

# Integrability of Maxwell's Equations

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## Abstract

This article adds to [4] some nearer explanations. It is shown again that Maxwell's equations are integrable, but I am doing it without quaternions, which is simpler, albeit more superficial. It suffices however to take insight into the very nature of classical electrodynamics and path integration in quantum electrodynamics.

## 1 Introduction

In its covariant form, Maxwell's equations are given as

$$\square A^\mu(x^0, \mathbf{x}) = j^\mu(x^0, \mathbf{x}), \quad (0 \leq \mu \leq 3),$$

along with two other conditions: the first is **charge conservation**,

$$\partial j^0 / \partial x^0 + \nabla \cdot \mathbf{j} = 0,$$

and the second equation is the so-called **Lorentz gauge**,

$$\partial A^0 / \partial x^0 + \nabla \cdot \mathbf{A} = 0$$

see: [3, II-18-6].

Both, Lorentz gauge and charge conservation are intimately related, and according to Poincaré's Lemma, they both state their local integrability on single connected regions in space time, on which they are continuously differentiable (see [1, Sec. 2-12 to 2-13]): Let  $\Omega \subset \mathbb{R}^4$  be single connected set contained in an open set  $U \subset \mathbb{R}^4$ , and let  $f_0, \dots, f_3$  be continuously differentiable function on  $U$  into either  $\mathbb{R}$  or  $\mathbb{C}$ . Then the following statements are equivalent:

1.  $\partial_0 f_0(x) + \dots + \partial_3 f_3(x) = 0 \quad \forall x \in \Omega$
2. For any two paths  $\gamma$  and  $\gamma'$  in  $\Omega$  joining  $a, b \in \Omega$ :  $\int f \cdot d\gamma = \int f \cdot d\gamma'$ .

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\*To the generous bookseller for Herman Kershaw, who once, at the book fair's last day in Frankfurt, ceded H. Cartan's book Differential Forms [1] to me, then a poor student, after having passed some checks that I really cared for it.

## 2 Problem Statement

So, obviously, one would like to condense the for Maxwell equation into a single equation  $\square L(x) = Q(x)$ , where  $L(x) = \int_a^x A \cdot d\gamma$  and  $Q(x) = \int_b^x j \cdot d\gamma'$ . However, there are two concerns as to this:

## 3 Concern 1

The first problem is that the domain of existence of  $j$  or  $A$  may not be simply connected as for instance a circle in two or more spatial dimensions: In case of a circle one might end in two different solutions, and in general, according to the number of holes and knots, that number may become any natural number.

In order to see that this concern can be overcome, let me note that for a continuously differentiable function  $F$  on  $U \subset \mathbb{R}^4$  and  $a, x \in U$  the integral  $\int_a^x \partial F \cdot d\gamma$  is invariant w.r.t. diffeomorphisms  $\Psi : U \rightarrow U$  (i.e. bijective mappings which are differentiable along with their inverse): This is so, because  $\partial F$  transforms contravariantly under  $\Psi$ , whereas  $d\gamma$  transforms covariantly, so that the Jacobi matrix cancels against its inverse. With this, let me explore the situation of two space and one time coordinate first: Given charges confined to a circle in the xy-plane, and let the time axis be perpendicular to it, there are two classes of paths one may go: to the side of the circle and those intersecting the circle. Those to the side contract homeomorphically to a point, so yield a zero result. So, what about the loops which intersect the circle one or more times? The point now is that with the time perpendicular to the circle in the xy-plane, the particles will not go in a circle, but in a spiral around the time axis, and the originally thought circle is being torn into a spiral. This spiral can be stretched homeomorphically into a line along the time axis. That leaves us with loops encircling the time axis one or several times. Now I observe that I can lift these loops further up the time axis without affecting the values of path integral along those loops. And, if the particles haven't stayed for eternity in that place - which may be assumed not to have happened, then, lifting the loops further towards  $t \rightarrow \infty$ , no charges will then be present and potential vector field will converge to zero. So, by the principle that all physical observable quantities are to be confined to a bounded region in spacetime, the path integrals along all loops (in the assumed 3-dimensional spacetime) give a zero value. The same argument now applies to 4 dimensional spacetime: all 3-dimensional holes in space can be stretched out towards  $t \rightarrow \infty$  and  $t \rightarrow -\infty$  resp., and since there both fields and charges are supposed to vanish, all integrals along closed paths again give zero.

## 4 Concern 2

The 2nd concern is that apart from the ranges of definition of  $j$  and  $A$ , the path integration must be taken not w.r.t. the Euclidean metrics, but w.r.t. the Minkowski metrics, i.e.: the path integration has to be restricted to regions on which the Lorentz matrix is invertible. Now, with one time and two space

coordinates, spacetime is divided into 3 disjoint such regions, separated by the forward and backward light cones. in 4 dimensions, however, in the space-like region  $\{(x^0, \mathbf{x}) \mid |x^0|^2 > \|\mathbf{x}\|^2\}$  the Lorentz metrics is twice degenerate, because the 3-dimensional Euclidean unit ball cuts into an upper and lower hemisphere: the Lorentz metrics simply does not distinct between corresponding points of either halves. The upper and lower hemispheres can be associated with the positive and negative sign of the determinant, or equivalently, a positive or negative sign of parity.

## 5 Conclusion

So, in 3 dimensions of space and time we end up with 3 connected regions in which Maxwell's equations can be integrated to one scalar equation, and in 4 dimensions of space and time, there are four regions in which Maxwell's equations can be integrated. The excluded regions in 4 dimensions, defined up to rotation in the 3-dimensional space, is the union of forward and backward light cone with  $\{(x^0, \dots, x^3) \mid x^3 = 0\}$ . The set  $\{(x^0, \dots, x^3) \mid x^3 = 0\}$  could be included to either be part of the upper or lower hemisphere. But a smarter solution would be the replacement of  $\mathbb{R}^3$  by the triple  $(\sigma_x, \sigma_y, \sigma_z)$  of Pauli matrices in which case the 2-fold coverage  $SU(2)$  of  $SO(3)$  would smoothly resolve the parity flip. In [4] just that is done. Anyhow, time inversion homeomorphically maps forward and backward light cones onto oneanother, and so does space inversion with the spacelike positive and negative parity cones. The four regions therefore come from the four combinations of the two discrete symmetries that the groups  $O(4)$  and  $U(4)$  possess: time and space inversion.

So, where did we reach? We showed that one can rewrite Maxwells equation into a quadrupel of wave equations of action integrals, and solving these, will give us the solution of Maxwell's equations in form of a quadruplet of actions. The general solution then will be any complex linear combination of the four component solutions up to the additions of constant complex vectors  $\chi$  in  $\mathbb{C}^4$ . And then, we get a  $U(4)$ -symmetry on top. (I could have chosen real linear combinations with  $O(4)$  on top, but since the fields  $A^\mu$  are complex, it's better to extend to the complex from scratch.) What is that symmetry group on top of the solutions good for? The interchange of any two of the four components maps one-to-one with the interchange of two of the four time and space coordinates  $x^0, \dots, x^3$ . In non-relativistic classical mechanics time and location coordinates are equivalent and part of one (Euclidean) symmetry group. We now achieved right that equivalence of space and time coordinates in Maxwell's relativistic theory, either.

## 6 Outlook

The integrability of Maxwell's equations offers an interesting perspective: It suffices to path integrate the sources,  $L(x) = \int j \cdot d\gamma$  (with a fixed starting

point) to get the action integral of the vector field  $A$ , and the differential of this action field will give  $A$  in turn.

That alludes to what R.P. Feynman said in [2]:

”..I was now convinced that since we had solved the problem of classical electrodynamics (and completely in accordance with my program from M.I.T., only direct interaction between particles, in a way that made fields unnecessary) that everything was definitely going to be all right. I was convinced that all I had to do was make a quantum theory analogous to the classical one and everything would be solved..”

Moreover, we touch quantum field theory by the following: Path integrating the vector field  $A$  in any of the four component regions, e.g. in the forward light cone gives a function  $F$  which has the dimension of energy by test charge. Next, an implicit additional charge factor enters from the charged sources that had been path integrated, plus we used  $dx^0 = cds$ , where  $c$  is the speed of light (that I tacitly set equal to 1). With  $e_0$  being the elementary electronic charge, I can write  $e_0 F = e_0 \int A \cdot d\gamma$  in units of  $e_0^2/c$  as a dimensionless function. But  $e_0^2/c = \alpha\hbar$ , where  $\alpha$  is known to be the dimensionless fine-structure constant. Add to this, so far the test charge is a constant  $e_0$  resting at some place  $(x^0, \mathbf{x})$ , but if we let  $e_0 = e_0(x)$  move in spacetime, then this makes  $F$  operate on  $e_0(x)$ .

That all is one side of the relations. The other one is that we never departed classical realms: It still holds that from  $F(x)$  alone, one can exactly determine the motion of the sources  $j(x)$ . It therefore repeats the canon from classical gravitation: The field is the equivalent dual of the particle view: The field holds the complete observability of the particles. In particular, this excludes energy-momentum transfer from the particles to field bosons by themselves: In it, energy-momentum exchange occurs only, when the electromagnetic field reaches a charged particle target. Or, as Feynman put it equivalently: photons are edges that connect two nodes: a charged particle source with a charged particle target.

## References

- [1] CARTAN, H. *Differential Forms*. Herman Kershaw, 1971.
- [2] FEYNMAN, R. P. *The Development of the Space-Time View of Quantum Electrodynamics*. [http://www.nobelprize.org/nobel\\_prizes/physics/laureates/1965/feynman-lecture.html](http://www.nobelprize.org/nobel_prizes/physics/laureates/1965/feynman-lecture.html), 1965.
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- [4] HUETTENBACH, H. D. *Disposing Classical Field Theory*. <http://vixra.org/pdf/1209.0027v1.pdf>, 2012.