

RIEMANN ZEROS AND AN EXPONENTIAL POTENTIAL

Jose Javier Garcia Moreta

Graduate student of Physics at the UPV/EHU (University of Basque country)

In Solid State Physics

Address: Practicantes Adan y Grijalba 2 5 G

P.O 644 48920 Portugalete Vizcaya (Spain)

Phone: (00) 34 685 77 16 53

E-mail: josegarc2002@yahoo.es

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ABSTRACT: We study a given exponential potential ae^{bx} on the Real half-line which is possible related to the imaginary part of the Riemann zeros. We extend also study also our WKB method to recover the potential from the Eigenvalue Staircase for the Riemann zeros, this eigenvalue staircase includes the oscillatory and smooth part of the Number of Riemann zeros.

In this paper and for simplicity we use units so $2m = 1 = \hbar$

- *Keywords:* = Riemann Hypothesis, WKB semiclassical approximation, exponential potential.

1. Exponential potential and Riemann zeros:

For $T \gg 1$, the number of Riemann zeros with imaginary part on the interval $[0, T]$ is given by [3]

$$N(T) = \frac{T}{2\pi} \ln \left(\frac{T}{2\pi e} \right) + \frac{7}{8} + O\left(\frac{1}{T}\right) + \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + iT \right) \quad (1)$$

Here $\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$ $\text{Re}(s) > 0$ is the Riemann zeta function [2] , and

the Branch of the logarithm is chosen, so the condition $N(0) = 0$ is satisfied

The Hilbert-Polya version for Riemann Hypothesis is the following ,can we find a Hamiltonian operator with positive and Real (since is a self-adjoint operator)

so their Energies satisfy $E_n = \gamma_n^2$ with $\rho_n = \frac{1}{2} + i\gamma_n$ a non-trivial zero of the

Riemann zeta function?.

For this Hamiltonian on the Real half-line $[0, \infty)$ in the form $H = p^2 + f(x)$,the potential should be positive $V(x) \geq 0$, so the energies would be also positive

$$E_n = \langle \Psi_n | H | \Psi_n \rangle = \langle p \Psi_n | p \Psi_n \rangle + \langle \Psi_n | V | \Psi_n \rangle \geq 0 \quad (2)$$

In order to obtain a Hamiltonian we will use the Bohr-Sommerfeld quantization conditions [5] in the form

$$2\pi N(E) = 2 \int_0^{a(E)} \sqrt{E_n - V(x)} dx = 2 \int_0^E \sqrt{E_n - x} \frac{df^{-1}}{dx} = 2\sqrt{\pi} D_x^{-\frac{1}{2}} f^{-1}(x) \quad (3)$$

Here 'a' inside $V(a) = E$ is a turning point of the classical Hamiltonian $H = p^2 + f(x)$, inside (3) we have used the definition of the half-derivative and the half-integral [7]

$$\frac{d^{\frac{1}{2}} f(x)}{dx^{\frac{1}{2}}} = \frac{1}{\Gamma(1/2)} \frac{d}{dx} \int_0^x \frac{dt f(t)}{\sqrt{x-t}} \quad \frac{d^{-\frac{1}{2}} f(x)}{dx^{-\frac{1}{2}}} = \frac{1}{\Gamma(1/2)} \int_0^x dt \frac{f(t)}{\sqrt{x-t}} \quad (4)$$

Also for our Hamiltonian we have imposed boundary conditions on the half line $[0, \infty)$ so the Eigenfunctions $H y_n(x) = E_n y(x)$ satisfy the boundary conditions $y_n(0) = 0 = y_n(\infty)$.

From (3) we obtain that the inverse of the potencial can be described implicitly in terms of the half-derivative of the Eigenvalue staircase (the smooth part)

function $N(E) = \sum_{n=0}^{\infty} H(E - \gamma_n^2)$ with $H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$ the Heaviside's step

function $f^{-1}(x) = 2\sqrt{\pi} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} n(x)$. For the case of the square of the Riemann

zeros, then the smooth part is given approximately by $N_{smooth}(E) = \frac{\sqrt{E}}{2\pi} \ln \left(\frac{\sqrt{E}}{2\pi e} \right)$

To compute the half-derivative we use the representation for the logarithm

$\ln(x) \approx \frac{x^\varepsilon - 1}{\varepsilon}$ $\varepsilon \rightarrow 0$, $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ in this case we get

$$f^{-1}(x) \approx \frac{(4\pi^2 e^2)^{-\varepsilon/2} A(\varepsilon) x^{\varepsilon/2} - B}{\sqrt{\pi} \varepsilon} \quad f(x) \approx 4\pi^2 e^2 \left(\frac{\varepsilon \sqrt{\pi} x + B}{A(\varepsilon)} \right)^{\frac{2}{\varepsilon}} \quad (5)$$

The constants are $A(\varepsilon) = \frac{\Gamma\left(\frac{3+\varepsilon}{2}\right)}{\Gamma\left(1+\frac{\varepsilon}{2}\right)}$ and $B = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ and we have used

the property of the half-derivative of powers of 'x' $\frac{d^{\frac{1}{2}} x^n}{dx^{\frac{1}{2}}} = \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} x^{n-\frac{1}{2}}$.

The last expression inside (5) is equal to $4\pi^2 \exp\left(2 - \frac{2}{\sqrt{\pi}} \frac{\partial F(s)}{\partial s} \Big|_{s=0}\right) e^{4x} = f(x)$

$F(s) = \frac{\Gamma\left(\frac{3}{2} + s\right)}{\Gamma(1+s)}$.So our toy model or approximate model for the Riemann

zeros is given by the Hamiltonian on the half line

$$E_n y(x) = -\frac{d^2 y(x)}{dx^2} + f(x) y(x) \quad y(0) = 0 = y(\infty) \quad E_n \approx \gamma_n^2 \quad \zeta\left(\frac{1}{2} + i\sqrt{E_n}\right) = 0 \quad (6)$$

$$f(x) = \begin{cases} 4\pi^2 \exp\left(2 - \frac{2}{\sqrt{\pi}} \frac{\partial F(s)}{\partial s} \Big|_{s=0}\right) e^{4x} & x > 0 \\ \infty & x \leq 0 \end{cases} \quad . \text{ the properties of (6) are}$$

- The potential inside (6) tends to ∞ in the limit $x \rightarrow \pm\infty$, so (6) has a discrete spectrum
- The potential inside (6) is always positive so the Energies will be always positive $\langle H \rangle = E_n > 0$
- The spectrum is approximately given by the imaginary part of the Riemann Zeros, Hamiltonian (6) reproduces approximately the imaginary part for the Riemann zeros
- The Bohr-sommerfeld conditions for the exponential potential inside (6) reproduces the smooth part of the spectral staircase for the square of the imaginary zeros $2 \int_0^{a=a(E)} \sqrt{E_n - ce^{bx}} dx \approx N_{smooth}(E) = \frac{\sqrt{E}}{2\pi} \ln\left(\frac{\sqrt{E}}{2\pi e}\right)$

- The factor $\frac{7}{8}$ may be viewed as a Maslov index inside the Bohr-

Sommerfeld quantization conditions $\pi\left(n(E) + \frac{7}{8}\right) = \int_0^{a=a(E)} p(x) dx$ with

$p = \sqrt{E - ae^{bx}}$ the momentum of the particle inside the potential.

- The square root of the Energies satisfy that $|\sqrt{E_{n+1}} - \sqrt{E_n}| \rightarrow 0$ in the limit of big quantum numbers $n \rightarrow \infty$.
- Berry and Keating [3] get a similar smooth density of states for their Hamiltonian $-i\left(x \frac{d}{dx} + \frac{1}{2}\right)\Psi(x) = E_n \Psi(x)$ however they do not know what boundary conditions to impose in order to get a discrete spectrum ,which is equal to the imaginary part of the zeros

Equation (6) can be immediatly solved [9] and [1] with a change of variable $x = e^{2u}$ the ODE (6) becomes a differential equation that can be solved in terms of the Bessel functions

$$u^2 \frac{d^2 y(u)}{du^2} + u \frac{dy}{du} + (u^2 - E_n) = 0 \quad y(u) = C_1 J_\mu \left(\frac{\sqrt{-\lambda}}{2} e^{2x} \right) + C_2 J_{-\mu} \left(\frac{\sqrt{-\lambda}}{2} e^{2x} \right) \quad (7)$$

With $\mu = i \frac{\sqrt{E_n}}{2}$ and $\lambda = 4\pi^2 \exp \left(2 - \frac{2}{\sqrt{\pi}} \frac{\partial F(s)}{\partial s} \Big|_{s=0} \right) = 157.9136704... = 16\pi^2$, the exact quantization condition (not the one coming from the Bohr-Sommerfeld rules) is then determined by the boundary condition on the half real line $[0, \infty)$ and it is

$$J_\mu \left(\frac{\sqrt{-\lambda}}{2} \right) = 0 = J_\mu (2\pi i) \quad \lambda = 4\pi^2 \exp \left(2 - \frac{2}{\sqrt{\pi}} \frac{\partial F(s)}{\partial s} \Big|_{s=0} \right) \quad \mu = \frac{i\sqrt{E_n}}{2} \quad (8)$$

Unfortunately, there is no exact analytic method to solve the equation (8) to obtain the energies of the Hamiltonian so we can only solve (8) by numerical methods, an approximate method to obtain the Energies for big values of the Quantum number n is to use the semiclassical method

$N_{smooth}(E) = \frac{\sqrt{E}}{2\pi} \ln \left(\frac{\sqrt{E}}{2\pi e} \right) = n + \frac{1}{2} \approx n$, this equation can be inverted to get the energies in term of the Lambert W-function

$$E_n \approx \frac{4\pi^2 n^2}{W^2(ne^{-1})} \quad W(x)e^{W(x)} = x \quad W(x) = \sum_{n=1}^{\infty} (-n)^{n-1} \frac{x^n}{n!} \quad (9)$$

If we use the asymptotic property for the Lambert W-function $\lim_{x \rightarrow \infty} \frac{W(x)}{\ln x} = 1$ and

take the positive square root we find $\sqrt{E_n} = k_n \approx \frac{2\pi n}{\ln n}$, this is precisely the imaginary part of the Riemann zeros in the limit $n \rightarrow \infty$

The Quantum condition for the energies inside (8) can be generalized to the half

line $[u_0, \infty)$ in the form $J_\mu \left(\frac{\sqrt{-\lambda}}{2} e^{2u_0} \right) = 0$, one of our conjecture is that

, depending on the value of u_0 , we should have different approximations to the

Riemann Xi-function $\xi(s) = \frac{s(s-1)}{2} \Gamma \left(\frac{s}{2} \right) \zeta(s)$ for positive and negative values of 'x', this is why the condition (8) should give an approximation for the square of the Riemann zeros.

$$\frac{\xi \left(\frac{1}{2} + ix \right)}{\xi \left(\frac{1}{2} \right)} \approx \frac{c J_{\frac{ix}{2}} \left(\frac{\sqrt{-\lambda}}{2} e^{2u_0} \right) + \bar{c} J_{-\frac{ix}{2}} \left(\frac{\sqrt{-\lambda}}{2} e^{2u_0} \right)}{(c + \bar{c}) J_0 \left(\frac{\sqrt{-\lambda}}{2} e^{2u_0} \right)} \approx \frac{\xi \left(\frac{1}{2} - ix \right)}{\xi \left(\frac{1}{2} \right)} \quad (10)$$

Here \bar{c} means the complex conjugate of the number c , the easiest expression inside (10) is when $c = \bar{c} = 1$ and $u_0 = 0$

2. An implicit equation for the potential $f^{-1}(x)$ on the real line $[0, \infty)$, Beyond the smooth part of the Riemann zeros

The main problem with our Hamiltonian operator (6) is that we have simply ignored the contribution of the sum of the primes to the eigenvalue staircase for the Riemann zeros defined by

$$N_{osc}(E) = -\frac{1}{\pi} \sum_p \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{p^{\frac{n}{2}}} \sin(n\sqrt{E} \ln p) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + i\sqrt{E}\right) \quad (11)$$

The EXACT equation for the potential (defined implicitly) is the following

$$f^{-1}(x) = 2 \sum_{n=0}^{\infty} \frac{H(x - \gamma_n^2)}{\sqrt{x - \gamma_n^2}} = \frac{2}{\sqrt{\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \arg \xi\left(\frac{1}{2} + i\sqrt{E}\right) \quad \xi(s) = \frac{s(s-1)}{2} \pi^{\frac{-s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \quad (12)$$

The sum $\sum_{n=0}^{\infty} \frac{H(x - \gamma_n^2)}{\sqrt{x - \gamma_n^2}}$ is made over the imaginary part of the Riemann zeros on the upper complex plane $\Im m(s) > 0$, this sum over zeros can be turned into a sum over primes and prime powers with the aid of the Riemann-Weil explicit formula [10]

$$\sum_{\gamma} h(\gamma) = 2h\left(\frac{i}{2}\right) - g(0) \ln \pi - 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} g(\ln n) + \frac{1}{2\pi} \int_{-\infty}^{\infty} ds h(s) \frac{\Gamma'(s)}{\Gamma(s)} \left(\frac{1}{4} + \frac{is}{2}\right) \quad (13)$$

Here, $g(k) = \frac{1}{2\pi} \int_0^{\infty} dx h(x) \cos(kx) = g(-k)$ $h(x)$ and $g(x)$ are test functions which

form a Fourier transform pair and $\Lambda(n) = \begin{cases} \ln p & n = p^k \\ 0 & \text{otherwise} \end{cases}$ is the Mangoldt function., see [2].

If we insert the expression $h(x, r) = \frac{H(x - r^2)}{\sqrt{x - r^2}}$ inside (12) and use the identity

for the Bessel function $\frac{1}{\pi} \int_0^x \frac{dt \cos(ut)}{\sqrt{x^2 - t^2}} = \frac{J_0(ux)}{2}$ then the expression for the

potential of our Hamiltonian (2) on the real half-line $[0, \infty)$ becomes

$$f^{-1}(x) = \frac{4H\left(x + \frac{1}{4}\right)}{\sqrt{4x+1}} + \frac{1}{2\pi} \int_{-\sqrt{x}}^{\sqrt{x}} \frac{dr}{\sqrt{x-r^2}} \left(\frac{\Gamma'\left(\frac{1}{4} + \frac{ir}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{ir}{2}\right)} - \ln \pi \right) - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} J_0(\sqrt{x} \ln n) \quad (14)$$

The last sum over primes and prime powers can be interpreted in terms of the half derivative of the argument of the Riemann zeta function on the critical line

$$\frac{2}{\sqrt{\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \arg \zeta \left(\frac{1}{2} + i\sqrt{x} \right) = - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} J_0(\sqrt{x} \ln n) \quad (15)$$

For $x \gg 1$ equation (13) becomes $\varepsilon \rightarrow 0$.

$$f^{-1}(x) \approx \frac{(4\pi^2 e^2)^{-\varepsilon/2} A(\varepsilon) x^{\varepsilon/2} - B}{\sqrt{\pi} \varepsilon} - \frac{2}{\sqrt[4]{x}} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{2\pi n \ln n}} \cos \left(\sqrt{x} \ln n - \frac{\pi}{4} \right) \quad (16)$$

Unfortunately the sum $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} J_0(\sqrt{x} \ln n)$ is DIVERGENT so we must truncate it, for example by summing only up to some finite number of primes and their prime powers to get some corrections to the exponential potential deduced for the Hamiltonian inside (6).

Another idea would be the following, the sum $\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{2\pi n \ln n}} \cos \left(\sqrt{x} \ln n - \frac{\pi}{4} \right)$ is related to the real part of the half-integral of the divergent series

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\frac{1}{2} + i\sqrt{x}}} =_{reg} - \frac{\zeta'}{\zeta} \left(\frac{1}{2} + i\sqrt{x} \right) \text{ for } x \gg 1, \text{ this last term may be computed by}$$

numerical method instead of evaluating a divergent series for positive $x > 0$.

Equations (15) and (16) can be improved a little more if we could prove (conjecture) that the quantity

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \arg \zeta \left(\frac{1}{2} + i\sqrt{x} \right) \rightarrow 0 \quad x \rightarrow \infty \quad (17)$$

In this case the potential $f(x) = 4\pi^2 \exp \left(2 - \frac{2}{\sqrt{\pi}} \frac{\partial F(s)}{\partial s} \Big|_{s=0} \right) e^{4x}$ would be almost

the exact potential, and the quantization condition $J_{\frac{i}{2}\sqrt{E_n}} \left(\frac{\sqrt{-\lambda}}{2} \right) = 0$ would give

the imaginary part for the Riemann zeros for big quantum number $n \gg 1$ with $E_n = \gamma_n^2$. In this approach the imaginary part of the Riemann zeros on the critical

line are not energies but allowed values of the quantized momenta for the exponential potential λe^{4x} so $J_{\frac{i\gamma_n}{2}}\left(\frac{\sqrt{-\lambda}}{2}\right)=0$ with $\zeta\left(\frac{1}{2}+i\gamma_n\right)=0$.

From the WKB method and the properties of the half-derivative operator we know for our model that the potential is related to the density of states by

$$f^{-1}(x) = 2\sqrt{\pi} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \sum_{n=0}^{\infty} H(x - E_n) \quad \frac{1}{2\sqrt{\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} f^{-1}(x) = \sum_{n=0}^{\infty} \delta(x - E_n) \quad (18)$$

Since $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \left(\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} f(x) \right) = \frac{df(x)}{dx}$, if we take the half derivative inside (14) and use

the identities for the Bessel function and the Dirac delta function

$$\sqrt{\pi} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} J_0(a\sqrt{x}) = \frac{\cos(a\sqrt{x})}{\sqrt{x}} \quad \delta(f(x)) = \sum_n \frac{\delta(x - x_n)}{|f'(x_n)|} \quad (19)$$

We obtain the distributional Riemann-Weil trace formula, so the density of states of our Hamiltonian, with the potential defined implicitly inside (14) is just the Riemann-Weil trace formula

$$\begin{aligned} \sum_{n=0}^{\infty} \delta(x - \gamma_n) + \sum_{n=0}^{\infty} \delta(x + \gamma_n) &= \frac{1}{2\pi} \frac{\zeta}{\zeta'} \left(\frac{1}{2} + ix \right) + \frac{1}{2\pi} \frac{\zeta'}{\zeta} \left(\frac{1}{2} - ix \right) - \frac{\ln \pi}{2\pi} \\ &+ \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + i \frac{x}{2} \right) \frac{1}{4\pi} + \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} - i \frac{x}{2} \right) \frac{1}{4\pi} + \frac{1}{\pi} \delta \left(x - \frac{i}{2} \right) + \frac{1}{\pi} \delta \left(x + \frac{i}{2} \right) \end{aligned} \quad (20)$$

If we take the integral inside (19) with respect to 'x' we obtain the eigenvalue staircase $N(x) = \sum_n H(x - E_n) = \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + i\sqrt{x} \right)$ the spectral eigenvalue staircase for the Riemann zeros.

In general for small 'x' we can evaluate the inverse of the potential numerically by computing the sum $\sum_{n=0}^{\infty} \frac{H(x - \gamma_n^2)}{\sqrt{x - \gamma_n^2}}$, from the properties of the Heaviside step function this sum is finite and easy to evaluate with a computer for $\gamma_n^2 \leq 10^4$, for big 'x' we can use the asymptotics $x \gg 1$

$$f^{-1}(x) \approx 2\sqrt{\pi} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \left(\frac{\sqrt{x}}{2\pi} \ln \left(\frac{\sqrt{x}}{2\pi e} \right) \right) + \frac{2}{\sqrt{\pi}} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \arg \zeta \left(\frac{1}{2} + i\sqrt{x} \right) + O \left(\frac{1}{\sqrt{x}} \right) \quad (21)$$

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