Abstract: A new concept of exponential-geometric mean is introduced and its properties are analyzed.

The concepts and properties of means of a set of numbers are well studied in calculus. In [1], a mean \( \mu \) of a set of numbers \( x_i, i=1,2,\ldots,n \) is defined as the value that satisfies the condition:

\[
\min(x_1,x_2,\ldots,x_n) \leq \mu \leq \max(x_1,x_2,\ldots,x_n).
\]

In this short note I introduce a new so-called exponential-geometric mean and give some of its properties.

Definition: A lower (upper) exponential-geometric mean \( \mu \) of two positive numbers \( a \) and \( b \) is:

\[
\mu_0 = \frac{a^b b^a}{\sqrt[7]{a^b b^a}} \quad \left( \mu_1 = \frac{a^b b^a}{\sqrt[7]{a^b b^a}} \right).
\]

We can see that the exponential-geometric mean \( \mu \) of any two positive numbers \( a \) and \( b \) conforms to the general definition of the mean. Indeed, without loss of generality, let’s assume \( a \leq b \), then:

\[
\begin{align*}
    a &\leq \frac{a^b b^a}{\sqrt[7]{a^b b^a}} \leq b \Rightarrow a^{a+b} \leq a^b b^a \leq b^{a+b} \Rightarrow \\
    a &\leq b
\end{align*}
\]

Note that the other two combinations \( a^b b^a \) and \( a^b b^a \) may not suit the general definition of the mean.

Examples: The lower and upper exponential-geometric means of numbers

1) \( a = 2 \) and \( b = 3 \) are \( \mu = \frac{2^{3.2}}{3} \approx 2.35 \) and \( \mu = \frac{3^{2.3}}{2} \approx 2.55 \).

2) \( a = 2 \) and \( b = 2.5 \) are \( \mu = \frac{2^{2.5}}{2.5} \approx 2.21 \) and \( \mu = \frac{2.5^{2}}{2.5} \approx 2.26 \).

3) \( a = 0.5 \) and \( b = 0.7 \) are \( \mu = \sqrt[26]{0.5^{0.7}} \cdot 0.7^{0.5} \approx 0.58 \) and \( \mu = \sqrt[5.3]{0.5^{0.7}} \cdot 0.7^{0.5} \approx 0.61 \).

It is well known the following relationships between harmonic, geometric and arithmetic means:

\[
\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2}.
\]

Lemma 1: For the lower and upper exponential-geometric means of two positive numbers \( a \) and \( b \) the following holds true:

\[
\frac{a^b b^a}{\sqrt[7]{a^b b^a}} \leq \frac{2ab}{a+b} \leq \frac{\sqrt{ab}}{\sqrt[7]{a^b b^a}} \leq \frac{a^b b^a}{\sqrt[7]{a^b b^a}}.
\]

Proof: It is sufficient to prove the leftmost inequality, since it is equivalent with the rightmost shown below.

\[
\frac{a^b b^a}{\sqrt[7]{a^b b^a}} \leq \frac{2ab}{a+b} \Rightarrow \frac{a^b b^a}{a+b} \leq \frac{2ab}{a+b} \Rightarrow \frac{a+b}{2} \leq \frac{a^b b^a}{a+b}.
\]

Now, without loss of generality, let us denote \( b = ac, c \geq 1 \). Then,

\[
\frac{a + ac}{2} \leq \frac{a^c}{a^c + ac} \Rightarrow \frac{a(1+c)}{2} \leq \frac{a^c}{1+c} \Rightarrow 1 \leq \frac{2c^c}{1+c}.
\]
For the function \( y(x) = \frac{2x^x}{1+x} \), \( y(1) = 1 \) and it is monotonously increasing at \( x \geq 1 \). Its limit for \( x \to +\infty \) is \( \lim_{x \to +\infty} \frac{2x^x}{1+x} = 2 \). □

**Lemma 2:** For the lower exponential-geometric mean (LEGM), harmonic mean (HM), geometric mean (GM), arithmetic mean (AM) and upper exponential-geometric mean (UEGM) the following relations hold true:

1) \( \text{LEGM} \cdot \text{UEGM} = \text{AM} \cdot \text{HM} = \text{GM}^2 \);
2) \( \text{LEGM} \cdot \text{AM} \leq \text{HM} \cdot \text{UEGM} \);
3) \( \text{UEGM} - \text{AM} \geq \text{HM} - \text{LEGM} \);
4) \( \text{UEGM} - \text{LEGM} \geq \text{AM} - \text{HM} \);
5) \( \text{LEGM} + \text{UEGM} \geq \text{GM} \).

**Proof:** The 1st is rather straightforward:

\[
\text{LEGM} \cdot \text{UEGM} = \frac{a+b}{e} \cdot \frac{a+b}{e} = ab = \frac{2ab}{a+b} = \text{AM} \cdot \text{HM} = \left( \sqrt{ab} \right)^2 = \text{GM}^2.
\]

The 2nd is obvious since \( \text{LEGM} \leq \text{HM} \) and \( \text{AM} \leq \text{UEGM} \).

The 3rd and 4th can be proved by the method used in Lemma 1.

The 5th is also obvious. If we take into account the 1st relation, then

\[
\text{LEGM} + \text{UEGM} \geq \sqrt{\text{LEGM} \cdot \text{UEGM}}.
\]

The exponential-geometric means and their properties allow to estimate cumbersome and inconvenient expressions (and their limits). For example:

1) \( \left( \sin^2 \alpha \right)^{\sin^2 \alpha} \cdot \left( \cos^2 \alpha \right)^{\cos^2 \alpha} \geq \frac{1}{2} \) for \( \alpha \neq \frac{\pi k}{2} \), \( k \in \mathbb{Z} \).

2) \( \tan x + \cot x \sqrt{\left( \tan x \right)^{\cot x} \cdot \left( \cot x \right)^{\tan x}} \leq 1 \) for \( x \in \left\{ \pi k, k \in \mathbb{Z} ; \frac{\pi}{2} \pm \pi k, k \in \mathbb{Z} \right\} \)

(\( \tan x \)^{\cot x} \cdot (\cot x)^{\tan x} \leq 1 \) for \( x \in \left\{ \pi k, k \in \mathbb{Z} ; \frac{\pi}{2} \pm \pi k, k \in \mathbb{Z} \right\} \).

3) \( x^{x+1} \cdot 1^x \geq \frac{x+1}{2} \) for \( x > 0 \) \( \Rightarrow x^{x+1} \geq \frac{x+1}{2} \) for \( x > 0 \).

4) \( x^{e^x} \cdot 1^{e^x} \leq \frac{2e^x}{e^x+1} \) for \( x > 0 \) \( \Rightarrow \exp \left( \frac{x}{e^x+1} \right) \leq \frac{2e^x}{e^x+1} \) for \( x > 0 \).

5) \( x^{x^a} \cdot 1^{x^a} \leq \frac{x^a+1}{2} \) for \( x, a > 0 \) \( \Rightarrow x^{a} \left( \frac{a}{x^a+1} \right) \leq \frac{x^a+1}{2} \) for \( x, a > 0 \).

**Reference**


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