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ULTRA–UNIVERSAL MODELS

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Abstract

The concept of ultra–universal algebras in varieties is generalized to models of first order theories. Characterizations of theories which have ultra–universal models are found and general examples of ultra–universal models are investigated. In particular we show that a theory has an ultra–universal model iff it is consistent and its class of models satisfies the joint embedding property.

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INTRODUCTION

The concept of an ultra-universal algebra in a variety was introduced by Bruyns and Rose in [2]. In this paper we generalize this concept to models of a first order theory. In section 1 we characterize the theories which have ultra-universal models (Theorem 1.3 and Corollary 1.5) and in section 2 we consider some general examples of ultra-universal models and sufficient conditions for a model to be ultra-universal.

NOTATION AND TERMINOLOGY

\( \mathcal{L} \) will denote a first order language with equality and \( \mathbf{V} \) will denote the set of all universal sentences in \( \mathcal{L} \). By a theory we simply mean a set of sentences in \( \mathcal{L} \) (not necessarily a closed set of sentences). If \( \Sigma \) is a theory then \( \operatorname{Mod} \Sigma \) will denote the set of all models of \( \Sigma \). If \( \mathcal{K} \) is a class of models for \( \mathcal{L} \) then \( \operatorname{Th} \mathcal{K} \) will denote the theory of \( \mathcal{K} \), i.e. the set of all sentences of \( \mathcal{L} \) which hold in all members of \( \mathcal{K} \). We say that \( \mathcal{K} \) satisfies the Joint Embedding Property (JEP) iff every pair of members of \( \mathcal{K} \) is embeddable in a third member of \( \mathcal{K} \).


**Definition 1.1** (cf. Bruyns and Rose [2])

Let \( \Sigma \) be a theory. A model \( \mathcal{M} \) of \( \Sigma \) is called an ultra-universal model of \( \Sigma \) iff every model of \( \Sigma \) is embeddable in an ultrapower of \( \mathcal{M} \). If \( \mathcal{K} \) is an elementary class, \( \mathcal{M} \) is said to be ultra-universal in \( \mathcal{K} \) iff \( \mathcal{M} \) is an ultra-universal model of \( \operatorname{Th} \mathcal{K} \).

The definition of an ultra-universal model of a theory \( \Sigma \) refers only to structural properties of the models of \( \Sigma \). However we can immediately obtain a semantic characterization of ultra-universal models:
Proposition 1.2
Let \( M \) be a model of \( \Sigma \). The following are equivalent:

i) \( M \) is an ultra-universal modal of \( \Sigma \).

ii) Every universal sentence holding in \( M \) holds in all models of \( \Sigma \).

iii) Every existential sentence holding in some model of \( \Sigma \) holds in \( M \).

(See [1] page 185–188.) □

Let \( \Sigma \) be a consistent closed theory. Then \( \Sigma \) has a model \( M \) and Th \( \{ M \} \) is then a complete consistent closed extension of \( \Sigma \). Moreover any complete consistent closed extension \( \Sigma' \) of \( \Sigma \) is of this form. Proposition 1.2 tells us that \( \Sigma \) has an ultra-universal model iff \( \Sigma' \) can be chosen so that \( \Sigma' \cap \forall = \Sigma \cap \forall \).

The following Theorem gives necessary and sufficient conditions for a theory to have an ultra-universal model:

Theorem 1.3
Let \( \Sigma \) be a theory. The following are equivalent:

i) \( \Sigma \) has an ultra-universal model.

ii) \( \Sigma \) is consistent and for all universal sentences \( \varphi \) and \( \psi \), \( \Sigma \vdash \varphi \lor \psi \) implies that \( \Sigma \vdash \varphi \) or \( \Sigma \vdash \psi \).

iii) Mod \( \Sigma \) is non-empty and satsifies the JEP.

iv) There is a complete consistent extension T of \( \Sigma \) such that for every universal sentence \( \varphi \), T \( \vdash \varphi \) implies \( \Sigma \vdash \varphi \).

Proof:

(i) \( \Rightarrow \) (iii): Assume (i). Let \( M \) be an ultra-universal model of \( \Sigma \). \( M \in \text{Mod} \Sigma \) and so \( \text{Mod} \Sigma \) is non-empty. Let \( A, B \in \text{Mod} \Sigma \). Then there are ultrafilters \( \mathcal{U} \) and \( \mathcal{E} \) with \( A \) embeddable in \( \Pi_{p}M \) and \( B \) embeddable in \( \Pi_{c}M \). Now \( \Pi_{p}M \) and \( \Pi_{c}M \) are both embeddable in \( \Pi_{p \times c}M \),

3
(see [4] page 382), and so A and B are both embeddable in $\Pi_{\mathcal{D}} \mathcal{X} \mathcal{M}$. Thus Mod $\Sigma$ satisfies the JEP. (iii) $\Rightarrow$ (ii): Assume (iii). $\Sigma$ is consistent since Mod $\Sigma$ is non-empty. Let $\varphi$ and $\psi$ be universal sentences with $\Sigma \vdash \varphi \lor \psi$. Suppose that $\Sigma \not\models \varphi$ and $\Sigma \not\models \psi$. Then there are models $A$ and $B$ of $\Sigma$ with $A \not\models \varphi$ and $B \not\models \psi$. There is a model $C$ of $\Sigma$ with both $A$ and $B$ embeddable in $C$. Then $C \models \varphi \lor \psi$ but $C \not\models \varphi$ and $C \not\models \psi$, a contradiction. Thus $\Sigma \vdash \varphi$ or $\Sigma \vdash \psi$ as required. (ii) $\Rightarrow$ (iv): Assume (ii). Put $\Gamma = \{ \neg \varphi : \varphi \in \forall \text{ but } \Sigma \not\models \varphi \}$. We show that $\Sigma \cup \Gamma$ is consistent. Let $\Delta$ be a finite subset of $\Sigma \cup \Gamma$. If $\Delta \subseteq \Sigma$ then obviously $\Delta$ is consistent, so suppose that there is an $n < \omega$ and $\sigma_0, \ldots, \sigma_n \in \forall$ such that $\Sigma \not\models \sigma_i$ for all $i \in \{ 0, \ldots, n \}$ and $\Delta \cap \Gamma = \{ \neg \sigma_0, \ldots, \neg \sigma_n \}$. Put $\sigma = \sigma_0 \lor \ldots \lor \sigma_n$. Suppose that $\Sigma \models \sigma$. Then applying (ii) we get $\Sigma \models \sigma_i$ for some $i \in \{ 0, \ldots, n \}$, a contradiction. Thus $\Sigma \not\models \sigma$ and so there is a model $N$ of $\Sigma$ such that $N \not\models \sigma$. Then $N \models \neg \sigma$. Now $\neg \sigma \equiv \neg \sigma_0 \land \ldots \land \neg \sigma_n$ and so $N$ is a model of $\Delta$. Hence $\Delta$ is consistent. By compactness $\Sigma \cup \Gamma$ is consistent and so it has a complete consistent extension $T$. Let $\varphi$ be a universal sentence with $T \models \varphi$. Then $\Sigma \models \varphi$ or else $\neg \varphi \in \Gamma \subseteq T$, a contradiction. (iv) $\Rightarrow$ (i). Assume (vi). Then $T = \text{Th} \{ M \}$ for some model $M$ of $\Sigma$. Let $\varphi$ be a universal sentence with $M \models \varphi$. Then $T \models \varphi$ and so $\Sigma \models \varphi$ whence $N \models \varphi$ for all models $N$ of $\Sigma$. Thus $M$ is an ultra-universal model of $\Sigma$. □

The equivalence of (ii) and (iii) was already found by A. Robinson in [5].

Corollary 1.4

Let $N$ be any model for $\mathcal{L}$. Then the universal theory of $N$, that is $\forall \cap \text{Th} \{ N \}$, has an ultra-universal model. □

We can characterize ultra-universal theories in terms of filters in a certain sublattice of the Lindenbaum algebra for $\mathcal{L}$. Let $\mathcal{B} = \{ B, \land, \lor, \neg, F, T \}$ be the Lindenbaum algebra for $\mathcal{L}$. Let $[\varphi]$ denote the equivalence class of a formula $\varphi$ under the equivalence relation $\equiv$ given by $\varphi \equiv \psi$ iff $\vdash (\varphi \leftrightarrow \psi)$. Put $A = \{ [\varphi] : \varphi \in \forall \}$. $F$ is the equivalence class of all universally
false sentences and \( T \) is the class of all universally true sentences. Hence \( F = [(\forall x)(x \neq x)] \) and \( T = [(\forall x)(x = x)] \) and so we see that \( F, T \in A \). Also \( A \) is a sublattice of \( B \). Thus we have a distributive 0,1–lattice \( A = \langle A, \wedge, \vee, F, T \rangle \). Now if \( \Sigma \) is a theory put \( \mathcal{P}_\Sigma = \{ [\varphi] : \Sigma \vdash \varphi \} \). Then \( \mathcal{P}_\Sigma \) is a filter in \( B \). ( Moreover all filters in \( B \) are of this form, since if \( \mathcal{F} \) is a filter in \( B \), putting \( \Sigma = \cup \mathcal{F} \) gives \( \mathcal{P}_\Sigma = \mathcal{F} \). ) Note that \( \mathcal{P}_\Sigma \) is proper iff \( \Sigma \) is consistent and \( \mathcal{P}_\Sigma \) is an ultrafilter iff \( \Sigma \) is complete and consistent.

**Corollary 1.5**

Let \( \Sigma \) be a theory. The following are equivalent:

i) \( \Sigma \) has an ultra–universal model.

ii) \( \mathcal{P}_\Sigma \) can be extended to an ultrafilter \( \mathcal{D} \) with \( \mathcal{D} \cap A = \mathcal{P}_\Sigma \cap A \).

iii) \( \mathcal{P}_\Sigma \cap A \) is a prime proper filter in the distributive 0,1–lattice \( A \). □

2. General examples of ultra–universal models.

**Theorem 2.1**

Let \( \Sigma \) be a theory and let \( M \) be a model of \( \Sigma \) with the property that every finitely generated substructure of a model of \( \Sigma \) is embeddable in \( M \). Then \( M \) is an ultra–universal model of \( \Sigma \).

**Proof:**

Let \( N \) be a model of \( \Sigma \). Let \( I \) be the collection of finite subsets of \( N \) and for all \( i \in I \) let \( N_i \) be the substructure of \( N \) generated by the finite subset \( i \). Then there is an ultrafilter \( \mathcal{D} \) over \( I \) such that \( N \) is embeddable in \( \prod_{\mathcal{D}} N_i \). ( See [3] page 213. ) Now for all \( i \in I \) \( N_i \) is embeddable in \( M \) and so we see that \( N \) is embeddable in \( \prod_{\mathcal{D}} M \). □

If \( X \) is a set let \( S_X \) denote the symmetric group on \( X \) ( i.e. the group of all permutations on \( X \) under composition ), let \( M_X \) denote the function monoid on \( X \) ( i.e. the monoid of all functions from \( X \) into \( X \) under composition ), and let \( N_X \) denote the function semigroup on
X ( i.e. the semigroup reduct of $M_X$).

**Corollary 2.2**

Let X be an infinite set.

i) $S_X$ is ultra-universal in the variety of groups.

ii) $M_X$ is ultra-universal in the variety of monoids.

iii) $N_X$ is ultra-universal in the variety of semigroups.

**Proof:**

(i): Let G be a finitely generated group with underlying set G. Now G is embeddable in $S_G$ (Cayley's Theorem) and G is countable whence we see that $S_G$ is embeddable in $S_X$. Thus G is embeddable in $S_X$. The result follows by Theorem 2.1. (ii) is proved similarly to (i) since Cayley's Theorem can be generalized to monoids. (iii): Let H be a finitely generated semigroup with underlying set H. Let $^* \not\in H$ and let L be the monoid with underlying set $L = H \cup \{ ^* \}$ obtained by appending $^*$ to H to as an identity element. Then L is embeddable in $M_L$. But L is countable and so $M_L$ is embeddable in $M_X$ whence L is embeddable in $M_X$. Now H is embeddable in the semigroup reduct of L and the latter is embeddable in $N_X$. Thus H is embeddable in $N_X$. The result follows by Theorem 2.1. □

**Corollary 2.3**

Let $\mathcal{K}$ be an elementary class of non-trivial Boolean algebras. If B is an infinite member of $\mathcal{K}$ then B is ultra-universal in $\mathcal{K}$.

**Proof:**

Any finitely generated Boolean algebra is finite and any non-trivial finite Boolean algebra is embeddable in any infinite Boolean algebra. □

The concept of a factor embeddable variety was introduced in [2]. We now generalize this concept to arbitrary classes of structures.
Definition 2.4

Let $\mathcal{K}$ be a class of structures of a fixed type. Let $\{ N_i : i \in I \}$ be a non-empty family of structures in $\mathcal{K}$. If $j \in I$ and $f : N_j \to \prod_i N_i$ is an embedding we say that $f$ is a factor embedding. If in addition $f$ satisfies the condition $\pi_j \circ f$ is the identity on $N_j$, where $\pi_j : \prod_i N_i \to N_j$ is the $j$th projection, we say that $f$ is a strong factor embedding. $\mathcal{K}$ is said to be factor embeddable iff for every non-empty family $\{ N_i : i \in I \}$ of structures in $\mathcal{K}$, and each $j \in I$, there is a factor embedding $f_j : N_j \to \prod_i N_i$. □

We have a nice characterization of factor embeddable classes:

Proposition 2.5

Let $\mathcal{K}$ be a class of structures. The following are equivalent:

i) $\mathcal{K}$ is factor embeddable.

ii) For all $M, N \in \mathcal{K}$ there is a homomorphism $f_{M, N} : M \to N$.

Proof:

(i) ⇒ (ii): Assume (i). Let $M, N \in \mathcal{K}$. Then there is a factor embedding $f : M \to M \times N$. Let $\pi : M \times N \to N$ be the projection onto $N$. Then $f_{M, N} = \pi \circ f$ is a homomorphism from $M$ to $N$. (ii) ⇒ (i): Assume (ii). Let $\{ N_i : i \in I \}$ be a non-empty family of structures in $\mathcal{K}$. Let $j \in I$. Define $f_j : N_j \to \prod_i N_i$ by: For all $a \in N_j$, $f_j(a) = (b_i)_i$ where $b_j = a$ and $b_i = f_{N, N_i}(a)$ for $i \neq j$. Then $f_j$ is a (strong) factor embedding. Hence (ii) implies (i). □

Note that the above proof shows that the factor embeddings given by the definition of factor embeddability can always be chosen to be strong factor embeddings.

A useful corollary to the above proposition is:
Corollary 2.6
Let $\mathcal{K}$ be a non-empty class of structures and suppose there is a structure $R$ which a retract of all members of $\mathcal{K}$. Then $\mathcal{K}$ is factor embeddable. □

Proof:
Let $M, N \in \mathcal{K}$. There is a surjective homomorphism $p : M \rightarrow R$ and an embedding $m : R \rightarrow N$. Then $m \circ p$ is a homomorphism from $M$ to $N$ and the result follows by Theorem 2.5. □

Theorem 2.7
Let $\mathcal{K}$ be a non-empty factor embeddable elementary class. Let $M$ be the product of all finitely generated substructures of members of $\mathcal{K}$. If $M \in \mathcal{K}$ then $M$ is ultra-universal in $\mathcal{K}$.

Proof:
Immediate from Theorem 2.1. □

Corollary 2.8
Let $\mathcal{K}$ be a factor embeddable universal Horn class. Then the product of all finitely generated members of $\mathcal{K}$ is ultra-universal in $\mathcal{K}$. □

In particular the above applies to factor embeddable varieties. Let $\mathcal{V}$ be a variety and suppose that every member of $\mathcal{V}$ has a one element subalgebra. Then the trivial member of $\mathcal{V}$ is a retract of all members of $\mathcal{V}$ and so by Corollary 2.6, $\mathcal{V}$ is factor embeddable. By Corollary 2.8, the product of all finitely generated members of $\mathcal{V}$ is ultra-universal in $\mathcal{V}$. We thus see how to construct ultra-universal members of any subvariety of groups, monoids or lattices.

Open Problems
Let $G$ be the group of all permutations on $\mathfrak{S}_0$ with finite support. Is $G$ ultra-universal in
the variety of groups? Since $S_{\aleph_0}$ is ultra-universal in the variety of groups, this is equivalent to asking if $S_{\aleph_0}$ is embeddable in an ultrapower of $G$. Now $G$ is embeddable in an ultraproduct of its finitely generated subgroups which are all finite, and every finite group is embeddable in $G$ and so the question is equivalent to: can every group (equivalently can $S_{\aleph_0}$) be embedded in an ultraproduct of finite groups. This in turn is equivalent to asking: is it true that for any universal sentence $\varphi$, all finite groups satisfy $\varphi$ iff all groups (equivalently $S_{\aleph_0}$) satisfy $\varphi$.

Notice that the variety of groups has a subdirectly irreducible ultra-universal member, $S_{\aleph_0}$, and satisfies the amalgamation property. This is also true of lattices. (Infinite partition lattices are ultra-universal and simple, see [2].) Is there some general theorem concerning amalgamation and subdirectly irreducible ultra-universal models? □

REFERENCES


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