{Mpc}} = 2.3 \times 10^{-18}\ \text{s}^{-1} \tag{44}$$

For historical reasons, the mass density is expressed in terms of a "critical density"

$$\rho_{cr} = \frac{3H_0^2}{8\pi G} = 9.5 \times 10^{-30}\,\text{g-cm}^{-3} \tag{45}$$

The mass density, including the missing mass, is estimated to be

$$\rho_0 = 0.27\,\rho_{cr} = 2.6 \times 10^{-30}\,\text{g-cm}^{-3} \tag{46}$$

Substitution into (41) yields a positive cosmological constant

$$\begin{aligned}\Lambda &= \frac{8\pi G}{c^2}\left(\frac{3H_0^2}{8\pi G} - \rho_0\right) \\ &= \frac{8\pi G}{c^2}(0.73\,\rho_{cr}) = 1.3\times 10^{-56}\,\mathrm{cm}^{-2} \qquad (47)\end{aligned}$$

According to formula (33), the acceleration will be positive, if

$$\rho_m < \frac{c^2}{4\pi G}\,\Lambda = 1.46\,\rho_{cr} = 1.4\times 10^{-29}\,\text{g-cm}^{-3} \qquad (48)$$

Therefore, the acceleration $\ddot{R}$ is now positive and will remain so in the future. It was apparently negative at times in the distant past. The total pressure is

$$p = p_g + p_\Lambda = \frac{c^4}{4\pi G}\Big(\frac{3H^2}{c^2} + \Lambda\Big) \qquad (49)$$

which is positive.

Appendix: Conservation of energy and momentum

The differential law of conservation is derived by summing the invariant expression $e_\mu T^{\mu\nu}\, dV_\nu$

$$\sum_{\delta R} e_\mu T^{\mu\nu}\, dV_\nu = \Big\{e_\mu\, \partial_\nu(\sqrt{-g}\, T^{\mu\nu}) + (\nabla_\nu e_\mu)\sqrt{-g}\, T^{\mu\nu}\Big\} d^4x \qquad (50)$$

The region δR is closed and infinitesimal, while dV_ν is the vector

$$dV_\nu = \sqrt{-g}\,(dx^1dx^2dx^3,\, dx^0dx^2dx^3,\, \ldots) \qquad (51)$$

By definition, $\nabla_\nu e_\mu = e_\lambda Q^\lambda_{\mu\nu}$, so that

$$\sum_{\delta R} e_\mu T^{\mu\nu}\, dV_\nu = e_\mu \Big\{\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\, T^{\mu\nu}) + Q^\mu_{\lambda\nu} T^{\lambda\nu}\Big\}\sqrt{-g}\, d^4x \qquad (52)$$

Energy and momentum are conserved, if

$$\mathrm{div}\, T^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\, T^{\mu\nu}) + Q^\mu_{\lambda\nu} T^{\lambda\nu} = 0 \qquad (53)$$

The $Q^\mu_{\lambda\nu}$ are related to the Christofel coefficients $\Gamma^\mu_{\lambda\nu}$ by the formula

$$Q^{\mu}_{\lambda\nu} = \Gamma^{\mu}_{\lambda\nu} + g^{\mu\alpha} g_{\nu\beta} Q^{\beta}_{[\lambda\alpha]} \tag{54}$$

Therefore, the divergence may be written in the form

$$\operatorname{div} T^{\mu\nu} = T^{\mu\nu}_{\ ;\nu} + g^{\mu\alpha} Q^{\beta}_{[\lambda\alpha]} T_{\beta}{}^{\lambda} \tag{55}$$

where $T^{\mu\nu}_{\ ;\nu}$ is the (contracted) covariant derivative. Similarly, the divergence of the mixed energy tensor is

$$\operatorname{div} T_{\mu}{}^{\nu} = T_{\mu}{}^{\nu}{}_{;\nu} + Q^{\beta}_{[\alpha\mu]} T_{\beta}{}^{\alpha} \tag{56}$$

References

1. K. Dalton, "Gravitational Waves", *Hadronic J.* 35(2012)209.

2. K. Dalton, "Gravity and the Fermion Mass", *Nuovo Cim.* 124B(2009)1175.

3. R. Adler, M. Bazin, M. Schiffer, *Introduction to General Relativity*, (McGraw-Hill, New York, 2nd ed., 1975) chap. 12, 13.