RIEMANN ZEROS AND AN EXPONENTIAL POTENTIAL

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ABSTRACT: We study a given exponential potential ae^{bx} on the Real half-line which is possible related to the imaginary part of the Riemann zeros. We study also how we could use the WKB method to recover the potential from the Eigenvalue Staircase for the Riemann zeros
In this paper and for simplicity we use units so $2m = 1 = \hbar$

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1. Exponential potential and Riemann zeros:

For $T >> 1$, the number of Riemann zeros with imaginary part on the interval $[0,T]$ is given by [3]

$$N(T) = \frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + O\left(\frac{1}{T}\right) + \frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right) \qquad (1)$$

Here $\zeta(s) = \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$ $\mathrm{Re}(s) > 0$ is the Riemann zeta function [2] .

The Hilbert-Polya version for Riemann Hypothesis is the following ,can we find a Hamiltonian operador with positive and Real (since is a self-adjoint operador) so their Energies satisfy $E_n = \gamma_n^2$ with $\rho_n = \frac{1}{2} + i\gamma_n$ a non-trivial zero of the Riemann zeta function?.

For this Hamiltonian on the Real half-line $[0,\infty)$ in the form $H = p^2 + f(x)$,the potential should be positive $V(x) \ge 0$, so the energies would be also positive

$$E_n = \langle \Psi_n | H | \Psi_n \rangle = \langle p\Psi_n | p\Psi_n \rangle + \langle \Psi_n | V | \Psi_n \rangle \ge 0 \qquad (2)$$

In order to obtain a Hamiltonian we will use the Bohr-Sommerfeld quantization conditions [5] in the form

$$2\pi N(E) = 2\int_0^{a=a(E)} \sqrt{E_n - V(x)}dx = 2\int_0^{E} \sqrt{E_n - x}\,\frac{df^{-1}}{dx} = 2\sqrt{\pi}D_x^{-\frac{1}{2}} f^{-1}(x) \quad (3)$$

Here 'a' inside $V(a) = E$ is a turning point of the classical Hamiltonian $H = p^2 + f(x)$, inside (3) we have used the definition of the half-derivative and the half-integral [7]

$$\frac{d^{\frac{1}{2}} f(x)}{dx^{\frac{1}{2}}} = \frac{1}{\Gamma(1/2)}\frac{d}{dx}\int_0^x \frac{dtf(t)}{\sqrt{x-t}} \qquad \frac{d^{-\frac{1}{2}} f(x)}{dx^{-\frac{1}{2}}} = \frac{1}{\Gamma(1/2)}\int_0^x dt\frac{f(t)}{\sqrt{x-t}} \quad (4)$$

Also for our Hamiltonian we have imposed boundary conditions on the half line $[0,\infty)$ so the Eigenfunctions $Hy_n(x) = E_n y(x)$ satisfy the boundary conditions $y_n(0) = 0 = y_n(\infty)$.

From (3) we obtain that the inverse of the potencial can be described implicitly in terms of the half-derivative of the Eigenvalue staircase (the smooth par) function $N(E) = \sum_{n=0}^{\infty} H(E - \gamma_n^2)$ with $H(x) = \begin{cases} 1 & \text{x>0} \\ 0 & \text{x<0} \end{cases}$ the Heaviside's step function $f^{-1}(x) = 2\sqrt{\pi}\,\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} n(x)$. For the case of the square of the Riemann zeros, then the smooth part is given approximately by $N(E) \approx \frac{\sqrt{E}}{2\pi}\ln\left(\frac{\sqrt{E}}{2\pi e}\right)$

To compute the half-derivative we use the representation for the logarithm $\ln(x) \approx \frac{x^{\varepsilon}-1}{\varepsilon}$ $\varepsilon \to 0$, $e = \sum_{n=0}^{\infty}\frac{1}{n!}$ in this case we get

$$f^{-1}{}_{smooth}(x) \approx \frac{\left(4\pi^2 e^2\right)^{-\varepsilon/2} A(\varepsilon)x^{\varepsilon/2} - B}{\sqrt{\pi}\varepsilon} \qquad f_{smooth}(x) \approx 4\pi^2 e^2\left(\frac{\varepsilon\sqrt{\pi}x + B}{A(\varepsilon)}\right)^{\frac{2}{\varepsilon}} \quad (5)$$

The constants are $A(\varepsilon) = \frac{\Gamma\left(\frac{3+\varepsilon}{2}\right)}{\Gamma\left(1+\frac{\varepsilon}{2}\right)}$ and $B = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ and we have used the property of the half-derivative of powers of 'x' $\frac{d^{\frac{1}{2}}x^n}{dx^{\frac{1}{2}}} = \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}x^{n-\frac{1}{2}}$.

The last expression inside (5) is equal to $4\pi^2 \exp\left(2-\frac{2}{\sqrt{\pi}}\frac{\partial F(s)}{\partial s}|_{s=0}\right)e^{4x} = f(x)$

$F(s) = \frac{\Gamma\left(\frac{3}{2}+s\right)}{\Gamma(1+s)}$ .So our toy model or approximate model for the Riemann zeros is given bye the Hamiltonian on the half line

$$E_n y(x) = -\frac{d^2 y(x)}{dx^2} + f(x)y(x) \quad y(0) = 0 = y(\infty) \quad E_n \approx \gamma_n^2 \quad \zeta\left(\frac{1}{2}+i\sqrt{E_n}\right) = 0 \quad (6)$$

$$f(x) = \begin{cases} 4\pi^2 \exp\left(2-\frac{2}{\sqrt{\pi}}\frac{\partial A(s)}{\partial s}|_{s=0}\right)e^{4x} & \text{x} > 0 \\ \infty & x \le 0 \end{cases}$$. the properties of (6) are

- The potential inside (6) tends to ∞ in the limit $x \to \pm\infty$, so (6) has a discrete spectrum
- The potential inside (6) is always positive so the Energies will be always positive $\langle H \rangle = E_n > 0$
- The spectrum is approximately given by the imaginary part of the Riemann Zeros, Hamiltonian (6) reproduces approximately the imaginary part for the Riemann zeros
- The Bohr-sommerfeld conditions for the exponential potential inside (6) reproduces the smooth part of the spectral staircase for the square of the imaginary zeros $2\int_0^{a=a(E)} \sqrt{E_n - ce^{bx}}\,dx \approx N_{smooth}(E) = \frac{\sqrt{E}}{2\pi}\ln\left(\frac{\sqrt{E}}{2\pi e}\right)$

Equation (6) can be inmediatly solved [9] and [1] with a change of variable $x = e^{2u}$ the ODE (6) becomes a differential equation that can be solved in terms of the Bessel functions

$$u^2 \frac{d^2 y(u)}{du^2} + u\frac{dy}{du} + \left(u^2 - E_n\right) = 0 \qquad y(u) = C_1 J_\mu\left(\frac{\sqrt{\lambda}}{2}e^{2x}\right) + C_2 J_{-\mu}\left(\frac{\sqrt{\lambda}}{2}e^{2x}\right) \quad (7)$$

With $\mu = \frac{\sqrt{E_n}}{2}$ and $\lambda = 4\pi^2 \exp\left(2-\frac{2}{\sqrt{\pi}}\frac{\partial F(s)}{\partial s}|_{s=0}\right)$, the exact quantization condition (not the one coming from the Bohr-Sommerfeld rules) is then determined by the boundary condition on the half real line $[0,\infty)$ and it is

$$J_\mu(\lambda) = 0 = g(E_n) \qquad \lambda = 4\pi^2 \exp\left(2-\frac{2}{\sqrt{\pi}}\frac{\partial F(s)}{\partial s}|_{s=0}\right) \quad \mu = \frac{\sqrt{E_n}}{2} \quad (8)$$

Unfortunately , there is no exact analytic method to solve the equation (8) to obtain the energies of the Hamiltonian so we can only solve (8) by numerical methods, an aproxímate method to obtain the Energies for big values of the Quantum number n is to use the semiclassical method $N_{smooth}(E) = \frac{\sqrt{E}}{2\pi} \ln\left(\frac{\sqrt{E}}{2\pi e} \right) = n + \frac{1}{2} \approx n$, this equation can be inverted to get the energies in term of the Lambert W-function

$$E_n \approx \frac{4\pi^2 n^2}{W^2(ne^{-1})} \qquad W(x)e^{W(x)} = x \qquad W(x) = \sum_{n=1}^{\infty} (-n)^{n-1} \frac{x^n}{n!} \qquad (9)$$

If we use the asymptotic property for the Lambert W-function $\lim_{x\to\infty} \frac{W(x)}{\ln x} = 1$ and take the positive square root we find $\sqrt{E_n} = k_n \approx \frac{2\pi n}{\ln n}$, this is precisely the imaginary part of the Riemann zeros in the limit $n \to \infty$

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