

THE POISSON REALIZATION OF $\mathfrak{so}(2, 2k + 2)$ ON MAGNETIC LEAVES

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ABSTRACT. Let $\mathbb{R}_*^{2k+1} = \mathbb{R}^{2k+1} \setminus \{\vec{0}\}$ ($k \geq 1$) and $\pi: \mathbb{R}_*^{2k+1} \rightarrow S^{2k}$ be the map sending $\vec{r} \in \mathbb{R}_*^{2k+1}$ to $\frac{\vec{r}}{|\vec{r}|} \in S^{2k}$. Denote by $P \rightarrow \mathbb{R}_*^{2k+1}$ the pullback by π of the canonical principal $SO(2k)$ -bundle $SO(2k+1) \rightarrow S^{2k}$. Let $E_{\sharp} \rightarrow \mathbb{R}_*^{2k+1}$ be the associated co-adjoint bundle and $E^{\sharp} \rightarrow T^*\mathbb{R}_*^{2k+1}$ be the pullback bundle under projection map $T^*\mathbb{R}_*^{2k+1} \rightarrow \mathbb{R}_*^{2k+1}$. The canonical connection on $SO(2k+1) \rightarrow S^{2k}$ turns E^{\sharp} into a Poisson manifold.

The main result here is that the real Lie algebra $\mathfrak{so}(2, 2k + 2)$ can be realized as a Lie subalgebra of the Poisson algebra $(C^{\infty}(\mathcal{O}^{\sharp}), \{, \})$, where \mathcal{O}^{\sharp} is a symplectic leaf of E^{\sharp} of special kind. Consequently, in view of the earlier result of the author, an extension of the classical MICZ Kepler problems to dimension $2k + 1$ is obtained. The hamiltonian, the angular momentum, the Lenz vector and the equation of motion for this extension are all explicitly worked out.

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1. INTRODUCTION

The Kepler problem is the physics problem about two bodies which attract each other by a force proportional to the inverse square of the distance. Historically this problem played a pivotal role in the development of both classical mechanics and quantum mechanics.

The Kepler problem has long been known to exist in all higher dimensions. A surprising discovery [1, 2] in the 1960s is that the magnetized versions of the Kepler problem, under the name of MICZ Kepler problem, also exist. For a while, these magnetized versions were thought [3] to exist only in dimension 3, 5 and possibly 9, corresponding to the division algebra \mathbb{C} , \mathbb{H} and \mathbb{Q} respectively. However, it was demonstrated in Ref. [4] that these magnetized versions exist quantum mechanically in all higher dimensions, and that leads us to believe that these magnetized versions also exist classically in all higher dimensions.

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In Ref. [5] the intimate relation between the conformal algebras of euclidean Jordan algebras and the generalized Kepler problems (i.e., integrable models which share the characteristic features of the Kepler problem) was discovered. In particular, this relation says that the “nice” Poisson realization of the conformal algebra on symplectic spaces and the classical generalized Kepler problems correspond to each other.

As an elaboration of the last sentence, let us take the Kepler problem as an example. Here, the euclidean Jordan algebra is $V = \mathbb{R} \oplus \mathbb{R}^3$ with this Jordan multiplication:

$$(\alpha, \mathbf{u})(\beta, \mathbf{v}) = (\alpha\beta + \mathbf{u} \cdot \mathbf{v}, \alpha\mathbf{v} + \beta\mathbf{u}),$$

and the conformal algebra is $\mathfrak{so}(2, 4)$. Let C_1 be the future light cone for the Minkowski space. Using the standard euclidean structure on $\Gamma(3)$, T^*C_1 can be viewed as a symplectic submanifold of T^*V . It is well-known that $\mathfrak{so}(2, 4)$ has a Poisson realization on the total cotangent space T^*V , hence a Poisson realization on T^*C_1 . This later Poisson realization on T^*C_1 , or equivalently on $T^*(\mathbb{R}^3 \setminus \{0\})$, is the “nice” Poisson realization of $\mathfrak{so}(2, 4)$ that corresponds to the classical Kepler problem. Though not emphasized in this article, the reformulation of the Kepler problem as a dynamic problem on C_1 is much more natural, cf. Ref. [6].

In section 2, we introduce the notion of **magnetic cone** for the Lie group $\mathrm{SO}(2k)$, a real algebraic set inside $\mathfrak{so}^*(2k)$ which is the union of certain special co-adjoint orbits of $\mathrm{SO}(2k)$. In section 3, we review the notion of Wong’s phase space and its symplectic leaves (i.e., Sternberg’s phase spaces) [7, 8, 9, 10], and then introduce the notion of **magnetic leaves**, which are certain special kind of Sternberg’s phase spaces. In section 4, we review the notion of generalized Dirac monopoles [11, 12], a concept that is crucially used in the introduction of the magnetic leaves. In section 5, we describe and prove our main result: the Poisson realization of the real Lie algebra $\mathfrak{so}(2, 2k + 2)$ on the magnetic leaves. In the last section, based on Ref. [13] and the main result here, we derive an extension of the classical MICZ Kepler problems to dimension $2k + 1$. The hamiltonian, the angular momentum, the Lenz vector and the equation of motion for this extension are all explicitly worked out.

2. MAGNETIC CONE

Let G be a compact semi-simple Lie group, \mathfrak{g} be its Lie algebra and \mathfrak{h} be (one of) its Cartan subalgebra. The Lie bracket shall be denoted by $[\cdot, \cdot]$, and the natural pairing $\mathfrak{g}^* \otimes \mathfrak{g} \rightarrow \mathbb{R}$ shall be written as $\langle \cdot, \cdot \rangle$. By convention, the elements of $i\mathfrak{g}$ (rather than \mathfrak{g}) shall be viewed as linear functions on \mathfrak{g}^* : for $X \in i\mathfrak{g}$ and $\xi \in \mathfrak{g}^*$ we have

$$(2.1) \quad X(\xi) := \langle \xi, iX \rangle.$$

It is well known that the Lie algebra structure on \mathfrak{g} defines a natural Poisson structure on \mathfrak{g}^* : Let $X, Y \in i\mathfrak{g}$, then the Poisson bracket of the linear function X with the linear function Y is the linear function $i[iX, iY]$, i.e.,

$$(2.2) \quad \{X, Y\} := i[iX, iY].$$

The adjoint action of G on \mathfrak{g} gives rise to the co-adjoint action of G on \mathfrak{g}^* . This co-adjoint action partitions \mathfrak{g}^* into a disjoint union of the co-adjoint orbits. Except for the trivial orbit $\{0\}$, the co-adjoint orbits are symplectic manifolds and are in fact the symplectic leaves of the Poisson manifold \mathfrak{g}^* .

The irreducible representations of G are all finite dimensional and unitarizable, and are parametrized by their highest weights, which are precisely the dominant integral weights

for G . A nice way to look at these representations is via the orbit theory of Kirillov [14]. Since G is a compact semi-simple Lie group, its co-adjoint orbits are closed and each of them intersects the fundamental Weyl chamber $\mathcal{C} \subset \mathfrak{h}^*$ in a single point. (Using the Killing metric on \mathfrak{g} , \mathfrak{h}^* can be naturally imbedded into \mathfrak{g}^* , so $\mathcal{C} \subset \mathfrak{g}^*$.) An orbit \mathcal{O} is called integral if the unique point in $\mathcal{O} \cap \mathcal{C}$ belongs to the weight lattice of G . The highest weight theory can be restated in the form of a bijection between the set of integral co-adjoint orbits and the set of equivalence classes of irreducible unitary representations of G : the highest weight representation $L(\lambda)$ with highest weight $\lambda \in \mathcal{C}$ corresponds to the integral co-adjoint orbit $G \cdot \lambda$.

In the remainder of this section, we take $G = \mathrm{SO}(2k)$, $\mathfrak{g} = \mathfrak{so}(2k)$, $\mathfrak{h} = \bigoplus_{i=1}^k \mathfrak{so}(2)$ so that an element of \mathfrak{h} in the defining representation of $\mathfrak{so}(2k)$ is a block-diagonal matrix whose diagonal blocks are 2×2 real skew-symmetric matrices. Hereafter, we shall use γ_{ab} ($1 \leq a, b \leq 2k$) to denote the element of $i\mathfrak{g}$ such that in the defining representation of \mathfrak{g} , $i\gamma_{ab}$ is represented by the skew-symmetric real symmetric matrix whose (a, b) -entry is -1 , (b, a) entry is 1 , and all other entries are 0 . For the invariant metric on \mathfrak{g} , we take the following one: Let ρ be the defining representation of \mathfrak{g} , for ξ, η in \mathfrak{g} , we have

$$(2.3) \quad (\xi, \eta) = -\frac{1}{2} \mathrm{tr} (\rho(\xi) \rho(\eta)^T).$$

Then $i\gamma_{ab}$ ($1 \leq a < b \leq 2k$) form an orthonormal basis for \mathfrak{g} . One can verify that the following Poisson bracket relations hold:

$$(2.4) \quad \{\gamma_{ab}, \gamma_{cd}\} = -\delta_{ac}\gamma_{bd} + \delta_{bc}\gamma_{ad} - \delta_{bd}\gamma_{ac} + \delta_{ad}\gamma_{bc}.$$

Let $\sigma_+ = \sum_{a=1}^k i\gamma_{2a-1, 2a} \in \mathfrak{h}$. Via the invariant metric (2.3), σ_+ is turned into an element of \mathfrak{g}^* , which shall be denoted by σ_+^* . Let $G' = G \times \mathbb{R}_+$. The co-adjoint action of G on \mathfrak{g}^* and the scaling action on \mathfrak{g}^* commute, so they gives rise to an action of $G' = G \times \mathbb{R}_+$ on \mathfrak{g}^* . The orbit $G' \cdot \sigma_+^*$ shall be referred to as the **positive magnetic cone** \mathcal{M}_+ for G .

Let $\sigma_- = i\gamma_{12} + \cdots + i\gamma_{2k-3, 2k-2} - i\gamma_{2k-1, 2k}$. Via the invariant metric (2.3), σ_- is turned into an element of \mathfrak{g}^* , which shall be denoted by σ_-^* . The orbit $G' \cdot \sigma_-^*$ shall be referred to as the **negative magnetic cone** \mathcal{M}_- for G .

One can check that \mathcal{M}_+ and \mathcal{M}_- are disjoint from each other, and $0 \in \mathfrak{g}^*$ is the unique extra limit point of \mathcal{M}_+ and also \mathcal{M}_- . We denote by \mathcal{M} the union of these three G' orbits: \mathcal{M}_+ , $\{0\}$, and \mathcal{M}_- . Then \mathcal{M} is topologically closed and shall be referred to as the **magnetic cone** for G .

We shall show that the magnetic cone for G is a real algebraic set inside \mathfrak{g}^* . To do that, we let Q be the homogenous quadratic polynomial function $\frac{1}{2k} \sum_{1 \leq a, b \leq 2k} (\gamma_{ab})^2$ on \mathfrak{g}^* . Then Q is invariant under the action of G : Let $g \in G$, then $g \cdot \gamma_{ab} = g_{aa'} g_{bb'} \gamma_{a'b'}$ where $[g_{aa'}] \in \mathrm{SO}(2n)$, so

$$\begin{aligned} g \cdot Q &= \frac{1}{2k} \sum_{1 \leq a, b \leq 2k} (g \cdot \gamma_{ab})^2 \\ &= \frac{1}{2k} \sum_{1 \leq a, b \leq 2k} g_{aa'} g_{bb'} \gamma_{a'b'} g_{aa''} g_{bb''} \gamma_{a''b''} \\ &= \frac{1}{2k} \delta_{a'a''} \delta_{b'b''} \gamma_{a'b'} \gamma_{a''b''} \\ &= \frac{1}{2k} \sum_{1 \leq a', b' \leq 2k} \gamma_{a'b'} \gamma_{a'b'} = Q. \end{aligned}$$

A similar computation shows that if $\sum_{a=1}^{2k} \gamma_{ab}\gamma_{ac} = \delta_{bc}Q$ hold for $1 \leq b, c \leq 2k$, then $\sum_{a=1}^{2k} (g \cdot \gamma_{ab})(g \cdot \gamma_{ac}) = \delta_{bc}Q$ hold for $1 \leq b, c \leq 2k$ and any $g \in G$, in fact, for any $g \in G'$.

Proposition 2.1. *The magnetic cone \mathcal{M} for $\text{SO}(2k)$ is the real algebraic set defined by the homogenous quadratic equations*

$$\sum_{a=1}^{2k} \gamma_{ab}\gamma_{ac} = \delta_{bc}Q \quad 1 \leq b, c \leq 2k.$$

Proof. 1) We need to check that the identities hold on \mathcal{M} . Let us do the checking on \mathcal{M}_+ . Due to the remark we made right before the statement of this proposition, it suffices to verify the identities at point σ_+ , i.e., the identities

$$\sum_{a=1}^{2k} (i\gamma_{ab}, \sigma_+)(i\gamma_{ac}, \sigma_+) = \frac{\delta_{bc}}{k} \sum_{1 \leq a < b \leq 2k} (i\gamma_{ab}, \sigma_+)^2.$$

These identities are clearly true because $\sigma_+ = i\gamma_{12} + i\gamma_{34} + \cdots + i\gamma_{2k-1, 2k}$. Similarly, the identities hold on \mathcal{M}_- . Therefore, the identities hold on the magnetic cone.

2) We need to check that the identities hold only on \mathcal{M} . It suffices to check that if the identities

$$(2.5) \quad \sum_{a=1}^{2k} (i\gamma_{ab}, \sigma)(i\gamma_{ac}, \sigma) = \frac{\delta_{bc}}{k} \sum_{1 \leq a < b \leq 2k} (i\gamma_{ab}\sigma)^2,$$

hold for $\sigma = x_1 i\gamma_{12} + x_2 i\gamma_{34} + \cdots + x_k i\gamma_{2k-1, 2k}$, then $|x_1| = |x_2| = \cdots = |x_k|$, so $\sigma^* \in \mathcal{M}$. This is indeed the case because, by letting $b = c = 2j$ in identities (2.5), we have

$$x_j^2 = \frac{1}{k}(x_1^2 + \cdots + x_k^2)$$

which is the same for all j between 1 and k . \square

We conclude this section by introducing a magnetic charge function $m: \mathcal{M} \rightarrow \mathbb{R}$. By definition, $|m(\xi)| = |\xi|$ and the sign of $m(\xi)$ is \pm if $\xi \in \mathcal{M}_{\pm}$. It is clear that m is invariant under the co-adjoint action of G and an orbit inside the magnetic cone is uniquely determined by the constant value that m takes. For this reason, if an orbit on which m takes the constant value μ , then this orbit is denoted by \mathcal{O}_{μ} and is referred to as the **magnetic orbit with magnetic charge μ** .

3. STERNBERG'S PHASE SPACES AND MAGNETIC LEAVES

This section is mainly a review of Sternberg's phase spaces, cf. Refs [7, 8, 9]. Let X be a smooth n -manifold, G a compact connected Lie group with Lie algebra \mathfrak{g} , and $P \rightarrow X$ a principal G -bundle over X with a fixed connection ∇ . Denote by $P^{\sharp} \rightarrow T^*X$ the pullback bundle of this principal G -bundle by the bundle projection map $T^*X \rightarrow X$, and

$$(3.1) \quad E^{\sharp} \rightarrow T^*X$$

be the co-adjoint bundle, i.e., the associated real vector bundle with \mathfrak{g}^* as its fiber.

Proposition 3.1. *With the set-up given above, the connection ∇ turns E^{\sharp} into a Poisson manifold.*

As shown in Ref. [9], the Hamilton's equation on E^\sharp for a natural Hamiltonian function is Wong's equation in Ref. [10], so E^\sharp is referred to as the Wong's phase space. However, the more relevant phase spaces to us are Sternberg's phase spaces $\mathcal{O}^\sharp := P^\sharp \times_G \mathcal{O} \subset E^\sharp$, which are known to be the symplectic leaves of the Wong's phase space E^\sharp , cf. Ref. [9].

We shall produce an indirect proof of this proposition by describing the Poisson bracket relations among the local coordinate functions. We use (U, ϕ) to denote a local coordinate chart for X and write $\phi = (x^1, \dots, x^n)$, then the cotangent frame dx^1, \dots, dx^n over U gives a local trivialization of the tangent bundle $T^*X \rightarrow X$ over U . This local trivialization shall always be assumed hereafter. By choosing U small enough if necessary, we can assume that the principal G -bundle $P \rightarrow X$ is trivial over U . We shall fix a trivialization of the bundle $P \rightarrow X$ over U , then the connection ∇ can be represented by a \mathfrak{g} -valued differential one-form A on U , and the curvature can be represented by a \mathfrak{g} -valued differential two-form F on U . We shall also fix a basis $\{T_a\}_{i=1}^D$ for \mathfrak{g} where $D = \dim \mathfrak{g}$. Since each T_a is a linear function on \mathfrak{g}^* by our convention, it is now clear that we have a local coordinate map from E^\sharp to $\phi(U) \times \mathbb{R}^n \times \mathbb{R}^D = \mathbb{R}^{2n+D}$. The domain of this coordinate map shall be referred to as a *good coordinate patch* for E^\sharp . On such a good coordinate patch we have the following local coordinate functions:

$$(x^1, \dots, x^n, \pi_1, \dots, \pi_n, T_1, \dots, T_D).$$

In terms of this local coordinate functions on E^\sharp , the basic Poisson bracket relations are

$$(3.2) \quad \begin{cases} \{x^j, x^k\} = 0, & \{x^j, \pi_k\} = \delta_k^j, & \{\pi_j, \pi_k\} = -F_{jk}, \\ \{T_\alpha, T_\beta\} = -C_{\alpha\beta}^\gamma T_\gamma, & \{T_\alpha, x_k\} = 0, & \{T_\alpha, \pi_k\} = -C_{\alpha\beta}^\gamma A_k^\beta T_\gamma, \end{cases}$$

where $C_{\alpha\beta}^\gamma$ s are the structure constants of \mathfrak{g} with respect to basis iT_α . Note that we have the vector potential $A = A_k dx^k$ with $A_k = A_k^\alpha T_\alpha$, the gauge field $F := dA + iA \wedge A = \frac{1}{2} F_{jk} dx^j \wedge dx^k$ with $F_{jk} = F_{jk}^\alpha T_\alpha$, so, if we use $p = p_k dx^k$ to denote the canonical momentum, we have $\pi_k = p_k + A_k$. Note also that A_k^α and F_{jk}^α are all real-valued functions of x^1, \dots, x^n . Finally, we remark that $iF = \Omega$ and $iA = \omega$ where Ω is the curvature form and ω is the connection form.

We conclude this section by introducing a few more spaces: $E_\sharp := P \times_G \mathfrak{g}^*$, $\mathcal{O}_\sharp := P \times_G \mathcal{O}$. In the case $G = \text{SO}(2k)$, X is $\mathbb{R}_*^{2k+1} := \mathbb{R}^{2k+1} \setminus \{0\}$, the co-adjoint orbit \mathcal{O} is a magnetic orbit, and the connection on $P \rightarrow X$ is a generalized Dirac monopole, a key concept which shall be reviewed in the next section, we let $\mathcal{M}^\sharp := P^\sharp \times_G \mathcal{M}$ and $\mathcal{M}_\sharp = P \times_G \mathcal{M}$. (recall that \mathcal{M} denotes the magnetic cone for $\text{SO}(2k)$.) It will be clear in the last section that there is a bijection between the symplectic leaves of \mathcal{M}^\sharp and the classical Kepler problems (with magnetic charges) in dimension $2k + 1$: the phase space of the classical Kepler problem with magnetic charge μ is the symplectic leaf \mathcal{O}_μ^\sharp , where \mathcal{O}_μ is the magnetic orbit with magnetic charge μ . A symplectic leaf of E^\sharp of the form \mathcal{O}_μ^\sharp shall be referred to as the **magnetic leave** of magnetic charge μ .

4. GENERALIZED DIRAC MONOPOLES

The Sternberg's phase spaces discussed in the above shall be specialized to our needs. In this specialization, $G = \text{SO}(n - 1)$, X is $\mathbb{R}_*^n := \mathbb{R}^n \setminus \{0\}$, the co-adjoint orbit \mathcal{O} is a magnetic orbit, and the connection on $P \rightarrow X$ is a generalized Dirac monopole [11, 12].

The discussion of generalized Dirac monopoles starts with the principal G bundle over S^{n-1} :

$$\begin{array}{c} \mathrm{SO}(n) \\ \downarrow \\ S^{n-1}. \end{array}$$

This bundle comes with a natural connection

$$\omega(g) := \mathrm{Pr}_{\mathfrak{so}(n-1)}(g^{-1}dg),$$

where $g^{-1}dg$ is the Maurer-Cartan form for $\mathrm{SO}(n)$, so it is an $\mathfrak{so}(n)$ -valued differential one form on $\mathrm{SO}(n)$, and $\mathrm{Pr}_{\mathfrak{so}(n-1)}$ denotes the orthogonal projection of $\mathfrak{so}(n)$ onto $\mathfrak{g} := \mathfrak{so}(n-1)$.

Under the map

$$(4.1) \quad \begin{array}{ccc} \pi : \mathbb{R}_*^n & \rightarrow & S^{n-1} \\ \vec{r} & \mapsto & \frac{\vec{r}}{|\vec{r}|}, \end{array}$$

the above bundle and connection are pulled back to a principal G -bundle

$$(4.2) \quad \begin{array}{c} P \\ \downarrow \\ \mathbb{R}_*^n \end{array}$$

with a connection. This connection, originally introduced in Ref. [11] and also independently in Ref. [12], extends the Dirac monopole to high dimensions, is referred to as the generalized Dirac monopole in dimension n .

We shall write $\vec{r} = (x_1, \dots, x_n)$ for a point in \mathbb{R}_*^n and r for the length of \vec{r} . The small Latin letters j, k , etc. be indices that run from 1 to n , and the small Latin letters a, b , etc. be indices that run from 1 to $n-1$. To do local computations, we need to choose a trivialization on U which is \mathbb{R}^n with the negative n -th axis removed and then write down the gauge potential explicitly. We have done that before in Eq. (10) of Ref. [11]. Note that, the gauge potential $A = A_k dx^k$ from Eq. (10) of Ref. [11] can be written as

$$(4.3) \quad A_n = 0, \quad A_b = -\frac{1}{r(r+x_n)} x^a \gamma_{ab}$$

where $\gamma_{ab} = \frac{i}{4}[\gamma_a, \gamma_b] \in i\mathfrak{g}$ with γ_a being the ‘‘gamma matrix’’ for physicists, $x^a \gamma_{ab}$ means $\sum_{a=1}^{n-1} x^a \gamma_{ab}$, something we shall assume whenever there is a repeated index. Note that $\gamma_a = ie_a$ with e_a being the element in the Clifford algebra that corresponds to the a -th standard coordinate vector of \mathbb{R}^{n-1} .

It is straightforward to calculate the gauge field strength $F_{jk} = \partial_j A_k - \partial_k A_j + i[A_j, A_k]$ and get

$$(4.4) \quad \begin{aligned} F_{nb} &= \frac{1}{r^3} x_a \gamma_{ab}, \\ F_{ab} &= -\frac{1}{r^2} \gamma_{ab} + \frac{x_a x_c \gamma_{cb} - x_b x_c \gamma_{ca}}{r^3(r+x_n)} \\ (4.5) \quad &= -\frac{1}{r^2} (\gamma_{ab} + x_a A_b - x_b A_a). \end{aligned}$$

Hereafter we assume that $n = 2k + 1$. The following lemma, whose quantum version first appeared in Ref. [4], is crucially used in this article.

Lemma 4.1. *Let $Q = \frac{1}{2k} \sum_{a,b} (\gamma_{ab})^2$ and $\nabla_\alpha = \partial_\alpha + iA_\alpha$. For the gauge potential A defined in Eq. (4.3), the following statements are true.*

1) *As functions on the good coordinate patch,*

$$(4.6) \quad x^k A_k = 0, \quad x^j F_{jk} = 0, \quad r^4 \sum_{i,j} (F_{ij})^2 = 2kQ$$

$$(4.7) \quad \nabla_l F_{jk} = \frac{1}{r^2} (-x^j F_{lk} - x^k F_{jl} - 2x^l F_{jk}), \quad \sum_j \nabla_j F_{jk} = 0$$

$$(4.8) \quad r^4 \{F_{jk}, F_{lm}\} = (r^2 \delta_{jl} - x_j x_l) F_{km} + \text{the remaining three terms.}$$

2) *The identities*

$$(4.9) \quad r^4 \sum_i F_{ij} F_{ij'} = Q \left(\delta_{jj'} - \frac{x_j x_{j'}}{r^2} \right)$$

hold at the point $[(p^\sharp, \xi)] \in E^\sharp$ if and only if the following set of homogeneous quadratic equations $\sum_a \gamma_{ab} \gamma_{ab'} = \delta_{bb'} Q$ hold at ξ , i.e., if and only if ξ is inside the magnetic cone \mathcal{M} .

Proof. Since $\gamma_{ab} = -\gamma_{ba}$, we have

$$\sum_j x_j A_j = \sum_b x_b A_b = -\frac{1}{r(r+x_n)} \sum_{a,b} x_a x_b \gamma_{ab} = 0.$$

The verification of the remaining identities is a direct and lengthy calculation. However, we note that the connection is rotational invariant, i.e., under the rotations of \mathbb{R}^{2k+1} , A (hence F) is invariant modulo gauge transformations. Combining with the transformation property of the remaining identities under rotations and gauge transformations, we just need to check the remaining identities at point $\vec{r}_0 = (0, \dots, 0, r)$, a much easier task. At this point \vec{r}_0 , since

$$(4.10) \quad A_i = 0, \quad F_{na} = 0, \quad F_{ab} = -\frac{1}{r^2} \gamma_{ab},$$

one can quickly finish the checking. \square

5. THE POISSON REALIZATION OF $\mathfrak{so}(2, 2k + 2)$ ON MAGNETIC LEAVES

The goal in this section is to present and prove our main result: the Poisson realization of $\mathfrak{so}(2, 2k + 2)$ on the magnetic leaf \mathcal{O}_μ^\sharp . We take the manifold X to be \mathbb{R}_*^{2k+1} , the coordinate patch U for X to be \mathbb{R}^{2k+1} with the negative $(2k + 1)$ -st axis removed, and the coordinate map $\phi: U \rightarrow \mathbb{R}^{2k+1}$ to be the inclusion, and the bundle $P \rightarrow X$ and its trivialization over U to be the one given in the last section. Note that this good coordinate patch for E^\sharp is dense in E^\sharp . We let $x_j = x^j$, $\vec{r} = (x_1, \dots, x_{2k+1})$ and $r = |\vec{r}|$, $\vec{\pi} = (\pi_1, \dots, \pi_{2k+1})$ and $\pi = |\vec{\pi}|$. Recall from section 3 that the basic Poisson bracket relations are

$$(5.1) \quad \left\{ \begin{array}{l} \{x_j, x_k\} = 0, \quad \{x_j, \pi_k\} = \delta_{jk}, \quad \{\pi_j, \pi_k\} = -F_{jk}, \\ \{T_\alpha, T_\beta\} = -C_{\alpha\beta}^\gamma T_\gamma, \quad \{T_\alpha, x_k\} = 0, \quad \{T_\alpha, \pi_k\} = -C_{\alpha\beta}^\gamma A_k^\beta T_\gamma. \end{array} \right.$$

Let us first introduce

$$(5.2) \quad \begin{cases} J_{i,0} & := r\pi_i, \\ J_{2k+2,0} & := \frac{1}{2}(r\pi^2 + \frac{Q}{r}) - \frac{1}{2}r, \\ J_{-1,0} & := \frac{1}{2}(r\pi^2 + \frac{Q}{r}) + \frac{1}{2}r. \end{cases}$$

where $Q = \frac{1}{2k} \sum_{a,b} (\gamma_{ab})^2$ as in Lemma 4.1. Then we let

$$(5.3) \quad \begin{cases} J_{i,j} & := -\{J_{i,0}, J_{j,0}\}, \\ J_{i,2k+2} & := -\{J_{i,0}, J_{2k+2,0}\}, \\ J_{i,-1} & := -\{J_{i,0}, J_{-1,0}\}, \\ J_{2k+2,-1} & := -\{J_{2k+2,0}, J_{-1,0}\}. \end{cases}$$

By a small computation based on the basic Poisson bracket relations (5.1), one arrives at the following explicit expression:

$$(5.4) \quad \begin{cases} J_{i,j} & = x_i\pi_j - x_j\pi_i + r^2 F_{ij}, \\ J_{i,2k+2} & = \frac{1}{2}x_i\pi^2 - \pi_i(\vec{r} \cdot \vec{\pi}) + r^2 F_{ij}\pi_j - \frac{Q}{2r^2}x_i - \frac{1}{2}x_i, \\ J_{i,-1} & = \frac{1}{2}x_i\pi^2 - \pi_i(\vec{r} \cdot \vec{\pi}) + r^2 F_{ij}\pi_j - \frac{Q}{2r^2}x_i + \frac{1}{2}x_i, \\ J_{2k+2,-1} & = \vec{r} \cdot \vec{\pi}. \end{cases}$$

Here we assume the repeated indices are dummy, i.e., are summed up.

Let the capital Latin letters such as A, B , etc. be indices that run from -1 to $2k+2$. Note that, as a real function on E^\sharp , F_{ij} factorizes through E_\sharp , moreover, for any point $[(p, \xi)] \in E_\sharp = P \times_G \mathfrak{g}^*$, we have

$$F_{ij}([(p, \xi)]) = \langle \xi, F_{ij}(\pi(p)) \rangle.$$

Here, $\pi: P \rightarrow X$ is the bundle projection. Therefore J_{AB} s are independent of the local trivialization we have fixed for the bundle $P \rightarrow X$, so they are defined on the whole E^\sharp , not just on a dense subset of E^\sharp . Recall that $\mathcal{M}^\sharp = P^\sharp \times_G \mathcal{M}$ is the union of all magnetic leaves of E^\sharp .

Theorem 1. *Viewing J_{AB} s as functions on \mathcal{M}^\sharp , we have the following two statements.*

1) J_{AB} s satisfy the following Poisson bracket relations:

$$(5.5) \quad \{J_{AB}, J_{A'B'}\} = -\eta_{AA'}J_{BB'} - \eta_{BB'}J_{AA'} + \eta_{AB'}J_{BA'} + \eta_{BA'}J_{AB'}$$

where the indefinite metric tensor η is $\text{diag}\{1, 1, -1, \dots, -1\}$ relative to the following order: $-1, 0, 1, \dots, 2k+2$ for the indices.

2) J_{AB} s satisfy the following quadratic relations

$$(5.6) \quad \eta^{AA'} J_{AB} J_{A'C} = \eta_{BC} Q.$$

Proof. One just needs to prove the two statements in a dense subset of \mathcal{M}^\sharp , such as the intersection of \mathcal{M}^\sharp with the dense good coordinate patch for E^\sharp mentioned in the beginning paragraph of this section.

The quadratic relations in part 2) of this theorem can be reduced to a single one (referred to as the primary quadratic relation in Ref. [5]):

$$(5.7) \quad \sum_{i=1}^{2k+1} J_{i,0}^2 + J_{2k+2,0}^2 - J_{-1,0}^2 = -Q$$

because of the following observation initially discovered in Ref. [5]: thanks to the Poisson bracket relations in part 1) of Theorem 1, all the other quadratic relations in part 2) can be

obtained by taking the Poisson bracket of the primary quadratic relation with some suitable J_{AB} . The checking of the primary quadratic relation (5.7) is very easy:

$$\sum_{i=1}^{2k+1} J_{i,0}^2 + J_{2k+2,0}^2 - J_{-1,0}^2 = r\pi_i r\pi_i - XY = r^2\pi^2 - (r^2\pi^2 + Q) = -Q.$$

Therefore part 2) is proved modulo part 1). However, the proof of part 1) is quite involved, and the next subsection is wholly devoted to it. \square

5.1. Proof of part 1) of theorem 1. The following lemma is quite useful in the proof of part 1) of Theorem 1.

Lemma 5.1.

$$(5.8) \quad \begin{cases} \{J_{ij}, x_k\} &= -x_i\delta_{jk} + x_j\delta_{ik}, \\ \{J_{ij}, \pi_k\} &= -\pi_i\delta_{jk} + \pi_j\delta_{ik}, \\ \{J_{ij}, F_{i'j'}\} &= \delta_{ii'}F_{jj'} + \delta_{jj'}F_{ii'} - \delta_{ij'}F_{ji'} - \delta_{ji'}F_{ij'}. \end{cases}$$

Proof.

$$\begin{aligned} \{J_{ij}, x_k\} &= \{x_i\pi_j - x_j\pi_i + r^2F_{ij}, x_k\} = \{x_i\pi_j - x_j\pi_i, x_k\} \\ &= -(x_i\delta_{jk} - x_j\delta_{ik}). \end{aligned}$$

$$\begin{aligned} \{J_{ij}, \pi_k\} &= \{x_i\pi_j - x_j\pi_i + r^2F_{ij}, \pi_k\} \\ &= \pi_j\delta_{ik} - \pi_i\delta_{jk} - x_iF_{jk} + x_jF_{ik} + \nabla_k(r^2F_{ij}) \\ &= -\pi_i\delta_{jk} + \pi_j\delta_{ik}. \quad \text{By identity (4.7)} \end{aligned}$$

$$\begin{aligned} \{J_{ij}, F_{i'j'}\} &= \{x_i\pi_j - x_j\pi_i + r^2F_{ij}, F_{i'j'}\} \\ &= x_i\{\pi_j, F_{i'j'}\} - x_j\{\pi_i, F_{i'j'}\} + r^2\{F_{ij}, F_{i'j'}\} \\ &= \frac{x_i}{r^2}(2x_jF_{i'j'} + x_{i'}F_{jj'} + x_{j'}F_{ij}) - \frac{x_j}{r^2}(2x_iF_{i'j'} + x_{i'}F_{ij'} + x_{j'}F_{i'i}) \\ &\quad + r^2\{F_{ij}, F_{i'j'}\} \quad \text{by identity (4.7)} \\ &= r^2\{F_{ij}, F_{i'j'}\} - \frac{1}{r^2}(-x_ix_{i'}F_{jj'} - x_ix_{j'}F_{ij'} + x_jx_{i'}F_{ij'} + x_jx_{j'}F_{i'i}) \\ &= \delta_{ii'}F_{jj'} + \delta_{jj'}F_{ii'} - \delta_{ij'}F_{ji'} - \delta_{ji'}F_{ij'}. \quad \text{by identity (4.8)} \end{aligned}$$

\square

As a quick corollary, we have $\{J_{ij}, r\} = \{J_{ij}, \pi^2\} = 0$, and

$$(5.9) \quad \{J_{ij}, J_{i'j'}\} = \delta_{ii'}J_{jj'} + \delta_{jj'}J_{ii'} - \delta_{ij'}J_{ji'} - \delta_{ji'}J_{ij'}.$$

So J_{ij} s satisfy the commutation relations among the standard basis elements of the real Lie algebra $\mathfrak{so}(2k + 1)$. Then Lemma 5.1 may be paraphrased as follows: under the action of J_{ij} s, x_i s and π_i s transform as $\mathfrak{so}(2k + 1)$ vectors, and F_{ij} s transform as $\mathfrak{so}(2k + 1)$ bi-vectors. It is then clear that $J_{2k+2,0}$, $J_{-1,0}$, and $J_{2k+2,-1}$ all transform as $\mathfrak{so}(2k + 1)$ scalars; $J_{i,2k+2}$, $J_{i,-1}$, and $J_{i,0}$ all transform as $\mathfrak{so}(2k + 1)$ vectors. Then it is clear that the Poisson relations (5.5) hold whenever J_{ij} appears on its left hand side.

By using the identity $x_iF_{ij} = 0$, one can check that $\{J_{2k+2,-1}, \vec{r}\} = -\vec{r}$, $\{J_{2k+2,-1}, r\} = -r$, $\{J_{2k+2,-1}, \frac{1}{r}\} = \frac{1}{r}$, $\{J_{2k+2,-1}, \vec{\pi}\} = \vec{\pi}$, $\{J_{2k+2,-1}, r^2F_{ij}\} = 0$. That is, $J_{2k+2,-1}$ is the *dimension operator* in physics. It is then clear that the Poisson relations (5.5) hold whenever $J_{2k+2,-1}$ appears on its left hand side.

The remaining verifications are divided into four cases.

Case 1.

$$\{J_{i,0}, J_{j,0}\} = -J_{ij}, \quad \{J_{i,0}, J_{2k+2,0}\} = -J_{i,2k+2},$$

$$\{J_{i,0}, J_{-1,0}\} = -J_{i,-1}, \quad \{J_{2k+2,0}, J_{-1,0}\} = -J_{2k+2,-1}.$$

But these are just the defining relations. So case 1 is done.

To check the remaining cases, it is convenient to introduce

$$\begin{aligned} X &:= r\pi^2 + \frac{Q}{r}, \quad Y := r, \quad W_i := \{Y, J_{i,0}\} = x_i \\ Z_i &:= \{X, J_{i,0}\} = x_i\pi^2 - 2\pi_i(\vec{r} \cdot \vec{\pi}) + 2r^2F_{ij}\pi_j - \frac{Q}{r^2}x_i. \end{aligned}$$

Case 2.

$$\begin{aligned} \{J_{i,-1}, J_{-1,0}\} &= J_{i,0}, & \{J_{i,0}, J_{2k+2,0}\} &= 0, \\ \{J_{i,2k+2}, J_{-1,0}\} &= 0, & \{J_{i,2k+2}, J_{2k+2,0}\} &= -J_{i,0}, \end{aligned}$$

or equivalently,

$$(5.10) \quad \{W_i, Y\} = \{Z_i, X\} = 0, \quad \{W_i, X\} = \{Z_i, Y\} = 2J_{i,0}.$$

Proof. It is clear that $\{W_i, Y\} = 0$. Now $\{W_i, X\} = r\{x_i, \pi^2\} = 2J_{i,0}$. Next, using the identity $F_{ij}x_j = 0$, we have

$$\begin{aligned} \{Z_i, Y\} &= \{x_i\pi^2 - 2\pi_i(\vec{r} \cdot \vec{\pi}) + 2r^2F_{ij}\pi_j - \frac{Q}{r^2}x_i, r\} \\ &= x_i\{\pi^2, r\} - 2\{\pi_i(\vec{r} \cdot \vec{\pi}), r\} - 2rF_{ij}x_j \\ &= -2\frac{x_i}{r}\vec{r} \cdot \vec{\pi} - 2\pi_i\{\vec{r} \cdot \vec{\pi}, r\} - 2\{\pi_i, r\}\vec{r} \cdot \vec{\pi} \\ &= -2\frac{x_i}{r}\vec{r} \cdot \vec{\pi} + 2\pi_i r + 2\frac{x_i}{r}\vec{r} \cdot \vec{\pi} \\ &= 2J_{i,0}. \end{aligned}$$

Consequently, since $\{Z_i, r\frac{1}{r}\} = 0$, we get $\{\frac{1}{r}, Z_i\} = \frac{2}{r}\pi_i$. Finally

$$\begin{aligned} \{r\pi^2, Z_i\} &= \{r, Z_i\}\pi^2 + r\{\pi^2, Z_i\} \\ &= -2J_{i,0}\pi^2 + r\{\pi^2, Z_i\} \\ &= -2J_{i,0}\pi^2 + r\{\pi^2, x_i\pi^2 - 2\pi_i(\vec{r} \cdot \vec{\pi}) + 2r^2F_{ij}\pi_j - \frac{Q}{r^2}x_i\} \\ &= -2J_{i,0}\pi^2 + r(\{\pi^2, x_i\}\pi^2 - 2\{\pi^2, \pi_i(\vec{r} \cdot \vec{\pi})\} \\ &\quad + 2\{\pi^2, r^2F_{ij}\pi_j\} - Q\{\pi^2, \frac{x_i}{r^2}\}) \\ &= -2r\pi_i\pi^2 + r(-2\pi_i\pi^2 - 2\{\pi^2, \pi_i\}(\vec{r} \cdot \vec{\pi}) + 4\pi_i\pi^2 \\ &\quad + 2\{\pi^2, r^2F_{ij}\pi_j\} - Q\{\pi^2, \frac{x_i}{r^2}\}) \\ &= r(-2\{\pi^2, \pi_i\}(\vec{r} \cdot \vec{\pi}) + 2\{\pi^2, r^2F_{ij}\}\pi_j \\ &\quad + 2r^2F_{ij}\{\pi^2, \pi_j\} - Q\{\pi^2, \frac{x_i}{r^2}\}) \\ &= r(-2\{\pi^2, \pi_i\}(\vec{r} \cdot \vec{\pi}) - 2\{\pi^2, x_i\pi_j - x_j\pi_i\}\pi_j) \\ &\quad + r\left(2r^2F_{ij}\{\pi^2, \pi_j\} - Q\{\pi^2, \frac{x_i}{r^2}\}\right) \\ &= r(4\pi_k F_{ki}\vec{r} \cdot \vec{\pi} - 2\{\pi^2, x_i\pi_j\}\pi_j + 2\{\pi^2, x_j\pi_i\}\pi_j) \\ &\quad + r\left(4r^2F_{ij}F_{jk}\pi_k + 2Q\left(\frac{\pi_i}{r^2} - \frac{2x_i}{r^4}\vec{r} \cdot \vec{\pi}\right)\right) \\ &= r(4\pi_k F_{ki}\vec{r} \cdot \vec{\pi} - 2x_i\{\pi^2, \pi_j\}\pi_j + 2x_j\{\pi^2, \pi_i\}\pi_j) \\ &\quad + r\left(4r^2F_{ij}F_{jk}\pi_k + 2Q\left(\frac{\pi_i}{r^2} - \frac{2x_i}{r^4}\vec{r} \cdot \vec{\pi}\right)\right) \\ &= r\left(-4r^2F_{ji}F_{jk}\pi_k + 2Q\left(\frac{\pi_i}{r^2} - \frac{2x_i}{r^4}\vec{r} \cdot \vec{\pi}\right)\right) \end{aligned}$$

$$= -\frac{2Q}{r}\pi_i. \quad \text{by identity (4.9)}$$

Therefore, $\{X, Z_i\} = \{r\pi^2 + \frac{Q}{r}, Z_i\} = 0$. □

Case 3.

$$(5.11) \quad \{J_{i,-1}, J_{j,0}\} = -\eta_{ij}J_{-1,0}, \quad \{J_{i,2k+2}, J_{j,0}\} = -\eta_{ij}J_{2k+2,0}$$

or equivalently

$$(5.12) \quad \{Z_i, J_{j,0}\} = -\eta_{ij}X, \quad \{W_i, J_{j,0}\} = -\eta_{ij}Y.$$

Proof. Half of the checking is easy: $\{W_i, J_{j,0}\} = \{x_i, r\pi_j\} = r\delta_{ij} = -\eta_{ij}Y$. To check the remaining half, we recall from step 2 that $\{Z_i, r\} = 2J_{i,0} = 2r\pi_i$, then

$$\begin{aligned} \{Z_i, J_{j,0}\} &= r\{Z_i, \pi_j\} + 2J_{i,0}\pi_j \\ &= r\{x_i\pi^2 - 2\pi_i(\vec{r} \cdot \vec{\pi}) + 2r^2F_{ik}\pi_k - \frac{Q}{r^2}x_i, \pi_j\} + 2r\pi_i\pi_j \\ &= r(\delta_{ij}\pi^2 - 2x_iF_{kj}\pi_k) + r(2F_{ij}\vec{r} \cdot \vec{\pi} - 2\pi_i\pi_j) \\ &\quad + r(-2r^2F_{ik}F_{kj} + \{2r^2F_{ik}, \pi_j\}\pi_k) + Qr\{\pi_j, \frac{x_i}{r^2}\} + 2r\pi_i\pi_j \\ &= \delta_{ij}r\pi^2 - 2rx_iF_{kj}\pi_k + 2rF_{ij}\vec{r} \cdot \vec{\pi} \\ &\quad + r(-2r^2F_{ik}F_{kj} + 4x_jF_{ik}\pi_k - 2r^2\{\pi_j, F_{ik}\}\pi_k) + Qr\{\pi_j, \frac{x_i}{r^2}\} \\ &= \delta_{ij}r\pi^2 - 2r^3F_{ik}F_{kj} + 2r(2x_jF_{ik} + x_iF_{jk} + x_kF_{ij} + r^2\nabla_j F_{ik})\pi_k \\ &\quad + Qr\{\pi_j, \frac{x_i}{r^2}\} \\ &= \delta_{ij}r\pi^2 + 2r^3F_{ki}F_{kj} - Qr\left(\frac{\delta_{ij}}{r^2} - 2\frac{x_ix_j}{r^4}\right) \quad \text{by identity (4.7)} \\ &= \delta_{ij}(r\pi^2 + \frac{Q}{r^2}) \quad \text{by identity (4.9)} \\ &= -\eta_{ij}X. \end{aligned}$$

□

Case 4.

$$\{J_{i,-1}, J_{j,-1}\} = -J_{ij}, \quad \{J_{i,2k+2}, J_{j,-1}\} = -\eta_{ij}J_{2k+2,0}, \quad \{J_{i,2k+2}, J_{j,2k+2}\} = J_{ij}$$

or equivalently,

$$(5.13) \quad \{Z_i, Z_j\} = \{W_i, W_j\} = 0, \quad \{Z_i, W_j\} = -2(\eta_{ij}J_{2k+2,-1} + J_{ij}).$$

Proof. It is clear that $\{W_i, W_j\} = 0$ because $W_i = x_i$. Next,

$$\begin{aligned} \{Z_i, W_j\} &= \{x_i\pi^2 - 2\pi_i\vec{r} \cdot \vec{\pi} + 2r^2F_{ik}\pi_k - \frac{Q}{r^2}x_i, x_j\} \\ &= x_i\{\pi^2, x_j\} - 2\{\pi_i\vec{r} \cdot \vec{\pi}, x_j\} + 2r^2F_{ik}\{\pi_k, x_j\} \\ &= -2x_i\pi_j - 2\pi_i\{\vec{r} \cdot \vec{\pi}, x_j\} - 2\{\pi_i, x_j\}\vec{r} \cdot \vec{\pi} - 2r^2F_{ij} \\ &= -2x_i\pi_j + 2\pi_ix_j + 2\delta_{ij}\vec{r} \cdot \vec{\pi} - 2r^2F_{ij} \\ &= -2(x_i\pi_j - x_j\pi_i + r^2F_{ij}) + 2\delta_{ij}\vec{r} \cdot \vec{\pi} \\ &= -2(\eta_{ij}J_{2k+2,-1} + J_{ij}). \end{aligned}$$

Finally, using results from case 2 and case 3, we have

$$\begin{aligned} -\{Z_i, Z_j\} &= \{\{J_{i,0}, X\}, Z_j\} \\ &= \{\{J_{i,0}, Z_j\}, X\} + \{J_{i,0}, \{X, Z_j\}\} \\ &= \{\eta_{ij}X, X\} + \{J_{i,0}, 0\} = 0. \end{aligned}$$

□

6. AN EXTENSION OF THE CLASSICAL MICZ KEPLER PROBLEMS

In view of the work done in Ref. [13], the Poisson realization of the conformal algebra $\mathfrak{so}(2, 2k+2)$ on a magnetic leave naturally yields a classical generalized Kepler problem associated with the Jordan algebra $\Gamma(2k+1)$. The goal here is to describe this new classical generalized Kepler problems, as envisaged in Ref. [4].

Recall from Ref. [13] that the **classical universal Hamiltonian**, **classical universal angular momentum**, and the **classical universal Lenz vector** are

$$(6.1) \quad \mathcal{H} = \frac{\frac{1}{2}\mathcal{X}_e - 1}{\mathcal{Y}_e}, \quad \mathcal{L}_{u,v} = \{\mathcal{L}_u, \mathcal{L}_v\}, \quad \mathcal{A}_u := \frac{1}{2} \left(\mathcal{X}_u - \mathcal{Y}_u \frac{\mathcal{X}_e}{\mathcal{Y}_e} \right) + \frac{\mathcal{Y}_u}{\mathcal{Y}_e}.$$

respectively. Here, u, e are the elements of the Jordan algebra $\Gamma(2k+1)$ with e being the identity element. In our Poisson realization of $\mathfrak{so}(2, 2k+2)$ on magnetic leaves,

$$(6.2) \quad \begin{cases} \mathcal{L}_{e_i} = J_{i,0} = r\pi_i, \\ \mathcal{Y}_e = Y = r, \\ \mathcal{X}_e = X = r\pi^2 + \frac{Q}{r}, \\ \mathcal{X}_{e_i} = -Z_i = -x_i\pi^2 + 2\pi_i(\vec{r} \cdot \vec{\pi}) - 2r^2 F_{ij}\pi_j + \frac{Q}{r^2}x_i, \\ \mathcal{Y}_{e_i} = W_i = x_i. \end{cases}$$

Here e_1, \dots, e_{2k+1} are the standard basis vectors for \mathbb{R}^{2k+1} . Therefore, we have a generalized classical Kepler problem for which, the hamiltonian is

$$(6.3) \quad \boxed{H = \frac{1}{2}\pi^2 + \frac{Q}{2r^2} - \frac{1}{r}},$$

the angular momentum $L = \frac{1}{2} \sum_{i,j} L_{ij} e_i \wedge e_j$ with $L_{ij} = L_{e_j, e_i}$ is

$$(6.4) \quad \boxed{L = \vec{r} \wedge \vec{\pi} + r^2 F},$$

and the Lenz vector $\vec{A} = \sum_i A_i e_i$ with $A_i = A_{e_i}$ is

$$(6.5) \quad \boxed{\vec{A} = \vec{\pi} \lrcorner L + \frac{\vec{r}}{r}}.$$

By using the basic Poisson bracket relations (5.1), the Hamilton's equation $f' = \{f, H\}$ for the basic functions x_i, π_i , and T_α becomes

$$(6.6) \quad \begin{cases} x'_i = \pi_i, \\ \pi'_i = -\frac{x_i}{r^3} + Q\frac{x_i}{r^4} + \pi_j F_{ji}, \\ T'_\alpha = \pi_k [iA_k, T_\alpha] = -C_{\alpha\beta}^\gamma \pi_k A_k^\beta T_\gamma. \end{cases}$$

So the equation of motion becomes

$$(6.7) \quad \begin{cases} \vec{r}' = -\frac{\vec{r}}{r^3} + Q\frac{\vec{r}}{r^4} + \vec{r}' \lrcorner F, \\ T'_\alpha = [i\vec{r}' \lrcorner A, T_\alpha]. \end{cases}$$

A coordinate-free and gauge-free formulation for the equation of motion, which should exist in the first place, can be obtained from the above equation of motion. To describe it, we let $\vec{r}: \mathbb{R} \rightarrow X := \mathbb{R}_*^{2k+1}$ be a smooth map, and ξ be a smooth section of the pullback bundle of $E_{\mathbb{H}} \rightarrow X$ under \vec{r} . Then the 2nd equation of (6.7) says that ξ is a covariantly constant section of the pullback bundle over \mathbb{R} , i.e.,

$$\frac{D}{dt}\xi = 0.$$

Since Q becomes $\frac{|\xi|^2}{k}$, Q is a constant, i.e., independent of the time. Then the equation of motion (6.7) can be reformulated as equation

$$(6.8) \quad \left\{ \begin{array}{l} \ddot{r} = -\frac{\dot{r}}{r^3} + \frac{|\xi|^2}{k} \frac{\dot{r}}{r^4} + \langle \xi, \dot{r} \lrcorner F \rangle, \\ \frac{D}{dt} \xi = 0. \end{array} \right.$$

Here \langle, \rangle refers to the pairing of the aforementioned pullback bundle and its dual vector bundle, and vectors are identified with one forms via the standard euclidean structure of \mathbb{R}^{2k+1} .

Eq. (6.8) can be viewed as a dynamical equation on E_{\sharp} , but then it is not super integrable. By restricting to the magnetic leave with magnetic charge μ , Eq. (6.8) defines a super integrable model which generalize the classical MICZ Kepler problem. This super integrable model shall be referred to as the **classical Kepler problem with magnetic charge μ in dimension $2k + 1$** . In dimension 5, it is essentially Iwai's $SU(2)$ -Kepler problem, cf. Ref. [3].

It is not hard to see that $L^2 - kQ = |\dot{r} \wedge \dot{r}^*|^2$, then $L^2 - kQ > 0$ for the non-colliding orbits. Note that $kQ = \mu^2$ on \mathcal{O}_{μ} . By using the quadratic relations (5.6) one can check that, for the non-colliding orbits of the classical Kepler problem with magnetic charge μ , the total energy

$$(6.9) \quad H = -\frac{1-A^2}{2(L^2-\mu^2)}.$$

Here, A^2 is the length square of the Lenz vector \vec{A} , i.e., $A^2 = \sum_i A_i^2$, and L^2 is the length square of L , i.e., $L^2 = \sum_{i<j} L_{ij}^2$. It is expected from Ref. [6] (though not from Ref. [3]) that even in this generalized model, a non-colliding orbit remains an ellipse or a parabola or a branch of hyperbola when the total energy H is negative or zero or positive respectively. The details will be presented elsewhere.

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