HYLOMORPHIC FUNCTIONS

ANTONY VAN DER MUDE

Abstract. Contemporary mathematicians since the time of Frege have hypothesized that the objects of mathematics exist in a Platonic universe. Mathematical Platonism is part of the Doctrine of the Forms, which was criticized by Aristotle, who stated that the Forms do not exist apart from the things of the real world - a theory known as Hylomorphism. This paper postulates that the observer in the Copenhagen Interpretation of Quantum Mechanics can be represented as a hylomorphic function. These functions, since they compute universal properties of real events, are not time invertible, like most theories of physics. Instead, they define the arrow of time and the information carried by physical media. They also represent the atomic Qualia of subjective experience. Since Church's thesis seems to be a universal property of reality, the effective procedures seem to be an underlying representation of hylomorphic functions over the integers. But due to quantum undecidability, the hylomorphic functions are not effective, but hypercomputations that cannot be computed by a finite procedure. An example of a hypercomputation is the task of Learning in the Limit (similar to Identification in the Limit) over recursively enumerable sets of inputs. The Kolmogorov set of incompressible numbers is an example of this class of functions - but they are random numbers. On the other hand, although reality has a random component, it is predictable within that randomness. We illustrate a hypercomputation that is Learnable in the Limit, such that there is a computable function that gets arbitrarily close in accuracy to this hypercomputation. It is an example of how hylomorphic functions can model physical observations. This implies that though there can be no axiomatic Theory of Everything, we can come up with theories that are more and more accurate the more we learn.

1. Introduction

Metaphysics is the study of the fundamental nature of reality. The field was named after the book by Aristotle that discusses what he termed 'First Philosophy'. In this book he asked about existence and change and how we can understand reality. A major concern of Metaphysics is Ontology, the question of the nature of being or existence, and what categories of things there are that exist.

One of the most important topics of metaphysics and ontology is the Problem of Universals. This is the question of how universal concepts - sometimes known as Forms - come to be associated with the different objects of reality. The notion of the Forms came from Aristotle's teacher Plato and Plato's teacher Socrates. This question also applies to mathematics: why does 1 plus 1 always equal 2 is a question of universals.

There are a variety of approaches to metaphysics. Most theories of metaphysics are often classified one of three ways: Realism, Idealism, or Nominalism. Realists consider the universals to have a real existence. Some realists claim the universals are separate from the world of experience, whose existence manifests itself in a dual reality of universal Forms coexisting with the objects of this world. Other Realists consider the Forms to have an existence that is part of reality. Idealists claim that reality is a construction of the mind. Nominalists deny the existence of universals as a separate existence except as the names of concepts - they are mere words. This paper will maintain that the universals have a real existence, and describe the nature of this existence. Therefore, the alternatives of nominalism and idealism will not be addressed.

Socrates and his student Plato were metaphysical realists. Their theory has come to be known as the Doctrine of the Forms. Plato discusses the Forms in the dialogues Phaedo, Phaedrus, Parmenides and the Republic[10]. The basic Doctrine of the Forms begins with the notion that everything that exists has certain abstract characteristics. The Forms are the universal properties the individual objects of existence share in common. Some Forms are qualities such as Justice and Beauty. Other Forms are more prosaic - an object painted blue has blueness as a Form. A chair can partake of the Form Chairness. At even a more basic level, the number of objects, such as the number of grapes in a bunch, is a Form. Mathematical objects besides number, such as the geometric properties of circle and triangle are also Forms.

To Socrates and Plato, the Forms have a separate existence. In the dialogue the Republic, Plato described humans as being chained inside a cave of reality, where reality is just the shadows cast upon the wall. The dialogue makes the claim that in actuality, the Forms are the images that cast those shadows. It is only a philosopher, who is freed from the bonds of reality who can actually experience the Forms first hand. The Forms exist in a separate world, that to them was even more real than the world around us, which is subject to error and illusion. In the Phaedo, Plato talks of Socrates' death as entering the realm of the Forms.

The idea of dual realms of existence has always been part of philosophy. This idea goes back to the beginning of human knowledge, as a spiritual reality that enlightened humans are in contact with. Hinduism was one of the earliest religions that postulated a heavenly realm, similar in concept to the Forms. Most other great religions since then have considered a heavenly realm.

When it comes to the ontology of mathematics, most contemporary mathematicians adhere to an updated Doctrine of the Forms called Mathematical Platonism[4]. Mathematical Platonism maintains that the objects of mathematics, such as numbers, exist in an ideal world independent of time and space, separate from their individual instantiations as objects. This theory does not have all of the properties of classical Platonism, but it does postulate a separate realm of existence for the Forms. This viewpoint was expressed in the modern form by Frege. Other famous mathematicians such as Kurt Godel have expressed a Mathematical Platonism. It has been generally acknowledged that most modern mathematicians are Platonists.

Frege's argument for mathematical Platonism is illustrated by the following quotes from his book "The Foundations of Arithmetic"[17][8]:

> Yet surely the number one looks like a definite particular object, with properties that can be specified, for example that of remaining unchanged when multiplied by itself? [Introduction, p xiv]
>
> [T]his kind of the mental processes which precede the forming of a judgment of number ... can ever take the place of a genuine definition of the concept. It can never be adduced in proof of any proposition of arithmetic; it acquaints us with none of the

> properties of numbers. For number is no whit more an object of psychology or a product of mental processes than, let us say, the North Sea is, ... (It) is something objective. [Section 26, p 34]
>
> We are faced therefore, with the following difficulty: If we try to produce the number by putting together different distinct objects, the result is an agglomeration in which the objects contained remain still in possession of precisely those properties which serve to distinguish them from one another; and that is not the number. [Section 39, p 50]
>
> Time is only a psychological necessity for numbering, it has nothing to do with the concept of number. We do represent objects which are non-spatial and non-temporal by spatial or temporal points, and this may perhaps be of advantage in carrying out the procedure of numbering; but it presupposes, fundamentally, that the concept of number is applicable to the non-spatial and the non-temporal. [Section 40, p 53]
>
> Number is not anything physical, but nor is it anything subjective (an idea). [Section 45, p 58]
>
> The concept of Number: Every individual number is a self-sufficient object. [Part IV Title, p 67]
>
> But, it will perhaps be objected, even if the Earth is really not imaginable, it is at any rate an external thing, occupying a definite place; but where is the number 4? It is neither outside us nor within us. And, taking those words in their spatial sense, that is quite correct. To give spatial co-ordinates for the number 4 makes no sense; but the only conclusion to be drawn from that is, that 4 is not a spatial object, not that it is not an object at all. Not every object has a place. [Section 61, p 72]
>
> It is only recently that infinite Numbers have been introduced, in a remarkable work by G. Cantor. I heartily share his contempt for the view that in principle only finite Numbers ought to be admitted as actual. Perceptible by the senses they are not, nor are they spatial - any more than fractions are, or negative numbers, or irrational or complex numbers; and if we restrict the actual to what acts on our senses or at least produces effects which may cause sense-perceptions as near of remote consequences, then naturally no number of any of these kinds is actual. [Section 85, p 97]
>
> [E]ven the mathematician cannot create things at will, any more than the geographer can; he too can only discover what is there and give it a name. [Section 96, pp 107-8]

In general, mathematical Platonism consists of the following three principles.[4]

- Existence: The objects of mathematics have a real existence.
- Abstractness: The objects of mathematics are objects that transcend the space-time continuum of physical objects and therefore are separate from the laws of physics.
- Independence: The mathematical objects are independent of thought and language, so they still exist even if there were no one to comprehend them.

Aristotle gave an alternative to Platonism. In his Metaphysics[1], he analyzed the Doctrine of the Forms, and concurred with Plato in the belief that the Forms are real: they provide a conceptual framework that we use to understand the objects of reality, and these concepts exist in their own right. But he had criticisms of the doctrine as Plato described it. These objections led him to a different theory where the Forms do not exist apart from Things.

One of the objections, called the Third Man argument, was actually first proposed in Plato's dialogue Parmenides. The objection comes from the observation that Forms can be combined to form more complex Forms. This means that it is possible to have a complex Form that sits between an object and another Form that the object possesses, and this complex Form expresses the relationship between the other Form and the object. In book Kappa of the Metaphysics, Aristotle summarizes the problem this way:

> But it is hard to say, even if one supposes them [the Forms] to exist, why in the world the same is not true of the other things of which there are Forms, as of the objects of mathematics. I mean that these thinkers place the objects of mathematics between the Forms and the perceptible things, as a kind of third set of things apart both from the Forms and from the things in this world; but there is not a third man or horse besides the ideal and the individuals. If on the other hand it is not as they say, with what sort of things must the mathematicians be supposed to deal? Certainly not with the things in this world; for none of these is the sort of thing which the mathematical sciences demand. [Book K, Section 1]

The idea that the Forms exist in a separate plane of existence leads to other problems. Also in book Kappa, Aristotle has this argument:

> Indeed, it is in general hard to say whether one must assume there is a separable substance besides the sensible substances (i.e. the substances in this world), or that these are the real things and wisdom is concerned with them. For we seem to seek another kind of substance, and this is our problem, i.e. to see if there is something which can exist apart by itself and belongs to no sensible thing. Further, if there is another substance apart from and corresponding to sensible substances, which kinds of sensible substances must be supposed to have this corresponding to them? Why should one suppose men or horses to have it, more than either the other animals or even all lifeless things? On the other hand to set up other and eternal substances equal in number to the sensible and perishable substances would seem to fall beyond the bounds of probability. But if the principle we now seek is not separable from corporeal things, what has a better claim to the name matter? This, however does not exist in actuality, but exists in potency. And it would seem rather that the form or shape is a more important principle than this; but the form is perishable, so that there is no eternal substance at all which can exist apart and independent. But this is paradoxical; for how is there to be order unless there is something eternal and independent and permanent? [Book K Section 2]

The Metaphysics ends with some arguments applied to mathematical objects in particular. Aristotle discusses the relationship between the mathematical Forms and reality, and the question of their independent existence:

> Again, it is not the ideal numbers that are the causes of musical phenomena and the like (for equal ideal numbers differ from one another in form; for even the units do); so that we need not assume Ideas for this reason at least ... These, then, are the results of the theory, and yet more might be brought together. The fact that our opponents have much trouble with the generation of numbers and can in no way make a system of them, seems to indicate that the objects of mathematics are not separable from the sensible things, as some say, and that they are not the first principles [Book N section 6].

So Aristotle has an ontology that is different from that of Plato and Socrates. Although he acknowledges the existence of universals - ideal Forms - they do not have a separate existence in an ideal world. As he put it in Book Zeta:

> Is there, then, a sphere apart from the individual spheres or a house apart from the bricks? Rather we may say that no 'this' would ever have been coming to be, if this had been so, but that the 'form' means the 'such', and is not a 'this' a definite thing; but the artist makes, or the father begets, a 'such' out of a 'this'; and when it has been begotten, it is a 'this such'. And the whole 'this', Callias or Socrates, is analogous to 'this brazen sphere', but man and animal to 'brazen sphere' in general. Obviously, then, the cause which consists of the Forms (taken in the sense in which some maintain the existence of the Forms, i.e. if they are something apart from the individuals) is useless, at least with regard to comings-to-be and to substances; and the Forms need not, for this reason at least, be self-subsistent substances. [Book Z section 8]

This applies also to the objects of mathematics:

> Regarding the objects of mathematics, why are the formulae of the parts not parts of the formulae of the wholes; e.g. why are not the semicircles included in the formula of the circle? It cannot be said, "because these parts are perceptible things"; for they are not. But perhaps this makes no difference; for even some things which are not perceptible must have matter; indeed there is some matter in everything which is not an essence and a bare form but a 'this'. The semicircles, then, will not be parts of the universal circles, as has been said before; for while one kind of matter is perceptible, there is another which is intelligible. [Book Z Section 11]

So he concludes:

> And it is evident that the objects of mathematics do not exist apart; for if they existed apart their attributes would not have been present in bodies. [Book N, Section 3]

That means that the mathematical Forms do not exist apart from things.

The idea that the Forms do not exist apart from things has been termed Hylomorphism[14], from the concept hylo - wood or matter - and the concept morph - form or spirit.

This terminology arose out of the nineteenth century's appreciation of St. Thomas Aquinas' analysis of Aristotle's thought as it applied to Christian philosophy. This paper will give a physical description of hylomorphism.

2. Definition of a Hylomorphic Function

Aristotle's hylomorphic ontology leaves open the question: if the Forms are a representation of the things of physical reality, and the Forms do not exist apart from Reality, then what is the physical representation of the Forms? I postulate that, instead of the Platonic concept of mathematical objects existing in an abstract universe, there exist hylomorphic functions that are a part of reality. The atomic Forms are the range of functions that are the result of a quantum measurement. This means that mathematical objects as Forms are not fundamental objects - they are the outputs of hylomorphic functions which are themselves fundamental. So the Forms themselves are contingent objects that arise from their relationship to the objects of physics as the end product of an observation.

In the Copenhagen interpretation of Quantum Mechanics[21], the properties of real objects can be described as a wave function that provides only a probabilistic description of these objects. The act of observation causes the probability distribution to converge to a particular value: this is known as the wave function collapse. The hylomorphic functions are that mechanism and their output is a hylomorphic Form.

Functional hylomorphism postulates that every quantum state is associated with a function that may or may not converge to a quantifiable result at any given time. A hylomorphic function could be represented as a selection of the maximum probability of a quantum mechanical event. In most cases, the computation of the probabilities will not converge to an answer. If the function does not converge, no observation is computed. When it does converge, the function is computed. Other hylomorphic functions could identify minimum energy states of a system such as the zero-point energy of a system.

The Forms that arise from hylomorphic functions are universal due to the fact that the same function gives the same result when the inputs are the same and the physical phenomenon of interest is instantiated by a physical computation. For example, Forms that relate to a physical quantity or measurement yield a result twice as big if the inputs are twice as much. Therefore mathematical concepts such as $1 + 1 = 2$ are experienced as universal. The conservation laws of hylomorphic functions are the mathematical Forms that define the basic object of mathematics.

I postulate that hylomorphic functions are ontologically separate from the time-symmetric laws of physics. In general, the laws of physics are invertible. There are exceptions, such as the Third law of Thermodynamics that states that entropy increases over time. But hylomorphic functions are typically uninvertible. Once a measurement is made, the inputs that produced that output cannot be reconstructed uniquely. This is because the object is a Form - a universal - and thus applies to an infinite set of inputs (at least in most cases). Since this is a many-to-one mapping, it is impossible to invert it.

The hylomorphic functions are not restricted to purely quantum mechanical observations of reality. Higher level universals are complex Forms that are decomposable into atomic Forms, similar to how a physical object such as a chair is composed of physical atoms. For example, the classical observations of gravitational motions of the planets are hylomorphic functions. Any observation of reality is the result of the output of a hylomorphic function, either a simple quantum observation or a complex of these observations. Note, though, that there is a difference between the physical existence of a table and its Form. The physical existence is composed of atoms, but the Form of the table is an observation composed of atomic Forms generated by hylomorphic functions.

3. Characteristics of Hylomorphic Functions

Hylomorphic functions, since they are the physical manifestation of the universal Forms, can be characterized in a number of interesting ways.

- Hylomorphic functions define the Arrow of Time.
  - Hylomorphic Functions are many to one and therefore not invertible. This means that they define the arrow of time. The many to one property also implies that they increase the number of accessible states, and therefore increase entropy.
- Hylomorphic Functions are the Observer in the Copenhagen Interpretation of Quantum Mechanics.
  - The observer is simply a measurement. For example, a measurement in physics can be a real value, such as a position, or an integer, such as the event of a positron-electron annihilation. In all cases, they are functions that map physical states into a mathematical object that is the value of the measurement.
  - Hylomorphic functions collapse the wave function into a single measurement, but this does not make the wave function deterministic from then on. The measurement of a universal Form is a property that characterizes the wave function at this time, but the wave function still maintains it non-determinacy.
- Hylomorphic Functions are the Qualia.
  - Qualia[11] are considered to be the fundamental unitary sensations of thought. Although they can be anything from the sensation of light and sound to the expression of an emotion, Qualia always involve some functional change. In this sense, Qualia are simply the most basic measurement possible. Qualia as hylomorphic functions give the impression of time passing.
  - Although Qualia are basic sensations, this does not imply that there is always a corresponding perception - let alone an awareness - that can react to the perception, nor need there be a consciousness that is self-aware. Like atoms that can be combined to form more complex structures from molecules up to things like rocks or animals, Qualia are combined to form more complex mental constructs. The hylomorphic functions occur for example as part of a neuron firing, since that is essentially the neural correlate of a measurement.
  - This concept is similar to Leibniz's Monadism[12], although there are significant differences. One major difference is that Leibniz considered consciousness to consist of a single monad. The theory of hylomorphic functions postulates that consciousness is a complex construct built out of atomic Qualia.

  - Hylomorphic functions can be considered to be a type of panpsychism, but only in the simplest sense. The universe does not consist of atomic consciousness, no more than a single machine instruction in a computer is a computer program.
  - This also implies that a measurement in Quantum Mechanics does not imply a conscious observer. A measurement is the end result of a hylomorphic function, but there may be no conscious observer to take note of this measurement. Hylomorphic functions are the basis of perception, but the sensation of that perception or the awareness of it requires some higher order processing. Self-awareness and consciousness are not fundamental - they arise out of these fundamental functions.
  - The separate nature of hylomorphic functions from the symmetrical laws of physics appears to lead to a duality. Dualism seems to be true because the observer is different from the physical waveform. But they are just separate processes in a single reality.
- Hylomorphic Functions are the basic units of Information.
  - Functional hylomorphism explains why information seems to be independent of the medium carrying the information. Information is a hylomorphic property, not a property of matter. Entropy, a measure of the randomness of a system, is also a measure of the carrying capacity of a communication medium. But the information - the message carried by the medium - is the particular value that the medium carries. This value comes from a hylomorphic function or functions, and is therefore an abstract Form. Since these abstract Forms are the outputs of measurements, they require a medium to carry the information, but being abstract, they are essentially independent of the medium.
  - This is why nominalism seems to be true because the Forms are Qualia. Words are Forms. The words carry information.

4. Hylomorphic Functions can be Integer or Real

Particle-wave duality in the Copenhagen interpretation is due to the fact that mathematical representations of physical phenomena can be expressed in terms of functions of reals or integers. Therefore, when it comes to hylomorphic functions, both integers and reals exist as Forms. Real functions describe the wave properties of quantum mechanics. The integer functions describe the quantum properties.

Although hylomorphic functions possess this duality, they are not limited to these fields. The fact that we have this duality in physics is an indication that the Cantor's Continuum Hypothesis is a fact of nature. Other fields beyond the integer and reals exist, but we are not able to interpret them due to the limitations of human comprehension. Therefore duality is a limitation of the human ability to interpret the computations of the hylomorphic functions. If we were able to perceive functions computed over fields that transcend the reals, the duality of quantum mechanics who lead to a trinary interpretation or even more.

Max Tegmark[20] and Edward Fredkin [7] and others have proposed that the laws of science can be represented as automata. But quantum undecidability makes this claim implausible. The fact that Church's thesis seems to be universally true indicates that the laws of physics have automata as a component. But that does not mean that the universe can be modeled as an effective procedure. The Arithmetic Hierarchy[18] shows how the primitive recursive functions can be used as a base set that can represent any number of levels of logical expressions where the higher levels cannot be represented as recursively enumerable functions.

The rest of this paper will be focused on functions over the integers because Church's thesis gives an intellectual basis for describing effective computations over the integers. Church's thesis indicates that general recursive functions (or their correlate such as Turing Machines) actually exist as a basis for describing hylomorphic functions over the integers. Another way of saying this is that Church's Thesis encapsulates a whole family of equivalent Forms that are a universal class of mathematical objects. Although conceptual frameworks such as Blum-Shub-Smale machines[3][6] extend the Turing Machine model to the reals, there is as yet no correlate to Church's thesis for functions over the reals. So we will make no general statements about them.

5. Hylomorphic Functions are Non Recursively Enumerable

Although Church's Thesis describes the universality of the concept of an effective procedure independent of any particular formal representation, the hylomorphic functions are not limited just to the effective procedures. The hylomorphic functions can be composed of primitive or general recursive functions, but not be effective. Functions such as these are known as hypercomputation[16][5] or supertasks. For example, the first two levels of the Arithmetic Hierarchy are the primitive recursive and general recursive functions, but the higher levels of the hierarchy are not effectively computable.

An important class of sets whose members are not the domain of a general recursive function is known as the immune sets[18]. Formally, an immune set A is an infinite set such that for every infinite recursively enumerable set B, there exists an infinite set of elements that are in B but not in A. This means that every recursively enumerable set is wrong for an infinite number of values x when it claims that x is an element of A when in fact it is not.

One of the natural examples of an immune set is the set of incompressible numbers in Kolmogorov Complexity[13]. Kolmogorov complexity $K(x) = i$ can be defined as the smallest index i of a Turing Machine TM_i that, starting with an empty tape, halts with x on its tape. Note that every value x has a corresponding Turing Machine TM_y that just encodes the digits of x in its state table. The value y corresponds to x by a uniform constant. If there is no shorter Turing Machine that outputs x, then $K(x) = y$. This is termed an incompressible number. The incompressible elements of K are also called Kolmogorov random numbers, since Per Martin-Löf[15] showed that they pass every test of randomness. This set is immune because, if there were an infinite recursively set whose domain is φ_x and are all incompressible numbers, then almost all elements of φ_x could be compressed. We can construct a series of Turing Machines TM_y where y is based uniformly on φ_x and an index i, such that TM_y runs φ_x and generates the value z, the i^{th} element of its domain. This compresses the value z since almost all encodings of TM_y and the index i are less than the corresponding value z. This is an explanation for why immune sets are random - there is no infinite subset where a pattern can be detected. Since first order logic is equivalent to the effective functions, another way of stating this is that for any axiom set, only a finite number of values can be proven to be Kolmogorov random.

Not every non-recursively enumerable set is completely random, though. Each non-R.E. set contains a certain amount of randomness, but there can be some underlying pattern that could be captured to a certain extent by a partial recursive function, although there are an infinite number of errors. There is no general way to describe the amount of randomness in a non-R.E. set. But it is possible to express hylomorphic functions in a way that uses the general recursive functions and still has the indeterminacy of quantum mechanics.

Sets such as K have a conceptual genesis in Machine Learning. Kolmogorov based his complexity theory on some previous work by Ray Solomonoff[19]. Solomonoff, and later Mark Gold[9], developed some initial results on a type of Machine Learning called Identification in the Limit. Identification in the Limit is a learning task where the learner is presented with a sequence of elements from a general recursive set and is given the task to determine an index x for the function φ_x whose domain is the set. Since, in general, holomorphic functions are not identifying the set itself, but instead properties of the set that need to be learned, I shall use the term "Learning in the Limit".

Definition 5.1. A property P is Learnable in the Limit on a sequence S, where S is an infinite sequence of inputs $S(n)$, and $P = \varphi_i$ is a partial recursive function where there exists a t such that for all $u >= t$

$$\varphi_i(< S(0), ...S(u) >) = \varphi_i(< S(0), ...S(t) >) = x$$

In this case, $P(S)$ learns x in the limit as a property of the sequence. If there is no final value x, P changes its mind an infinite number of times and so does not converge. We use the notation $P(S) = x$ to indicate that $P(S)$ is Learnable in the Limit as x.

As an example that relates to hylomorphic functions, the sequence could be a well ordering of a set of quantum states. The sequence can of course be countably infinite. So the values in the sequence could be probability values, or even pairs $S =<< 0, p_0 >, < 1, p_1 >, ... >$ where $P(S) = x$ means that P has learned in the limit the state x with highest probability p_x. This state x would be the observation that the hylomorphic function generates.

Although the sequences that are given to P can be anything, for the sake of simplicity, our examples will assume that the sequences are recursively enumerable: the sequence S is the output of a total recursive function φ_s.

Theorem 5.2. *If S is recursively enumerable by a total function $\varphi_s(n) = S(n)$, then P is a partial recursive function φ_j where there exists a t such that for all $u >= t$*

$$\varphi_j(u) = \varphi_j(t) = x$$

Proof. Trivial. $\varphi_j = \varphi_i(\varphi_k(n))$ where

$$\varphi_k(n) =< \varphi_k(n-1), S(n) >=< \varphi_k(n-2), S(n-1), S(n) >= ... =< S(0), ..., S(n) >$$

and the $S(n)$ values are generated by the function φ_s. □

If S is the output of a total recursive function φ_s then we can replace the sequence input in the input of the property $P(S)$ with the index of the total recursive function: $P(s)$.

Definition 5.3. If S is recursively enumerable by a total function $\varphi_s(n) = S(n)$, and $P(S)$ is Learnable in the Limit, then we use the notation $P(s) = x$ where P is a partial recursive function $\varphi_p(s, y)$ where there exists a t such that for all $u >= t$ $\varphi_p(s, u) = \varphi_p(s, t) = x$.

The properties that are Learnable in the Limit are universal in the sense that there can be a whole class of sequences that have the same learnable property P.

Definition 5.4. A class A of sequences is Learnable in the Limit for a property P if every $S \in A$ is Learnable in the Limit for P.

If the class of sequences is a recursively enumerable class of recursively enumerable sequences, then we can replace the sequence in $P(S)$ with $P(s)$, where for all n, $\varphi_s(n) = S(n)$.

Definition 5.5. Assume that A is a recursively enumerable class of recursively enumerable sequences S, where the range of φ_a are values s such that if $S_s \in A$ then for all n, $\varphi_s(n) = S(n)$. Then for all $\varphi_a(n) = s$, if $P(S) = y$ then $P(s) = y$.

Note that the set of all shortest algorithmic descriptions of Kolmogorov complexity is expressed by a such property P. P takes the input x and generates a sequence $< x, x, x, ... >$ at time n of size n. Given that TM_y just encodes the digits of x in its state table, then for each initial segment of the sequence of size n, $P(x)$ runs TM_0 through TM_y each for n steps and outputs i if there exist a value $i < y$ where TM_i halts in less than n steps and outputs x. Otherwise it outputs y. Then P converges to y only if x is incompressible, and is therefore Kolmogorov random.

If the sequence S is recursively enumerable, a quantum measurement can be represented using the theorem above. In most cases, these functions are hypercomputations. If S comes from a non recursively enumerable set, then the function must be a hypercomputation, since the input is.

Learning in the Limit seems to be a good model for many hylomorphic functions. Learning in the Limit functions can look for either maximum or minimum values, for instance. So a quantum measurement could search for a maximum probability out of a sequence of value/probability pairs $< x, p >$ and return the smallest x with the highest probability. For this formulation, the sum of the probabilities does not need to be normalized to 1, since only maxima or minima are output. Also, a quantum measurement could search for the minimum energy out of a sequence of value/energy pairs $< x, e >$.

The hylomorphic functions define a universe of hypercomputations that can be represented as Turing Machines, although with a different convergence criterion. Some Turing machines may not halt in one time quantum: this means that there exist computations with infinite steps that diverge. Besides Turing computability existing as part of the hylomorphic functions, infinitesimals and infinity exist also. The functions are analyzing an infinite number of possibilities, and picking the best answer in the limit. That means that in each quantum of time, there are an infinite number of infinitesimal computations.

These functions cannot be predicted by effective functions, so they appear random. This is why quantum mechanics appears random but is subject to universal laws.

6. Hylomorphic Functions may be Learnable

Since hylomorphic functions are not recursively enumerable, no complete axiomatization of physics is possible. Therefore, there is no axiomatic Theory of Everything. But that does not necessarily mean that the universe is indescribable - in fact, the laws of physics show that an approximation is possible.

Reality seems to be describable by better and better approximations. This is the basis of perturbation theory. This makes it likely that the hylomorphic functions as not immune, since immune sets such as K are completely random. Since they are not effective either, they are somewhere in between.

Every non-immune hypercomputation is to some degree random. If it is 100% predictable it is recursively enumerable. But every hypercomputation has an infinite number of errors compared to any given effective procedure. This means it appears to be random to some extent.

A hypercomputation is approximable if there is a partial recursive function where the error rate on the hypercomputation converges to 0. That is the closest we can get to a Theory of Everything.

It is possible to come up with examples of hypercomputations that can be learned in the limit with a vanishing number of errors. An example is given in the next theorem. We first need to define an error bound in the limit.

Definition 6.1. Given two functions F and G, the error of G on F (or F on G) is the $\lim_{x\to\infty}(|\{y \leq x|F(y) \neq G(y)\}|/x)$. If $F(y)$ converges and $G(y)$ diverges, or vice versa, then $F(y) \neq G(y)$.

Theorem 6.2. *There exists a hypercomputation H such that there is a partial recursive function φ_t such that the error of φ_t on H in the limit is* 0.

Proof. Let $P(s)$ be a property that is Learnable in the Limit for a recursively enumerable class of recursively enumerable sequences S. φ_i is a partial recursive function for P. Assume P is a hypercomputation. Define $H(z) = P(n)$ where for all z the binary representation of z ends in n zeros and the rest of the number except the last 1 is x. So $z = (2x + 1) * 2^n$, i.e. $(10101000, n = 3, x = 1010)$. Therefore every second value of H is $P(0)$, every fourth value of H is $P(1)$ and so on.

Define the recursive function φ_t such that if $z = (2x + 1) * 2^n$, then $\varphi_t(z)$ runs $\varphi_i(n)$ for x steps. Since $P(0)$ is learnable in the limit, then as x goes to infinity, the error goes to $1/2$ even if all of the odd values are in error. In turn, $P(1)$ is learnable in the limit, so the error goes to $1/4$. So in the limit, the error is zero. □

This represents a kind of perturbation theory: the percentage of errors relative to the number of correct values can be arbitrarily small. The hypercomputation can also be modeled by an infinite series of general recursive functions that in the limit have an arbitrarily small error in their predictions.

This theorem can be extended where instead of $H(z) = P(n)$, we have $H(z) = \varphi_v(x)$ where $P(n) = v$. This is closer to the notion of a physical law, where the first index $P(0) = v$ we are learning is a first approximation - computed by φ_v - of the data represented by H. Then we learn the next approximation $P(1)$ and so on. It is typically true that all of these learning tasks are running in parallel, with the most important tasks (say $P(0)$) given the most priority.

This poses the question: if the list of sequences S is not recursively enumerable, would it be possible to learn P in the limit? This is possible even if the universe is not an effective procedure. The hylomorphic functions are an oracle for the universe: through experimentation, we use the observations generated by the hylomorphic functions to discover which properties to learn. This makes even human cognition a hypercomputation even if it could be modeled as a primitive recursive representation of a neural network. This is the power of the Aristotelian Doctrine of the Forms: the Forms do not exist in a separate world of ideals, but are a property of a single universe that contains both the symmetrical laws of physics and the hylomorphic functions as a representation of the Forms. These hylomorphic forms are the Qualia we use to understand the universe, and being hypercomputations, allow non-effective computations.

7. The Bekenstein bound and hylomorphic functions

Finally, we need to address the Bekenstein bound in limiting computations[2]. If the hylomorphic functions are considered as Turing Machines with an infinite number of infinitesimal computational steps in each time quantum, there can be an infinitely complex computation as long as their output is an integer. But there is a limit placed on the possible precision of expressing a real measurement that is given by the Bekenstein Bound. This limit expresses the amount of information in the number of bits in a given volume of space with a given mass.

The Bekenstein bound is the relationship between hylomorphic functions over the reals versus those functions over the integers. The bound limits the number of bits of information possible from the outputs of integer hylomorphic functions. The bound does not, though, limit a hylomorphic function over the reals to a fixed precision. The reason is that a real number is a single datum, just like an integer is. If this value had a fixed precision, then it would be a rational. Wave functions over the rationals would be Turing computable. This would make quantum mechanics decidable, instead of being undecidable. The conversion of a real to an integer is where the bound applies. That is, a hylomorphic function that is an integer approximation to a real hylomorphic function is limited in the number of bits of accuracy. This conversion does not have to meet the Bekenstein bound because of the separate domains of real and integer functions.

But for the hylomorphic functions over the integers by themselves, the limit applies. It is quite likely that the hylomorphic functions have a bound on the computation of their outputs, even though the intermediate computations can be of arbitrary complexity. An example of this is the upper limit on incompressible values in K. Since $K(x)$ must always be smaller than its encoding y, the Turing Machine TM_y that simply encodes x in its states then the function is bounded by the size of its input. It is quite likely that in general, the hylomorphic functions are bounded in a similar way, so that the computations fit in the Bekenstein bound.

8. Conclusions

The hylomorphic functions unify the Platonic duality into a single reality, where the laws of physics determine the objects of reality and the hylomorphic functions instantiate the conceptual qualities of these objects. But there still remains the problem that the physical world and the objects of cognition seem to be different. The hylomorphic functions provide an answer to this. They collapse the quantum mechanical wave function into a single observable value. This value is one of the Universal Forms of Metaphysics - a property of the wave function at that time. These forms are the Qualia that form the basis of cognition, and lead to the sense of consciousness experience.

This still leaves open the question of how the concepts and ideas we think about are composed of atomic hylomorphic functions. Although the objects of our perception are composed of atomic observations, such as when light impinges on the retina, these make up the total experience of an object such as a chair. But there is still the single identification of the chair as its Form. This, of course, is a hylomorphic function - an observation - that is the end product of this identification.

There are a number of concepts that are fundamental to physics and mathematics, such as the existence of integers and reals and the reality of the universal basis of effective computation that is expressed in Church's Thesis. These concepts should be considered to be a hylomorphic basis of reality - their universality has not been disproved, so they are probably have a real ontological existence. In this sense they are Learnable in the Limit.

Quantum mechanics has shown that there is a fundamental indeterminacy in physical measurement. The hylomorphic functions reflect this indeterminancy due to the fact that they are not effectively computable. What still needs to be determined is under what conditions a hylomorphic function converges to an observation and when they do not converge.

That the hylomorphic functions are hypercomputations that can be defined as extensions of effective procedures means that the laws of physics can possibly be expressed in formal logic, but ultimately not be axiomatizable. This means that the universe is not capable of being expressed as a Theory of Everything, even though we may know the universe more and more accurately. Although there may be a single Theory of Everything, it would not be computational or axiomatic: any computational representation would be subject to error. But this leaves open the question of how our theories are structured in such a way that, as we learn more and more, the accuracy of our observations get better and better. This poses the question - can the hylomorphic functions be expressed non-deterministically but completely, or is there a never-ending series of approximations? Insight into this question determines whether the laws of physics have an actual basis in reality or are just a notational convenience, and whether an equivalent intellectual basis can be constructed from an entirely different foundation.

References

[1] Aristote and W.D. Ross. *Aristotle's Metaphysics, a Revised Text with Introduction and Commentary by WD Ross,...* Clarendon Press, 1924.
[2] J.D. Bekenstein. Black holes and the second law. *Lettere Al Nuovo Cimento (1971–1985)*, 4(15):737–740, 1972.
[3] L. Blum, M. Shub, and S. Smale. On a theory of computation over the real numbers; np completeness, recursive functions and universal machines. In *Foundations of Computer Science, 1988., 29th Annual Symposium on*, pages 387–397. IEEE, 1988.
[4] J.C. Cole. Platonism, mathematical. *Internet Encyclopedia of Philosophy*, 2010.
[5] P. Cotogno. Hypercomputation and the physical church-turing thesis. *The British journal for the philosophy of science*, 54(2):181–223, 2003.
[6] Gilles Dowek. Non deterministic computation over the real numbers. In Press, 2013.
[7] E. Fredkin. An introduction to digital philosophy. *International Journal of Theoretical Physics*, 42(2):189–247, 2003.

[8] Frege and Austin. *The Foundations of Mathematics. A LogicoMathematical Enquiry into the Concept of Numbers.* Harper Torchbooks, 1953.
[9] E.M. Gold. Language identification in the limit. *Information and Control*, 10(5):447–474, 1967.
[10] E. Hamilton, H. Cairns, et al. *The collected dialogues of Plato: Including the letters*, volume 71. Bollingen, 1963.
[11] Amy Kind. Qualial. *Internet Encyclopedia of Philosophy*, 2008.
[12] R. Latta. Leibniz: The monadology and other philosophical writings, 1898.
[13] M. Li and P.M.B. Vitanyi. *An Introduction to Kolmogorov Complexity and its Applications.* Springer Verlag, 1997.
[14] Gideon Manning. The history of 'hylomorphism'. In Press, April 2013.
[15] P. Martin-Löf. The definition of random sequences. *Information and Control*, 9(6):602–619, 1966.
[16] T. Ord. The many forms of hypercomputation. *Applied mathematics and computation*, 178(1):143–153, 2006.
[17] E.H. Reck. Frege on numbers: Beyond the platonist picture. *Harvard Review of Philosophy*, 13(2):25, 2005.
[18] H. Rogers Jr. Theory of recursive functions and effective computability.
[19] R.J. Solomonoff. A formal theory of inductive inference part i. *Information and Control*, 7(1):1–22, 1964.
[20] M. Tegmark. The mathematical universe. *Foundations of Physics*, 38(2):101–150, 2008.
[21] J.A. Wheeler, W.H. Zurek, et al. *Quantum Theory and Measurement*, volume 10. Princeton University Press Princeton, NJ, 1983.

131 Wingate Drive, Hackettstown NJ, 07840
E-mail address: vandermude@acm.org
URL: www.vandermude.com