Some Solutions to the Clifford Space Gravitational Field Equations

Carlos Castro
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, Georgia. 30314, perelmanc@hotmail.com

October 2012

Abstract

We continue with the study of Clifford-space Gravity and find some solutions to the Clifford space (C-space) generalized gravitational field equations which are obtained from a variational principle based on the generalization of the Einstein-Hilbert-Cartan action. The C-space connection requires *torsion* and the field equations in C-space are *not* equivalent to the ordinary gravitational equations with torsion in higher 2^D-dimensions. We find specific metric solutions in the most simple case and discuss their difference with the metrics found in ordinary gravity.

1 Introduction

In the past years, the Extended Relativity Theory in C-spaces (Clifford spaces) and Clifford-Phase spaces were developed [1], [2]. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory whose generalized coordinates are Clifford polyvector-valued quantities which incorporate the lines, areas, volumes, and hyper-volumes degrees of freedom associated with the collective dynamics of particles, strings, membranes, p-branes (closed p-branes) moving in a D-dimensional target spacetime background. C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing. Our theory has 2 fundamental parameters : the speed of a light c and a length scale which can be set equal to the Planck length. The role of "photons" in C-space is played by *tensionless* branes. An extensive review of the Extended Relativity Theory in Clifford spaces can be found in [1]. The polyvector valued coordinates $x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, ...$ are now linked to the basis vectors generators γ^μ, bi-vectors generators $\gamma_\mu \wedge \gamma_\nu$, tri-vectors generators

$\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$, ... of the Clifford algebra, including the Clifford algebra unit element (associated to a scalar coordinate). These polyvector valued coordinates can be interpreted as the quenched-degrees of freedom of an ensemble of p-loops associated with the dynamics of closed p-branes, for $p = 0, 1, 2, ..., D-1$, embedded in a target D-dimensional spacetime background.

The C-space poly-vector-valued momentum is defined as $\mathbf{P} = d\mathbf{X}/d\Sigma$ where $\mathbf{X}$ is the Clifford-valued coordinate corresponding to the $Cl(1,3)$ algebra in four-dimensions, for example,

$$\mathbf{X} \ = \ s\,\mathbf{1} + x^{\mu}\,\gamma_{\mu} + x^{\mu\nu}\,\gamma_{\mu}\wedge\gamma_{\nu} + x^{\mu\nu\rho}\,\gamma_{\mu}\wedge\gamma_{\nu}\wedge\gamma_{\rho} + x^{\mu\nu\rho\tau}\,\gamma_{\mu}\wedge\gamma_{\nu}\wedge\gamma_{\rho}\wedge\gamma_{\tau} \quad (1)$$

it can be generalized to any dimensions, including $D = 0$.

The component s is the Clifford scalar component of the polyvector-valued coordinate and $d\Sigma$ is the infinitesimal C-space proper "time" interval which is *invariant* under $Cl(1,3)$ transformations which are the Clifford-algebra extensions of the $SO(1,3)$ Lorentz transformations [1]. One should emphasize that $d\Sigma$, which is given by the square root of the quadratic interval in C-space

$$(d\Sigma)^2 \ = \ (ds)^2 \ + \ dx_{\mu}\,dx^{\mu} \ + \ dx_{\mu\nu}\,dx^{\mu\nu} \ + \ \ldots \quad (2)$$

is *not* equal to the proper time Lorentz-invariant interval $d\tau$ in ordinary spacetime $(d\tau)^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = dx_{\mu}dx^{\mu}$. In order to match units in all terms of eqs-(1,2) suitable powers of a length scale (say Planck scale) must be introduced. For convenience purposes it is set to unity. For extensive details of the generalized Lorentz transformations (poly-rotations) in flat C-spaces and references we refer to [1].

Let us now consider C-space [1]. A basis in C-space is given by

$$E_A = \gamma, \ \gamma_{\mu}, \ \gamma_{\mu}\wedge\gamma_{\nu}, \ \gamma_{\mu}\wedge\gamma_{\nu}\wedge\gamma_{\rho}, \ \ldots \quad (3)$$

where γ is the unit element of the Clifford algebra that we label as $\mathbf{1}$ from now on. In (3) when one writes an r-vector basis $\gamma_{\mu_1}\wedge\gamma_{\mu_2}\wedge\ldots\wedge\gamma_{\mu_r}$ we take the indices in "lexicographical" order so that $\mu_1 < \mu_2 < \ldots. < \mu_r$. An element of C-space is a Clifford number, called also *Polyvector* or *Clifford aggregate* which we now write in the form

$$X \ = \ X^A E_A \ = \ s\,\mathbf{1} \ + \ x^{\mu}\gamma_{\mu} \ + \ x^{\mu\nu}\gamma_{\mu}\wedge\gamma_{\nu} + \ldots \quad (4)$$

A C-space is parametrized not only by 1-vector coordinates x^{μ} but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\alpha}$, ..., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops,..., onto the coordinate planes . By p-loop we mean a closed p-brane; in particular, a 1-loop is closed string. In order to avoid using the powers of the Planck scale length parameter L_p in the expansion of the polyvector X (in order to match units) we can set it to unity to simplify matters.

In a *flat* C-space the basis vectors E^A, E_A are *constants*. In a *curved* C-space this is no longer true. Each E^A, E_A is a function of the C-space coordinates

$$X^A \ = \ \{ \ s, \ x^{\mu}, \ x^{\mu_1\mu_2}, \ \ldots.., \ x^{\mu_1\mu_2\ldots..\mu_D} \ \} \quad (5)$$

which include scalar, vector, bivector,..., p-vector,... coordinates in the underlying D-dim base spacetime and whose corresponding C-space is 2^D-dimensional since the Clifford algebra in D-dim is 2^D-dimensional.

In curved C-space one introduces the $\mathbf{X}$-dependent basis generators γ_M, γ^M defined in terms of the beins E_M^A, inverse beins E_A^M and the flat tangent space generators γ_A, γ^A as follows $\gamma_M = E_M^A(\mathbf{X})\gamma_A, \gamma^M = E_A^M(\mathbf{X})\gamma^A$. The curved C-space metric expression $g_{MN} = E_M^A E_N^B \eta_{AB}$ also agrees with taking the scalar part of the Clifford geometric product $< \gamma_M \gamma_N >= g_{MN}$.

The covariant derivative of $E_M^A(\mathbf{X}), E_A^M(\mathbf{X})$ involves the generalized connection and spin connection and is defined as

$$\nabla_K E_M^A \ = \ \partial_K E_M^A \ - \ \Gamma_{KM}^L \ E_L^A \ + \ \omega_{KB}^A \ E_M^B \qquad (6a)$$

$$\nabla_K E_A^M \ = \ \partial_K E_A^M \ + \ \Gamma_{KL}^M \ E_A^L \ - \ \omega_{KA}{}^B \ E_B^M \qquad (6b)$$

If the nonmetricity is zero then $\nabla_K E_M^A = 0, \ \nabla_K E_A^M = 0$ in eqs-(6).

One of the salient features in [3] is that the C-space connection requires torsion in order to have consistency between the Clifford algebraic structure and the zero nonmetricity condition $\nabla_K g^{MN} = 0$. In the case of nonsymmetric connections with torsion, the curvature obeys the following relations under the exchange of indices

$$\mathbf{R}_{MNJK} \ = \ - \ \mathbf{R}_{NMJK}, \ \ \mathbf{R}_{MNKJ} \ = \ - \ \mathbf{R}_{MNJK}, \ \ \textit{but} \ \mathbf{R}_{MNJK} \ \neq \ \mathbf{R}_{JKMN} \qquad (7)$$

and is defined in terms of the connection components Γ_{KM}^L as follows

$$\mathbf{R}_{MNJ}{}^K \ = \ \partial_M \ \Gamma_{NJ}{}^K \ - \ \partial_N \ \Gamma_{MJ}{}^K \ + \ \Gamma_{ML}^K \ \Gamma_{NJ}^L \ - \ \Gamma_{NL}^K \ \Gamma_{MJ}^L \qquad (8)$$

If the anholonomy coefficients$f_{MN}^K \neq 0$ one must also include them into the definition of curvature (8) by adding terms of the form $-f_{MN}^L \Gamma_{LJ}^K$.

The standard Riemann-Cartan curvature tensor in ordinary spacetime is *contained* in C-space as follows

$$\mathcal{R}_{\mu_1\mu_2\rho_1}{}^{\rho_2} \ = \ \partial_{\mu_1}\Gamma^{\rho_2}_{\mu_2\rho_1} \ - \ \partial_{\mu_2}\Gamma^{\rho_2}_{\mu_1\rho_1} \ + \ \Gamma^{\rho_2}_{\mu_1\sigma} \ \Gamma^{\sigma}_{\mu_2\rho_1} \ - \ \Gamma^{\rho_2}_{\mu_2\sigma} \ \Gamma^{\sigma}_{\mu_1\rho_1} \ \subset$$

$$\mathbf{R}_{\mu_1\mu_2\rho_1}{}^{\rho_2} \ = \partial_{\mu_1}\Gamma^{\rho_2}_{\mu_2\rho_1} \ - \ \partial_{\mu_2}\Gamma^{\rho_2}_{\mu_1\rho_1} \ + \ \Gamma^{\rho_2}_{\mu_1 \ \mathbf{M}} \ \Gamma^{\mathbf{M}}_{\mu_2\rho_1} \ - \ \Gamma^{\rho_2}_{\mu_2 \ \mathbf{M}} \ \Gamma^{\mathbf{M}}_{\mu_1\rho_1} \qquad (9)$$

due to the contractions involving the polyvector valued indices $\mathbf{M}$ in eq-(9) There is also the crucial difference that $\mathbf{R}^{\rho_2}_{\mu_1\mu_2\rho_1}(s, x^\nu, x^{\nu_1\nu_2}, ...)$ has now an *additional* dependence on all the C-space polyvector valued coordinates $s, x^{\nu_1\nu_2}, x^{\nu_1\nu_2\nu_3}, ...$ besides the x^ν coordinates. The curvature in the presence of torsion does not satisfy the same symmetry relations when there is no torsion, therefore the Ricci-like tensor is no longer symmetric in general

$$\mathbf{R}_{MNJ}{}^N \ = \ \mathbf{R}_{MJ}, \quad \mathbf{R}_{MJ} \ \neq \ \mathbf{R}_{JM}, \ \ \mathbf{R} \ = \ g^{MJ} \ \mathbf{R}_{MJ} \qquad (10)$$

Denoting the Clifford scalar $\mathbf{s}$ component by the index 0, and that must *not* be confused with the temporal coordinate t, the C-space Ricci-like tensor is

$$\mathbf{R}_M^N \;=\; \sum_{j=1}^{D} \mathbf{R}_M{}^{N\,[\nu_1\nu_2\ldots\nu_j]}{}_{[\nu_1\nu_2\ldots\nu_j]} \;+\; \mathbf{R}_M{}^N{}_{\mathbf{0}}{}^{\mathbf{0}} \tag{11}$$

and the C-space curvature scalar is

$$\mathbf{R} = \sum_{j=1}^{D}\sum_{k=1}^{D} \mathbf{R}_{[\mu_1\mu_2\ldots\mu_j]\;[\nu_1\nu_2\ldots\nu_k]}{}^{[\mu_1\mu_2\ldots\mu_j]\;[\nu_1\nu_2\ldots\nu_k]} + \sum_{j=1}^{D} \mathbf{R}_{[\mu_1\mu_2\ldots\mu_j]\;\mathbf{0}}{}^{[\mu_1\mu_2\ldots\mu_j]\;\mathbf{0}} \tag{12}$$

The physical applications of C-space gravity to higher curvature theories of gravity were studied in [3]. In particular we have shown the relationship to Lanczos-Lovelock-Cartan gravity (with torsion) [5] and to $f(R)$ extended theories of gravity [7].

One may construct an Einstein-Hilbert-Cartan like action based on the C-space curvature scalar

$$\frac{1}{2\kappa^2} \int ds \prod dx^{\mu} \prod dx^{\mu_1\mu_2} \;\ldots\; dx^{\mu_1\mu_2\ldots\mu_D} \;\mu_m(g_{MN})\;\; \mathbf{R} \tag{13}$$

where $\mu_m(g_{MN})$ is a suitable integration measure.

For simplicity we shall set the nonmetricity Q^L_{MN} to zero from now on, so that

$$\Gamma^L_{MN} \;=\; \{^{\;L}_{MN}\} \;+\; K^L_{MN} \tag{14}$$

In [3] it was shown that a metric compatible connection (zero nonmetricity case) which is consistent with eq-(14) is given by

$$\Gamma^K_{MN} \;=\; \frac{1}{2}\, g^{KL}\, \partial_M g_{NL}, \quad \Gamma_{MNL} \;=\; \frac{1}{2}\, \partial_M g_{NL}, \tag{15}$$

and has torsion $T^K_{MN} = \Gamma^K_{MN} - \Gamma^K_{NM}$. The contorsion tensor is in this case

$$K_{MNK} = \Gamma_{MNK} - \{_{MNK}\} \;=\; \frac{1}{2}(\partial_K g_{NM} - \partial_N g_{KM}) \;=\; -\,\Gamma_{[NK]M} \;=\; -\,T_{NKM} \tag{16}$$

The contorsion tensor is defined in terms of the components of the torsion tensor as

$$K_{JMN} \;=\; \frac{1}{2}\,(T_{JMN} \;-\; T_{MNJ} \;+\; T_{NJM}) \tag{17a}$$

$$K^L_{MN} \;=\; g^{LJ}\, K_{JMN}, \quad K_{JMN} \;=\; -\; K_{JNM}, \quad T_{JMN} \;=\; -\; T_{MJN} \tag{17b}$$

The results (15) were obtained in the so-called "diagonal gauge" [3] where in a given coordinate system (generalized Lorentz frame) the mixed-grade components of the metric g_{MN}, g^{MN}, and beins E^A_M, inverse beins E^M_A, can be set to zero in order to considerably simplify the calculations; namely due to the very large diffeomorphism symmetry in C-space, one may choose a frame ("diagonal gauge") such that the mixed grade components of the metric, beins, inverse beins are zero. In this case, the Clifford algebra associated to the curved space basis generators $\gamma_M = E^A_M \gamma_A$ assumes the *same* functional form as it does in the flat tangent space, and obeys the (graded) Jacobi identities. The metric, beins, inverse beins, admit a decomposition into their irreducible pieces like in eq-(2.22) below. Only a restricted set of poly-coordinate transformations (generalized Lorentz transformations in the tangent space) will preserve such zero mixed-grade condition, namely the grade-preserving transformations.

The connection (15) with torsion $\Gamma^K_{MN} = \frac{1}{2} g^{KL} \partial_M g_{NL}$ required that the same-grade metric components $g^{[\mu_1\mu_2...\mu_k]\ [\nu_1\nu_2...\nu_k]}$ are decomposed into its irreducible factors as antisymmetrized sums of products of $g^{\mu\nu}$ as follows [3]

$$det \begin{pmatrix} g^{\mu_1\nu_1} & \ldots & \ldots & g^{\mu_1\nu_k} \\ g^{\mu_2\nu_1} & \ldots & \ldots & g^{\mu_2\nu_k} \\ ------------ & & -------------- & \\ g^{\mu_k\nu_1} & \ldots & \ldots & g^{\mu_k\nu_k} \end{pmatrix} \tag{18}$$

inserting the metric decomposition (18), before performing the variation of the action, leads to

$$\delta S \ = \ \frac{\delta S}{\delta g^{00}}\ \delta g^{00} \ + \ \frac{\delta S}{\delta g^{\mu\nu}}\ \delta g^{\mu\nu} \ + $$
$$\frac{\delta S}{\delta g^{[\mu_1\mu_2]\ [\nu_1\nu_2]}}\ \frac{\delta g^{[\mu_1\mu_2]\ [\nu_1\nu_2]}}{\delta g^{\mu\nu}}\ \delta g^{\mu\nu} \ + \ .... \tag{19}$$

with

$$\frac{\delta(\sqrt{|g|}\mathcal{L})}{\delta g^{\mu\nu}} \ - \ \partial(.......) \ \leftrightarrow \ \mathbf{R}_{(\mu\nu)} \ - \frac{1}{2}\ g_{\mu\nu}\ \mathbf{R} \tag{20}$$

the other remaining contributions of the polyvector-components denoted by the hatted indices give

$$\frac{\delta(\sqrt{|g|}\mathcal{L})}{\delta g^{\hat{M}\hat{N}}} \ - \ \partial(.........) \ \leftrightarrow \ \mathbf{R}_{(\hat{M}\hat{N})} \ - \frac{1}{2}\ g_{\hat{M}\hat{N}}\ \mathbf{R} \tag{21}$$

such that the $g_{\mu\nu}$ vacuum field equations in (19) acquire now the *extra* terms given by

$$\mathbf{R}_{(\mu\nu)} \ - \frac{1}{2}\ g_{\mu\nu}\ \mathbf{R} \ + \ \left(\mathbf{R}_{(\hat{M}\hat{N})} \ - \frac{1}{2}\ g_{\hat{M}\hat{N}}\ \mathbf{R} \right) \frac{\delta g^{\hat{M}\hat{N}}}{\delta g^{\mu\nu}} \ = \ \kappa^2\ \mathbf{T}_{\mu\nu} \tag{22}$$

These extra terms to eqs-(22) have the same role as an $effective$ stress energy tensor term $\kappa^2 \mathbf{T}^{eff}_{\mu\nu}$ contribution, up to a minus sign. A thorough discussion on the implications to dark energy, multi-metric theories of gravity and higher spin theories [8] were provided in [3]. Given

$$\mathbf{R}_L{}^K{}_{MN} \ = \ \partial_M\ \Gamma_{LN}^{\ K} \ - \ \partial_N\ \Gamma_{LM}^{\ K} \ + \ \Gamma^K_{MJ}\ \Gamma^J_{LN} \ - \ \Gamma^K_{NJ}\ \Gamma^J_{LM} \tag{23}$$

and

$$\mathbf{R}_{MNL}{}^{K} \;=\; \partial_M\ \Gamma^K_{NL} \;-\; \partial_N\ \Gamma^K_{ML} \;+\; \Gamma^J_{NL}\ \Gamma^K_{JM} \;-\; \Gamma^J_{ML}\ \Gamma^K_{JN} \qquad (24)$$

one has that if, and only if, the connections are *symmetric* then $\mathbf{R}_{MNL}{}^{K} = \mathbf{R}_{L}{}^{K}{}_{MN}$. Since in our case the connections are *not* symmetric then $\mathbf{R}_{MNL}{}^{K} \neq \mathbf{R}_{L}{}^{K}{}_{MN}$. Therefore we shall examine both cases when one has a conformally flat metric

$$g_{00} = e^{2\Omega}\eta_{00}, \quad g_{\mu\nu} = e^{2\Omega}\eta_{\mu\nu} \qquad (25a)$$

and

$$g_{[\mu_1\mu_2]\ [\nu_1\nu_2]} \;=\; e^{4\Omega}\ (\ \eta_{\mu_1\nu_1}\ \eta_{\mu_2\nu_2} \;-\; \eta_{\mu_1\nu_2}\ \eta_{\mu_2\nu_1}\ ),\ etc..... \qquad (25b)$$

the connection components $\Gamma^K_{MN} = \frac{1}{2} g^{KL}\partial_M g_{NL}$ (15) are

$$\Gamma^\rho_{\mu\nu} \;=\; (\partial_\mu\Omega)\ \delta^\rho_\nu, \quad \Gamma^0_{00} \;=\; (\partial_0\Omega)\ \delta^0_0, \quad \Gamma^{[\rho_1\rho_2]}_{\mu\ [\tau_1\tau_2]} \;=\; 2(\partial_\mu\Omega)\ \delta^{[\rho_1\rho_2]}_{[\tau_1\tau_2]}, \quad .... \qquad (26)$$

To simplify matters and without loss of generality we shall write down the C-space gravitational field equations associated with the Clifford algebra in one temporal dimension $Cl(0,1)$. The C-space when $D = 1$ is $2^D = 2$-dimensional. The Ricci-like and scalar curvatures associated with $R_{MNL}{}^{K}$ are

$$\mathbf{R}^{(1)}_{00} \;=\; -\ (\partial_0\Omega)\ (\partial_0\Omega)\ \frac{1}{L^2}, \quad \mathbf{R}^{(1)}_{11} \;=\; -\ (\partial_1\Omega)\ (\partial_1\Omega)\ \frac{1}{L^2} \qquad (27)$$

In this very special case involving conformally flat metrics (25), the Ricci-like tensor turned out to be symmetric, but this does not happen in general when there is torsion. The scalar curvature is

$$\mathbf{R}^{(1)} \;=\; -\ e^{-2\Omega}\ [\ (\partial_0\Omega)^2 \;-\; (\partial_1\Omega)^2\ ]\ \frac{1}{L^2} \qquad (28)$$

Whereas, the Ricci-like and scalar curvatures associated with R^K_{LMN} are

$$\mathbf{R}^{(2)}_{00} \;=\; -\ [\ (\partial_0\partial_0\Omega) \;+\; (\partial_0\Omega)(\partial_0\Omega)\ ]\ \frac{1}{L^2} \qquad (29a)$$

$$\mathbf{R}^{(2)}_{11} \;=\; -\ [\ (\partial_1\partial_1\Omega) \;+\; (\partial_1\Omega)(\partial_1\Omega)\ ]\ \frac{1}{L^2} \qquad (29b)$$

$$\mathbf{R}^{(2)} \;=\; -\ e^{-2\Omega}\ [\ (\partial^2\Omega) \;+\; (\partial\Omega)^2\ ]\ \frac{1}{L^2} \qquad (30)$$

where

$$(\partial^2\Omega) \;\equiv\; \partial_0^2\Omega \;-\; \partial_1^2\Omega; \quad (\partial\Omega)^2 \;\equiv\; (\partial_0\Omega)^2 \;-\; (\partial_1\Omega)^2 \qquad (31)$$

We will show how the solutions $differ$ from those of ordinary gravity in 2-dimensions. Our variables are the Clifford scalar $s \equiv x^0$ and the vectorial coordinate $x^1 \equiv t$. Inserting the metric (25) and connection (26) into the vacuum field equations associated with the $Cl(0,1)$ algebra and given by

$$R^{(1)}_{00} \ - \frac{1}{2} g_{00} \ R^{(1)} \ = \ 0, \ \ R^{(1)}_{11} \ - \frac{1}{2} g_{11} \ R^{(1)} \ = \ 0 \qquad (32)$$

yields after some algebra the following equations

$$- \frac{1}{2} \ (\partial_1 \Omega)^2 \ - \frac{1}{2} \ (\partial_0 \Omega)^2 \ = \ 0, \quad \eta_{00} = 1, \ \eta_{11} = -1 \qquad (33)$$

There are *no mixed* curvature components involving the scalar index s and the vector index $x^1 = t$. The most general solutions to (33) are of the form

$$\Omega \ = \ f(s + it), \quad f(s - it) \qquad (34)$$

for arbitrary functions. This result is the C-space analog of having conformally flat metric solutions in ordinary gravity in 2-dimensions. A *particular* solution to eq-(33) leading to a *constant* scalar curvature is

$$\Omega \ = \ - \ ln(s + it) \ \Rightarrow \ e^{2\Omega} \ = \ \frac{1}{(s + it)^2} \qquad (35)$$

so that the metric interval in C-space is

$$(d\Sigma)^2 \ = \ g_{MN} \ dX^M \ dX^N \ = \ \frac{ds^2 \ - \ dt^2}{(\ s \ + i \ t \ )^2} \qquad (36)$$

This metric (36) is the C-space extension of the hyperbolic upper complex pane metric (Siegel-plane $\mathbf{H}$) and corresponding to the Clifford algebra $Cl(0,1)$ in one-dimension. Since the latter algebra is isomorphic to the algebra of complex numbers it is not surprising to have a complex-valued metric. We should compare this metric (36) with the hyperbolic upper complex plane metric and the Poincare disc metric given, respectively, by

$$(d\sigma)^2 \ = \ L^2 \ \frac{dx^2 \ - \ dt^2}{t^2}, \quad (d\sigma)^2 \ = \ \frac{dx^2 \ + \ dy^2}{(1 - (\frac{x^2+y^2}{L^2}))^2} = \frac{dz d\bar{z}}{(\ 1 \ - \ z\bar{z}/L^2)^2} \qquad (37)$$

Real-valued metrics are are obtained in the *Euclidean* case $\eta_{11} = 1$ as we shall display below. The scalar curvature associated with the metric (36), and the connection with torsion (14), is constant and negative

$$R^{(1)}(\Gamma) \ = \ - \ e^{-2\Omega} \ (\ (\partial_s \Omega)^2 - (\partial_t \Omega)^2 \ ) \ = \ - \ \frac{2}{L^2} \qquad (38)$$

as one should expect. In ordinary 2-dim gravity the Einstein tensor is identically zero, but the scalar curvature is not necessarily zero. This is due to the topological invariant character of the Einstein-Hilbert action in 2-dimensions.

Real valued metric solutions are obtained in the *Euclidean* case $\eta_{11} = 1$. In this case the field equations yield

$$- \frac{1}{2} \ (\partial_1 \Omega)^2 \ + \frac{1}{2} \ (\partial_0 \Omega)^2 \ = \ 0, \quad \eta_{00} = 1, \ \eta_{11} = 1 \qquad (39)$$

The most general solutions to (39) are

$$\Omega \;=\; f(s+x), \quad f(s-x) \tag{40}$$

for arbitrary functions f. A particular solution to eq-(39) is

$$\Omega \;=\; -\,ln(s+x) \;\Rightarrow\; e^{2\Omega} \;=\; \frac{1}{(s+x)^2} \tag{41}$$

so that the *real*-valued metric interval in the C-space corresponding to Clifford algebra $Cl(1,0)$ is

$$(d\Sigma)^2 \;=\; g_{MN}\; dX^M\; dX^N \;=\; \frac{ds^2 \;+\; dx^2}{(\;s\;+\;x\;)^2} \tag{42}$$

By inspection we can verify that the metrics (36,42) are also solutions to the field equations

$$R^{(2)}_{00} \;-\frac{1}{2}g_{00}\; R^{(2)} \;=\; 0, \quad R^{(2)}_{11} \;-\frac{1}{2}g_{11}\; R^{(2)} \;=\; 0 \tag{43}$$

associated with the R^K_{LMN} curvatures. In this case the scalar curvature $R^{(2)}(\Gamma)$ is twice as $R^{(1)}(\Gamma)$. For example, when $\eta_{11} = -1$, one has

$$R^{(2)}(\Gamma) \;=\; -\; e^{-2\Omega}\; [\; (\partial_s\Omega)^2 - (\partial_t\Omega)^2 \;+\; (\partial^2_s\Omega) \;-\; (\partial_t\Omega)^2\;] = \;-\; \frac{4}{L^2} \tag{44}$$

The scalar curvature corresponding to a connection with torsion $\Gamma^L_{MN} = \{^{\,L}_{MN}\} + K^L_{MN}$ can be rewritten in terms of a purely Riemannian-like part plus torsion terms in the form [6], [4]

$$\mathbf{R}(\Gamma) \;=\; \mathbf{R}(\{\}) \;+a_1\; T_{MNL}\; T^{MNL} \;+a_2\; \nabla_M(\{\})T^M \tag{45}$$

for suitable numerical coefficients a_1, a_2 and whose values depend on which expressions for the scalar curvature $\mathbf{R}^{(1)}(\Gamma), \mathbf{R}^{(2)}(\Gamma)$ are chosen. When $2^D = 2$, for a conformally flat metric (25a), one has

$$\mathbf{R}(\{\}) \;=\; -\,2\; e^{-2\Omega}\; (\; \partial^2_0\Omega \;-\; \partial^2_1\Omega\;) \;=\; -\,(4/L^2), \quad \eta_{11} = -1 \tag{46}$$

Because the torsion is *not* zero, from eqs-(38,44) one can infer that the coefficients a_1, a_2 are not zero in the first case ($\mathbf{R}^{(1)} \;=\; -2/L^2$) while they are vanishing in the second case ($\mathbf{R}^{(2)} = -4/L^2$).

The field equations (22) corresponding to higher dimensional Clifford algebras $Cl(p,q)$ (with $p+q = D > 1$) are far more complicated. One of the key differences between C-space metric solutions to the generalized field equations and those in ordinary gravity is that in the latter case we have de Sitter and Anti de Sitter solutions to Einstein's equations with a cosmological constant which can be written in the form $e^{2\Omega}\eta_{\mu\nu}$ (in a certain coordinate system), whereas in the former C-space case there are *no* solutions of this form for $D > 1$ (for a nonvanishing cosmological constant). This is a direct consequence of the form of the metric components displayed in eqs-(25a,25b). There are $different$ weights $e^{2\Omega}, e^{4\Omega}$, ... associated with the $different$ grades of the metric components.

As mentioned earlier, C-space gravity is more closely related to Lanczos-Lovelock-Cartan higher curvature theories of gravity with torsion [3]. This is mainly due to the antisymmetric property of the polyvector-valued Clifford indices of eqs-(11,12). Finding nontrivial solutions to the C-space field equations eqs-(22) in $D = 4$ and corresponding to the Clifford algebra in $4D$ is a daunting task since the Clifford space is $2^4 = 16$-dimensional. In particular, finding the analogs of the static spherically symmetric vacuum Schwarzschild solutions and black-holes, black-branes, Hawking radiation, ...

Acknowledgements

We thank M. Bowers for very kind assistance.

References

[1] C. Castro and M. Pavsic, Progress in Physics **vol 1** (April 2005) 31. Phys. Letts **B 559** (2003) 74. Int. J. Theor. Phys **42** (2003) 1693.

[2] C. Castro, Foundations of Physics **35**, no.6 (2005) 971. Prog. in Phys. **1** (April 2005) 20.

[3] C. Castro, "Progress in Clifford Space Gravity", to appear to Advances in Applied Clifford Algebras (2012). "Lanczos-Lovelock and f (R) Gravity from Clifford Space Geometry", to appear in Int. J. Theor. Physics (2012). "Lanczos-Lovelock-Cartan Gravity from Clifford Space Geometry , to appear in the Int. J. Geom. Meth. in Mod. Phys.

[4] S. Vacaru, J.Math.Phys. **47** (2006) 093504

[5] C. Lanczos, Annals Math. 39, 842 (1938). D. Lovelock, J. Math. Phys **12** (1971) 498.

[6] F. Hehl, J. McCrea, E. Mielke and Y. Ne'eman, Physics Reports **258** (1995) 1-171.

[7] S. Capozziello and M. De Laurentis, "Extended Theories of Gravity" [arXiv:gr-qc/1108.6266] Phys.Rept. **509** (2011)167-321.

[8] M. Gaberdiel and M. Vasiliev, *Higher Spin Theories and AdS/CFT*, to appear in a special edition of J.Phys. **A**.