CLAY NAVIER-STOKES PROBLEM CORRECTLY SOLVED - CMI OFFERS ITS REPLY

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ABSTRACT. The Clay Navier-Stokes problem is correctly solved. The answer from CMI is included. The article discusses why and how the Clay Navier-Stokes problem should be corrected.

1. INTRODUCTION

Has the Clay Navier-Stokes problem been solved? Yes, the exact formulation of the problem as it is written in the official problem statement was correctly solved over two years ago in [1]. But that is not what we want! That is not the correct problem: CMI meant something different!

This may be so, but the real story is still rather interesting. I discussed [1] with an expert of the Clay Navier-Stokes problem, Claes Johnson from KTH in September 2012. Johnson has written an article questioning if the CMI Navier-Stokes problem is well-posed, so he should have read the official problem statement. Johnson's first argument against [1] was that strong solutions for Navier-Stokes equations are unique and therefore [1] must be wrong. Johnson gave the Fefferman quote as a support of his words. This quote is in the official problem statement on page 2 of [2]: it is known that (A) and (B) hold (also for $\nu = 0$) if the time interval $[0, \infty)$ is replaced by a small time interval $[0, T)$. However, looking at (A) and (B), that is Statements A and B in [2] on page 2, we can see that these statements do not make any claim that solutions can be continued uniquely. Furthermore, when we search for any versions of the official problem statement in the web, we do not find any version where (A) and (B) would ever have claimed that solutions are unique.

This reminded me of the time 2008 and 2009 when [1] was being reviewed in journals. Journal referees made the same reference to the Fefferman quote as a justification of their claim that strong solutions are unique. I reminded myself of having looked up Statements A and B at that time, and they did claim that solutions are unique. Several emails were sent to the author of the official problem statement and to CMI that the problem statement must be corrected. No answer was ever received, but in 2010 people reading [1] after it was published told me that there is nothing of uniqueness in the official problem statement. And that there never has been. The problem statement has not been corrected. It never made any such claim, but strangely enough Johnson still remembered that it did.

Now the official problem statement says, or should we say, has always stated, that either (A) and (B) hold, or there is a blowup solution. This apparently means that either (A) holds or (C) holds, and either (B) holds or (D) holds.

Key words and phrases. Navier-Stokes; Clay problems; CMI; mathematics today.

I will show by repeating the argument in [1] that this claim is wrong, if [2] ever made such a claim, but logically it must be either (A) of (C), and either (B) or (D), or else [2] has no internal logic in posing the Statements. We can very well have the situation that (B) holds and (D) also holds. Statement B says that for initial data $u^0(x)$ and zero external force f there exists a smooth solution that can be continued to the whole space-time. Statement D says that there exists initial data u^0 and a force f such that all solutions blow up. These statements are not exact opposites. There can be u^0 such that for identically zero f there exists a smooth solution in the whole space-time, but there also exists a force f, that is not identically zero but has zero value everywhere, such that there is only one solution under this force and that solution blows up. Such a force can be defined as a feedback control force. If there are many solutions, there can be one solution that can be continued to the whole space-time, and one blowup solution. We can select a feedback control force that steers all solutions to join the blowup solution.

Earlier, when the official problem statement claimed in (A) and (B) that solutions are locally unique (if it ever claimed this, we cannot prove it) it was clearer that the solution in [1] is a solution of the CMI Navier-Stokes problem, but even now when (A) and (B) make no claim of uniqueness, the solution of [1] is still a valid one. The official problem statement should still be corrected as [1] demands.

2. The solution in [1]

Let us look at the solution in [1]. It is very short and simple. First we need some definitions: Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ be the position, $t \geq 0$ the time, $p(x,t) \in \mathbb{R}$ the pressure, $u(x,t) = (u_i(x,t))_{1\leq i\leq 3} \in \mathbb{R}^3$ the velocity vector, and $f_i(x,t)$ the external force. The Navier-Stokes equations for incompressible fluids filling all of $\mathbb{R}^3$ for $t \geq 0$ are in [2] given as

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = v\Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t), \quad x \in \mathbb{R}^3,\ t \geq 0,\ 1 \leq i \leq 3 \tag{1}$$

$$\operatorname{div} u = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0, \quad x \in \mathbb{R}^3,\ t \geq 0 \tag{2}$$

with initial conditions

$$u(x,0) = u^0(x), \quad x \in \mathbb{R}^3. \tag{3}$$

Here v is a positive coefficient, $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the space variables, and $u^0(x)$ is $C^\infty(\mathbb{R}^3)$ vector field on $\mathbb{R}^3$. It is required to be divergence-free; i.e., satisfying $\operatorname{div} u^0 = 0$. The time derivative $\frac{\partial u_i}{\partial t}$ at $t = 0$ in (1) is taken to mean the limit when $t \to 0^+$.

Writing down the conditions (8), (9), (10) and (11) from [2], Statement D in the official problem statement is as follows (we already have required that $v > 0$ and $n = 3$):

Statement D: There exists a smooth, divergence-free vector field $u^0(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ and a number $C_{\alpha,m,K} > 0$ satisfying

$$u^0(x + e_j) = u^0(x), \quad f(x + e_j, t) = f(x,t), \quad 1 \leq j \leq 3.$$

(here e_j is the unit vector), and

$$|\partial_x^\alpha \partial_t^m f(x,t)| \leq C_{\alpha m K}(1+|t|)^{-K}$$

for any α, m and K, such that there exist no solutions (p,u) of (1), (2), (3) on $\mathbb{R}^3 \times [0,\infty)$ satisfying

$$u(x,t) = u(x+e_j,t)$$

on $\mathbb{R}^3 \times [0,\infty)$ for $1 \leq j \leq 3$, and

$$p, u \in C^\infty(\mathbb{R}^3 \times [0,\infty)).$$

In order to prove Statement D above, let us take the following $u^0(x)$:

$$\begin{aligned} u_1^0 &= 2\pi \sin(2\pi x_2) + 2\pi \cos(2\pi x_3), \\ u_2^0 &= 2\pi \sin(2\pi x_3) + 2\pi \cos(2\pi x_1), \\ u_3^0 &= 2\pi \sin(2\pi x_1) + 2\pi \cos(2\pi x_2), \end{aligned}$$

and let $f_i(x,t)$ be chosen identically zero for $1 \leq i \leq 3$. The following family of functions u and p satisfy (1)-(3):

$$\begin{aligned} u_1 &= e^{-\beta t} 2\pi \left(\sin(2\pi x_2) + \cos(2\pi x_3)\right), \\ u_2 &= e^{-\beta t} 2\pi \left(\sin(2\pi x_3) + \cos(2\pi x_1)\right), \\ u_3 &= e^{-\beta t} 2\pi \left(\sin(2\pi x_1) + \cos(2\pi x_2)\right), \\ p &= -e^{-2\beta t}(2\pi)^2 \sin(2\pi x_1)\cos(2\pi x_2) \\ &\quad - e^{-2\beta t}(2\pi)^2 \sin(2\pi x_2)\cos(2\pi x_3) \\ &\quad - e^{-2\beta t}(2\pi)^2 \sin(2\pi x_3)\cos(2\pi x_1). \end{aligned}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function with $g(0) = g'(0) = 0$. There is the following transform of (1):

$$\begin{aligned} u(x,t) &\to u(x',t') - g'(t), \qquad (4) \\ p(x,t) &\to p(x',t') + g''(t)\sum_{j=1}^{3} x_j, \end{aligned}$$

where $x'_j = x_j + g(t)$, $j = 1,2,3$, and $t' = t$. It keeps the initial values $u_i^0(x)$ and $f(x,t)$ fixed if $g(0) = g'(0) = 0$. It is a transform that can be done to any solution of (1)-(3) and the transformed equation still satisfies (1)-(3).

Let $\beta = (2\pi)^2 v$. Using the transform (4) we get the following family of functions u and p that satisfy (1)-(3) for zero external force:

$$\begin{aligned} u_1 &= e^{-\beta t} 2\pi \left(\sin(2\pi(x_2 + g(t))) + \cos(2\pi(x_3 + g(t)))\right) - g'(t), \\ u_2 &= e^{-\beta t} 2\pi \left(\sin(2\pi(x_3 + g(t))) + \cos(2\pi(x_1 + g(t)))\right) - g'(t), \\ u_3 &= e^{-\beta t} 2\pi \left(\sin(2\pi(x_1 + g(t))) + \cos(2\pi(x_2 + g(t)))\right) - g'(t), \end{aligned}$$

$$\begin{aligned} p = &-e^{-2\beta t}(2\pi)^2 \sin(2\pi(x_1+g(t)))\cos(2\pi(x_2+g(t))) \\ &- e^{-2\beta t}(2\pi)^2 \sin(2\pi(x_2+g(t)))\cos(2\pi(x_3+g(t))) \\ &- e^{-2\beta t}(2\pi)^2 \sin(2\pi(x_3+g(t)))\cos(2\pi(x_1+g(t))) + g''(t)\sum_{j=1}^{3} x_j. \end{aligned}$$

We found a whole set of new solutions.

We see that we can take any valid solution with $f = 0$ that fills the requirements of Statement D and do the transform (4). It yields a set of valid solutions that have a linearly growing term in the pressure. The function $g(t)$ can be freely selected as long as $g(0) = g'(0) = 0$.

Statement D has in the title Breakdown of Navier-Stokes Solutions on $\mathbb{R}^3/\mathbb{Z}^3$ but the statement is given on $\mathbb{R}^3$. It is the statement that has to be shown. It can be given any title. The statement lists all conditions (1), (2), (3), (8), (9), (10), (11) in [2]. It is stated in $\mathbb{R}^3$. Statement D does not require the pressure to be space-periodic.

Let us notice that we do not need to have $g(t)$ smooth everywhere. It can have a singularity. Let us select the function $g(t)$ as follows:

$$g(t) = \frac{1}{2}ct^2\frac{1}{a-t} \tag{5}$$

where $c \neq 0$ and $a > 0$.

With this $g(t)$ we obtain a solution that cannot be continued to the whole space-time, a blowup solution.

If you very much object to concluding that Statement D does not require space-periodicity for p and think that of course p must be space-periodic since the title of Statement D says $\mathbb{R}^3/\mathbb{Z}^3$, and that this counterexample only works because of this minimal formal error in the problem statement, please consider Statement C. Take a solution filling the conditions of Statement A for zero external force and make the transform (4) to that solution. Statement C does not set any growth conditions to the pressure. The external force has value zero, so there is no problem from it. The only additional condition you have to check is (7) in [2], bounded energy. In (4) your $u(x,t)$ has been transformed to $u(x',t) - g'(t)$ where $x'_j = x_j + g(t)$. The bounded energy condition in [2] requires that

$$\int_{\mathbb{R}}^3 |u(x,t)|^2 dx$$

is bounded by some number. You select $g(t)$ that is almost zero and has a very thin spike that goes to infinity. When $g(t)$ grows, x'_j grows and $u(x',t)$ vanishes in infinity. You easily find a function $g(t)$ that yields bounded energy. Statement C clearly says $\mathbb{R}^3$ in the title and we can create in almost exactly the same way a counterexample proving Statement C.

We still have to remove the other solutions, other choices of $g(t)$, so that there is only left the blowup solution. This can be done as follows. We select the time derivatives of u at $t = 0$ to be the time derivatives of this blowup solution at $t = 0$, then the solution is unique. Then the zero external force yields only this one solution, this particular blowup solution that cannot be continued to the whole space-time.

Can we select the time derivatives of u at $t = 0$? Look carefully at Statement B and Statement D in [2]. Clearly our solution fills Statement B. If $g(t) = 0$, we can continue the solution to the whole space-time, and we have the force identically zero. What is there in Statement D that forbids us from selecting the time derivatives of $u(x,t)$ at $t = 0$? Nothing. Statement D does not say anything about that. We are asked to select the initial data u^0, that is u at $t = 0$ and the external force. We can select parameters. Surely we also can select all such terms and parameters that the problem statement does not explicitly mention. In a counterexample you can select everything.

The solutions are required to be physically reasonable. There is the physicality condition (7) in [2] but it is not required in Statement D. In Statement D physical reasonability is (10) and (11) of [2], which we already included in Statement D. The initial data is u^0. That is physical here and in Theorem 2.4 of [1]. Our time derivatives of u are not physical at $t = 0$, but they are not required to be physical.

Thus, we have a proof of Statement D. It was as simple as that, and it is correct.

In [1] the last step is done a bit differently, more elegantly in a way, and we select only the external force, not the time derivatives of $u(x, t)$ at $t = 0$. This way follows exactly what Statement D asks us to do. We can obtain a unique solution by using a feedback control force as the external force.

The force in Theorem 2.4 of [1] is a feedback control force of the type as the steering force steering a car moving on a straight road. If the car moves to the direction of the road, is stays on the road without steering. If it moves to other directions, it drops off the road. A steering force turns the car to the direction of the road and then keeps it there. When the car is in the direction of the road, the force has zero value. How much work the force has to do to get the car on the correct direction depends on what was the original position of the car. If it already was in the correct direction, the force has always zero value. This last case is the situation in Theorem 2.4.

Mathematically we can consider a linear system $du/dt = ku + c + f$ where k and c can have many values and one solution for $f = 0$ is $dv/dt = av + b$. A feedback force $f = du/dt - dv/dt$ gives a solution $f = 0$, $u = v$ in case $k = a$ and $c = b$. So, if a linear system is started in correct direction, the feedback force always has the value zero. If the system has some other starting values k, c, the force f, which in this case is not the same as given above, first moves the system to the correct direction and then becomes zero. If the system is nonlinear, like the Navier-Stokes equation, we linearize it at a vicinity of $t = 0$ and the control force behaves in the same way. After some time the feedback control force sets $u = v$ and f becomes zero. After that point $u = v$ for all time because strong solutions are unique when all (also higher order) time derivatives of u are set, i.e., after $u = v$ for an interval.

What then is the initial position in Theorem 2.4? Does the external force f first need to move the system into correct direction? No: we are free to select the initial position. Nothing in Statement D, or any other place in the official problem statement, forbids us from assigning suitable values for the time derivatives of u at $t = 0$ so that the initial position is that of v, the blowup solution. Then f has value zero from the beginning $t = 0$ and $u = v$ all the time as in Theorem 2.4.

Which one selects the initial position? We or the external force? This is a matter of taste. We can say that the force does the selection. In the car example it is more natural to think that we have to leave the car in the initial position, but there are other similar examples. Consider an egg standing on its sharp head. A small knock pushes it to any side we want. Here it is clearly the external force that selects the direction. We can quite well say as [1] that it is the external force that selects the initial position.

For anybody having driven a car or taken a course in control theory, all this is obvious. Feedback control forces are not especially exotic creatures. But is a feedback control force allowed in Statement D? The official problem statement does not deny such a force. Therefore it is allowed. The problem statement only says: f is a given, externally applied force (e.g. gravity). Given means that the term f is not an unknown like u and p, but given. In Statement D it can be selected. A given force can very well be given either explicitly or implicitly as a function of u. External forces in NSE often have feedback from u. Gravity, the only example mentioned, has feedback from the distribution of the fluid mass. Fluid has mass or gravity had no effect. Mass creates gravity. We do not know if the whole space is filled with fluid. Thus, the mass distribution is a function of u. The gravitation force has an additive component given by a function of u. In general, the interest in fluid turbulence is largely caused by research on propulsion systems. Propulsion systems are a part of a steering system and thus, they have feedback from water motion. As the CMI problems are intended for a wide public and they explain even simple things to the reader, they certainly are intended to contain all assumptions. As there are no explicit limitations to the external force, there are no implicit assumptions that the external force must fill. Thus, f can be a feedback control force.

This is the whole solution in [1]. Did we see any misinterpretation of the problem description? No, everything was according to the official problem statement [2].

3. The need to correct the problem statement

We conclude that Theorem 2.4 in [1] is indeed a valid proof of Statement D. Theorem 2.4 shows that there exists a space-periodic solution that cannot be continued to the whole space. The initial conditions and the external force in this solution confirm fully to the requirements of the problem statement. The solution is given explicitly.

But we do not like the solution.

Thus, the problem statement should be corrected. Let us make some observations concerning this issue. The counterexample in Theorem 2.4 was possible because Statement D does not require the pressure to be space-periodic and because it does not forbid feedback control forces. Let us see what can be done to fix the problem.

Forbidding feedback control forces does not help. We already showed that it suffices to select the time derivatives of the velocity vector u in order to have a unique solution. We set the unique solution to be the blowup solution. In a counterexample we are asked to select the initial data, the external force, the parameters. Surely we are asked also to select all data that was not mentioned by Statement D. In a counterexample we can select everything. It is enough to show that one counterexample exists.

How about space-periodicity? It does not help to require space-periodicity for p in Statement D. If we add a condition that the pressure be space-periodic in Statement D, the solution of [1] still exists. Only it does not fill the required conditions and is therefore a counterexample to existence. Proving Statement D requires finding a counterexample to existence. If we reformulate Statement D in a torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ instead of $\mathbb{R}^3$, Statement D is fixed, but a similar counterexample can be made for Statement C. There are no restrictions to pressure in Statement C and we really cannot put restrictions on pressure: pressure can be eliminated from the equation at least for $f = 0$ (see [1], there it is done), and pressure is calculated from u. If we require p to behave nicely, we assume u to behave nicely. Then there is no sense to prove that u behaves nicely.

We could change the problem statement in such a way that the transform is not allowed, but are we sure that it is the only transform? Where is a proof of that?

The easiest change is to make Statement D the exact opposite of Statement B. That is, to state that the external force is identically zero in Statement D, but this is not what we want.

It is of course possible to state the official problem statement correctly in some way but it is not obvious what way CMI will use. Therefore the problem statement must first be given a correct formulation before one can know what exactly should be proven. [1] proves exactly what was stated in Statement D. If that is not what should have been proven, then the problem statement is incorrectly formulated.

4. The reply from CMI

The solution [1] is published in a peer-reviewed journal of good reputation two years ago. The article has not been refuted or disputed in the scientific forums. A paper published in a peer-reviewed journal is generally accepted by the scientific community unless it is refuted or disputed in the scientific forums. This means that [1] is generally accepted by the mathematical community. Thus, both conditions stated in the millennium prize challenge rules are filled. When both conditions are filled, CMI promises in the rules to set up a group to check the solution. The rules do not grant CMI or the Scientific Advisor Board the right to reject a solution without setting up a group to check the solution. For three months CMI tried to ignore [1] completely, but finally they had to answer. Below is the reply from CMI:

Dear Dr Jormakka,

Your paper has been considered by the Scientific Advisory Board. In the judgement of the board, the claim that the paper solves the Navier-Stokes problem is based on a misinterpretation of the problem description. Accordingly the board has concluded that it does not merit detailed consideration for the award of a Millennium Prize and it has therefore decided not to appoint a special advisory committee. The Clay Mathematics Institute will not enter into further correspondence on this matter.

With regards, Clay Mathematics Institute

We see that CMI refuses to set up a group to check the solution. They also refuse to explain what they mean by a misinterpretation.

There is no misinterpretation in the steps taken in [1]. The problem in Statement D is simply: find a function filling the listed conditions. It is the official problem statement that is poorly formulated and should be corrected.

CMI also refuses to admit that the official problem statement has ever been corrected. Strangely, experts like Johnson remember the Fefferman quote incorrectly. Let this be as it is. Electronic documents have no dates that cannot be changed. There is no way to prove that the problem statement ever made a claim of uniqueness of solutions. However, the official problem statement is still not correct now, October 2012. Statement D is as it always was, and it has been proven.

5. Conclusions

The solution may seem so simple that you may wonder if this was not known to the experts of the field. It does not seem to be the case. When [1] was in journal review, experts of PDEs, with one exception, started with the counterargument that strong solutions to the Navier-Stokes equation are known to be unique, thus the solution in [1] must be wrong. That is, with the wrong counterargument, because the strong solutions are not unique in the case of the Clay problem. The only exception did not know any better: he thought that I mean non-uniqueness of weak solutions and referred to [4]. That, of course, is well-known and not the same thing as in [1]. There are theorems that strong solutions are unique, but these theorems have some conditions that are not filled in Statement D. The closest goes a proof in Temam in [3]. However, that proof makes an implicit assumption that everything is space-periodic, while in the CMI problem the pressure is not required to be space-periodic. Thus, this proof does not apply.

The solution in [1] is very simple. The official problem statement does not specify how difficult the solution must be, all it demands is a complete proof of one of the four Statements. Actually the solution is on the very highest border of difficulty to be reviewed and published by a peer-reviewed journal in the normal manner in case the paper presents a solution of a known difficult problem and comes from an unknown source. It must be like the proof Fermat did not write to the margin for the Fermat Last Theorem, i.e., short and simple, rather than the Wiles proof of the theorem, i.e., long and difficult, since no expert will read the long and difficult. The normal manner to publish a journal article is to submit a manuscript to a journal and to get it accepted. This will not succeed if the problem is one of those famous problems: the editor will not accept the submission, or cannot find a referee, or the referees will not want to accept the paper. Getting a manuscript checked by world-leading experts before publication requires that you must have good connections within the mathematical community. You need inside friends or supporters. Otherwise experts will not read a manuscript you send to them. Of course they would not come to any seminar that you might try to arrange. This would be absurd, but even if you in some way manage to get into a seminar to present your solution to experts, they will refuse to say anything.

The rules of the Clay challenge do state that the solution should get a million, but apparently nobody in the mathematical community wants to give this solution a million, and CMI will not follow the rules it has stated in the web. Therefore, one should correct the official problem statement, and give a smaller prize for [1] for correcting the problem statement. The solution is simple, but it is neither easy nor fast to find such easy solutions to known difficult problems. Additionally CMI should set up a journal that receives submissions that try to solve the seven problems (indeed, there can be different proofs even to the problems that already were solved) and reviews them correctly.

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