RIEMANN ZEROS AND AN EXPONENTIAL POTENTIAL

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ABSTRACT: We study a given exponential potential ae^{bx} on the Real half-line which is possible related to the imaginary part of the Riemann zeros. We extend alsostudy also our WKB method to recover the potential from the Eigenvalue Staircase for the Riemann zeros, this eigenvalue staircase includes the oscillatory and smooth part of the Number of Riemann zeros.

In this paper and for simplicity we use units so $2m = 1 = \hbar$

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1. Exponential potential and Riemann zeros:

For $T >> 1$, the number of Riemann zeros with imaginary part on the interval $[0,T]$ is given by [3]

$$N(T) = \frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + O\left(\frac{1}{T}\right) + \frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right) \qquad (1)$$

Here $\zeta(s) = \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$ $\mathrm{Re}(s) > 0$ is the Riemann zeta function [2] , and the Branco of the logarithm is chosen, so the condition $N(0) = 0$ is satisfied

The Hilbert-Polya version for Riemann Hypothesis is the following ,can we find a Hamiltonian operador with positive and Real (since is a self-adjoint operador) so their Energies satisfy $E_n = \gamma_n^2$ with $\rho_n = \frac{1}{2} + i\gamma_n$ a non-trivial zero of the Riemann zeta function?.

For this Hamiltonian on the Real half-line $[0,\infty)$ in the form $H = p^2 + f(x)$,the potential should be positive $V(x) \ge 0$, so the energies would be also positive

$$E_n = \langle \Psi_n | H | \Psi_n \rangle = \langle p\Psi_n | p\Psi_n \rangle + \langle \Psi_n | V | \Psi_n \rangle \ge 0 \qquad (2)$$

In order to obtain a Hamiltonian we will use the Bohr-Sommerfeld quantization conditions [5] in the form

$$2\pi N(E)=2\int_0^{a=a(E)}\sqrt{E_n-V(x)}dx=2\int_0^E\sqrt{E_n-x}\,\frac{df^{-1}}{dx}=2\sqrt{\pi}D_x^{-\frac{1}{2}}f^{-1}(x) \quad (3)$$

Here 'a' inside $V(a)=E$ is a turning point of the classical Hamiltonian $H=p^2+f(x)$, inside (3) we have used the definition of the half-derivative and the half-integral [7]

$$\frac{d^{\frac{1}{2}}f(x)}{dx^{\frac{1}{2}}}=\frac{1}{\Gamma(1/2)}\frac{d}{dx}\int_0^x\frac{dtf(t)}{\sqrt{x-t}} \qquad \frac{d^{-\frac{1}{2}}f(x)}{dx^{-\frac{1}{2}}}=\frac{1}{\Gamma(1/2)}\int_0^x dt\frac{f(t)}{\sqrt{x-t}} \quad (4)$$

Also for our Hamiltonian we have imposed boundary conditions on the half line $[0,\infty)$ so the Eigenfunctions $Hy_n(x)=E_ny(x)$ satisfy the boundary conditions $y_n(0)=0=y_n(\infty)$.

From (3) we obtain that the inverse of the potencial can be described implicitly in terms of the half-derivative of the Eigenvalue staircase (the smooth par) function $N(E)=\sum_{n=0}^{\infty}H(E-\gamma_n^2)$ with $H(x)=\begin{cases}1 & \text{x>0}\\ 0 & \text{x<0}\end{cases}$ the Heaviside's step function $f^{-1}(x)=2\sqrt{\pi}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}n(x)$. For the case of the square of the Riemann zeros, then the smooth part is given approximately by $N(E)\approx\frac{\sqrt{E}}{2\pi}\ln\left(\frac{\sqrt{E}}{2\pi e}\right)$

To compute the half-derivative we use the representation for the logarithm $\ln(x)\approx\frac{x^{\varepsilon}-1}{\varepsilon}$ $\varepsilon\to 0$, $e=\sum_{n=0}^{\infty}\frac{1}{n!}$ in this case we get

$$f^{-1}(x)\approx\frac{\left(4\pi^2e^2\right)^{-\varepsilon/2}A(\varepsilon)x^{\varepsilon/2}-B}{\sqrt{\pi}\varepsilon} \qquad f(x)\approx 4\pi^2e^2\left(\frac{\varepsilon\sqrt{\pi}x+B}{A(\varepsilon)}\right)^{\frac{2}{\varepsilon}} \qquad (5)$$

The constants are $A(\varepsilon)=\frac{\Gamma\left(\frac{3+\varepsilon}{2}\right)}{\Gamma\left(1+\frac{\varepsilon}{2}\right)}$ and $B=\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}$ and we have used the property of the half-derivative of powers of 'x' $\frac{d^{\frac{1}{2}}x^n}{dx^{\frac{1}{2}}}=\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}x^{n-\frac{1}{2}}$.

The last expression inside (5) is equal to $4\pi^2 \exp\left(2-\frac{2}{\sqrt{\pi}}\frac{\partial F(s)}{\partial s}|_{s=0}\right)e^{4x}=f(x)$

$F(s)=\frac{\Gamma\left(\frac{3}{2}+s\right)}{\Gamma(1+s)}$ .So our toy model or approximate model for the Riemann zeros is given bye the Hamiltonian on the half line

$$E_n y(x)=-\frac{d^2y(x)}{dx^2}+f(x)y(x) \quad y(0)=0=y(\infty) \quad E_n\approx\gamma_n^2 \quad \zeta\left(\frac{1}{2}+i\sqrt{E_n}\right)=0 \quad (6)$$

$$f(x)=\begin{cases}4\pi^2\exp\left(2-\frac{2}{\sqrt{\pi}}\frac{\partial A(s)}{\partial s}|_{s=0}\right)e^{4x} & \mathrm{x}>0\\ \infty & x\le 0\end{cases}$$. the properties of (6) are

- The potential inside (6) tends to ∞ in the limit $x\to\pm\infty$, so (6) has a discrete spectrum
- The potential inside (6) is always positive so the Energies will be always positive $\langle H\rangle=E_n>0$
- The spectrum is approximately given by the imaginary part of the Riemann Zeros, Hamiltonian (6) reproduces approximately the imaginary part for the Riemann zeros
- The Bohr-sommerfeld conditions for the exponential potential inside (6) reproduces the smooth part of the spectral staircase for the square of the imaginary zeros $2\int_0^{a=a(E)}\sqrt{E_n-ce^{bx}}\,dx\approx N_{smooth}(E)=\frac{\sqrt{E}}{2\pi}\ln\left(\frac{\sqrt{E}}{2\pi e}\right)$
- The factor $\frac{7}{8}$ may be viewed as a Maslov index inside the Bohr-Sommerfeld quantization conditions $\pi\left(n(E)+\frac{7}{8}\right)=\int_0^{a=a(E)}p(x)dx$ with $p=\sqrt{E-ae^{bx}}$ the momentum of the particle inside the potential.

Equation (6) can be inmediatly solved [9] and [1] with a change of variable $x=e^{2u}$ the ODE (6) becomes a differential equation that can be solved in terms of the Bessel functions

$$u^2\frac{d^2y(u)}{du^2}+u\frac{dy}{du}+\left(u^2-E_n\right)=0 \qquad y(u)=C_1J_\mu\left(\frac{\sqrt{\lambda}}{2}e^{2x}\right)+C_2J_{-\mu}\left(\frac{\sqrt{\lambda}}{2}e^{2x}\right) \quad (7)$$

With $\mu=\frac{\sqrt{E_n}}{2}$ and $\lambda=4\pi^2\exp\left(2-\frac{2}{\sqrt{\pi}}\frac{\partial F(s)}{\partial s}|_{s=0}\right)$, the exact quantization condition (not the one coming from the Bohr-Sommerfeld rules) is then determined by the boundary condition on the half real line $[0,\infty)$ and it is

$$J_{\mu}\left(\frac{\sqrt{\lambda}}{2}\right)=0=g(E_n) \qquad \lambda=4\pi^2\exp\left(2-\frac{2}{\sqrt{\pi}}\frac{\partial F(s)}{\partial s}|_{s=0}\right) \qquad \mu=\frac{\sqrt{E_n}}{2} \quad (8)$$

Unfortunately , there is no exact analytic method to solve the equation (8) to obtain the energies of the Hamiltonian so we can only solve (8) by numerical methods, an aproxímate method to obtain the Energies for big values of the Quantum number n is to use the semiclassical method

$N_{smooth}(E)=\frac{\sqrt{E}}{2\pi}\ln\left(\frac{\sqrt{E}}{2\pi e}\right)=n+\frac{1}{2}\approx n$, this equation can be inverted to get the energies in term of the Lambert W-function

$$E_n \approx \frac{4\pi^2 n^2}{W^2(ne^{-1})} \qquad W(x)e^{W(x)}=x \qquad W(x)=\sum_{n=1}^{\infty}(-n)^{n-1}\frac{x^n}{n!} \qquad (9)$$

If we use the asymptotic property for the Lambert W-function $\lim_{x\to\infty}\frac{W(x)}{\ln x}=1$ and take the positive square root we find $\sqrt{E_n}=k_n\approx\frac{2\pi n}{\ln n}$, this is precisely the imaginary part of the Riemann zeros in the limit $n\to\infty$

2. An implicit equation for the potential $f^{-1}(x)$ on the real line $[0,\infty)$:

The main problem with our Hamiltonian operator (6) is that we have simpli ignored the contribution of the sum of the primes to the eigenvalue staircase for the Riemann zeros defined by

$$N_{osc}(E)=-\frac{1}{\pi}\sum_{p}\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{p^{\frac{n}{2}}}\sin\left(n\sqrt{E}\ln p\right)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+i\sqrt{E}\right) \quad (10)$$

The EXACT equation for the potential (defined implicitly) is the following

$$f^{-1}(x)=2\sum_{n=0}^{\infty}\frac{H(x-\gamma_n^2)}{\sqrt{x-\gamma_n^2}}=\frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\arg\xi\left(\frac{1}{2}+i\sqrt{E}\right) \quad \xi(s)=\frac{s(s-1)}{2}\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad (11)$$

The sum $\sum_{n=0}^{\infty}\frac{H(x-\gamma_n^2)}{\sqrt{x-\gamma_n^2}}$ is made over the imaginary part of the Riemann zeros on the upper complex plane $\Im m(s)>0$, this sum over zeros can be turned into a sum over primes and prime powers with the aid of the Riemann-Weil explicit formula [10]

$$\sum_{\gamma} h(\gamma) = 2h\left(\frac{i}{2}\right) - g(0)\ln\pi - 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\ln n) + \frac{1}{2\pi}\int_{-\infty}^{\infty} ds\, h(s)\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{is}{2}\right) \quad (12)$$

Here, $g(k) = \frac{1}{2\pi}\int_{0}^{\infty} dx\, h(x)\cos(kx) = g(-k)$ h(x) and g(x) are test functions which form a Fourier transform pair and $\Lambda(n) = \begin{cases} \ln p & n = p^k \\ 0 & \text{otherwise} \end{cases}$ is the Mangoldt function., see [2].

If we insert the expression $h(x,r) = \frac{H(x-r^2)}{\sqrt{x-r^2}}$ inside (12) and use the identity for the Bessel function $\frac{1}{\pi}\int_0^x \frac{dt\cos(ut)}{\sqrt{x^2-t^2}} = \frac{J_0(ux)}{2}$ then the expression for the potential of our Hamiltonian (2) on the real half-line $[0,\infty)$ becomes

$$f^{-1}(x) = \frac{4H\left(x+\frac{1}{4}\right)}{\sqrt{4x+1}} + \frac{1}{2\pi}\int_{-\sqrt{x}}^{\sqrt{x}}\frac{dr}{\sqrt{x-r^2}}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right) - \ln\pi\right) - \sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}J_0\left(\sqrt{x}\ln n\right) \quad (13)$$

The last sum over primes and prime powers can be interpreted in terms of the half derivative of the argument of the Riemann zeta function on the critical line

$$\frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\arg\zeta\left(\frac{1}{2}+i\sqrt{x}\right) = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}J_0\left(\sqrt{x}\ln n\right) \quad (14)$$

For x>>>1 equation (13) becomes $\varepsilon \to 0$.

$$f^{-1}(x) \approx \frac{\left(4\pi^2 e^2\right)^{-\varepsilon/2} A(\varepsilon)x^{\varepsilon/2} - B}{\sqrt{\pi}\varepsilon} - \frac{2}{\sqrt[4]{x}}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{2\pi n\ln n}}\cos\left(\sqrt{x}\ln n - \frac{\pi}{4}\right) \quad (15)$$

Unfortunately the sum $\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}J_0\left(\sqrt{x}\ln n\right)$ is DIVERGENT so we must truncate it , for example by summing only up to some finite number of primes and their prime powers to get some corrections to the exponential potential deduced for the Hamiltonian inside (6).

From the WKB method and the properties of the half-derivative operator we know for our model that the potential is related to the density of states by

$$f^{-1}(x) = 2\sqrt{\pi}\,\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\sum_{n=0}^{\infty} H(x-E_n) \qquad \frac{1}{2\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} f^{-1}(x) = \sum_{n=0}^{\infty}\delta(x-E_n) \quad (16)$$

Since $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left(\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\right) f(x) = \frac{df(x)}{dx}$, if we take the half derivative inside (14) and use the identities for the Bessel function and the Dirac delta function

$$\sqrt{\pi}\,\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} J_0\left(a\sqrt{x}\right) = \frac{\cos\left(a\sqrt{x}\right)}{\sqrt{x}} \qquad \delta\left(f(x)\right) = \sum_n \frac{\delta\left(x-x_n\right)}{|f'(x_n)|} \quad (17)$$

We obtain the distributional Riemann-Weil trace formula, so the density of states of our Hamiltonian , with the potential defind implicitly inside (14) is just the Riemann-Weil trace formula

$$\begin{aligned}\sum_{n=0}^{\infty}\delta\left(x-\gamma_n\right) + \sum_{n=0}^{\infty}\delta\left(x+\gamma_n\right) &= \frac{1}{2\pi}\frac{\zeta}{\zeta}\left(\frac{1}{2}+ix\right) + \frac{1}{2\pi}\frac{\zeta'}{\zeta}\left(\frac{1}{2}-ix\right) - \frac{\ln\pi}{2\pi} \\ &+ \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+i\frac{x}{2}\right)\frac{1}{4\pi} + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}-i\frac{x}{2}\right)\frac{1}{4\pi} + \frac{1}{\pi}\delta\left(x-\frac{i}{2}\right) + \frac{1}{\pi}\delta\left(x+\frac{i}{2}\right)\end{aligned} \quad (18)$$

If we take the integral inside (18) with respect to 'x' we obtain the eigenvalue staircase $N(x) = \sum_n H\left(x-E_n\right) = \frac{1}{\pi}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$

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