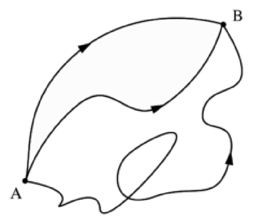

The Physical Origin of the Feynman Path Integral

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Abstract

The Feynman path integral is an essential part of our mathematical description of fundamental nature at small scales. However, what it seems to say about the world is very much at odds with our classical intuitions, and exactly why nature requires us to describe her in this way is currently unknown. We will describe here a possibility according to which the path integral may be the spacetime manifestation of objects existing in a lower-dimensional analog of spacetime until they give rise to the emergence of spacetime objects under a process that is currently labeled a 'Quantum Measurement'. This idea is based on a mathematical distinction which at present does not appear to be widely appreciated.

A Mathematical Distinction

An elementary fact is that every vector is characterized by a direction and magnitude. Here, we propose that it is also characterized by a third property, likely ignored so far because it is very subtle and in most contexts it has no appreciable relevance.

Definition: Intrinsic Dimensionality
The number of orthogonal components into which a vector can be decomposed in the highest-dimensional space in which it actually exists.

It is easy to confuse this with dimensionality of vectors in embedded spaces.

Example 1: the vector (u, v) in R^2, where u and v are along orthogonal directions has intrinsic dimensionality 2.

Example 2: the vector (x, y) in R^2 embedded in R^3 at $z = z_0$ looks like an intrinsically 2-dimensional vector, but due to the embedding has intrinsic dimensionality 3: In R^3 it is (x, y, z_0).

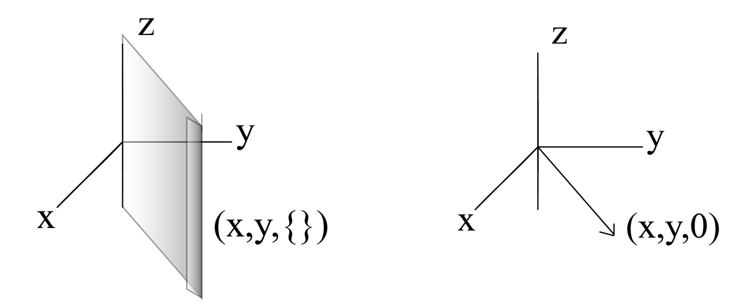
Example 3: *How to represent an intrinsically 2-dimensional vector in 3-space (i.e. when its vector space is not embedded in R^3)?*

Write it as $(x, y, \{\})$, where $\{\}$ indicates the absence of a value, in contrast to a variable which denotes the presence of an unknown value.

Note: This non-standard notation is used to symbolize a concept which appears not to have been articulated previously. It captures the situation in which a vector is represented in a space with a higher dimensionality than the vector's intrinsic dimensionality. The vector does not actually exist in that space, but because the $\{\}$ can be filled in with a value, the representation has the capacity to turn into a vector that actually exists in that space. We will define the representation $(x, y, \{\})$ and as actualizable, the term signifying capacity for becoming actual.

A Graphical Representation

Even more instructive is the graphical comparison of the actualizable vectors representation with actual vectors:



The object on the left represents a vector of intrinsic dimensionality 2 and must be represented as a "superposition" of all actualizable vectors in R^3 with common x and y coordinates. In contrast, even though the object on the right can be thought of as a vector in R^2 where R^2 is embedded in R^3 at $z = 0$, it is by virtue of the embedding a vector with intrinsic dimensionality 3, and must be represented as a single actual vector in R^3. Note that the assignment of a definite value for the third coordinate "collapses" the superposition of actualizable vectors to a single actual one in R^3:

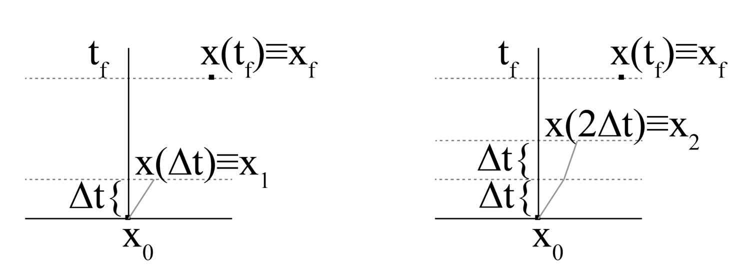
$$(x, y, \{\}) \xrightarrow{assign\ z} (x, y, z) \qquad (1)$$

Application to Spacetime

The above discussion immediately suggests the question of how to represent the object

$$(\{\}, \mathbf{r}) = (\{\}, \mathbf{r}(t_f) - \mathbf{r}(t_0)) \equiv (\{\}, \mathbf{r}_f - \mathbf{r}_0) \qquad (2)$$

in spacetime. This represents two spacetime events $\mathbf{r}(t_0)$ and $\mathbf{r}(t_f)$, not connected by a spacetime worldline but one traceable to a lower-dimensional level, since the intrinsic dimensionality of the vector it represents is 3. For simplicity, consider the $1 + 1$ analog $(\{\}, x_f - x_0)$. Since now instead of the absence of a single coordinate value we have the absence of a worldline, we implement the above superposition representation by applying it to each point on the interval between t_0 and t_f. Subdivide it into N small increments Δt, define $x(n\Delta t) \equiv x_n$ and symbolize the contribution of each path from x_{n-1} to x_n by $A(x_n, x_{n-1}, \Delta t) \equiv A_n$, integrating over all possible values for x_n. The first two steps from x_0 to x_2 for a sample path are shown below:



The superposition from x_0 to x_1 yields $\int_{-\infty}^{\infty} dx_1 A_1$ and from x_0 to x_2 it yields $\int_{-\infty}^{\infty} dx_1 A_1 \int_{-\infty}^{\infty} dx_2 A_2$. To $x_f \equiv x_N$ and taking the limit $N \to \infty$, $\Delta t \to 0$ this becomes

$$\lim_{N\to\infty,\Delta t\to 0} A_N \prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dx_n A_n \qquad (3)$$

Now, if we identify $A(x_n, x_{n-1}, \Delta t)$ with the free particle probability amplitude $\langle x_n \,|\, x_{n-1}\rangle$, we recover to first order the Feynman Path Integral in 1+1 dimensions!

Proof

Substituting $\langle x_n \,|\, x_{n-1}\rangle = \langle x_n|\, \hat{U}(\Delta t)\, |x_{n-1}\rangle$, where $\hat{U}(\Delta t) = e^{-i\frac{\hat{H}_n \Delta t}{\hbar}}$ is the time evolution operator, and then inserting a complete set of N momentum eigenstates $\int_{-\infty}^{\infty} dp_n \,|p_n\rangle \langle p_n| = 1$ gives

$$\lim_{N\to\infty,\Delta t\to 0} \int_{-\infty}^{\infty} dp_N \,\langle x_N \,|\, p_N\rangle \langle p_N|\, e^{-i\frac{\hat{H}_N \Delta t}{\hbar}} \,|x_{N-1}\rangle \prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dx_n \int_{-\infty}^{\infty} dp_n \,\langle x_n \,|\, p_n\rangle \langle p_n|\, e^{-i\frac{\hat{H}_n \Delta t}{\hbar}} \,|x_{n-1}\rangle \qquad (4)$$

substitute the free particle Hamiltonian $\hat{H}_n = \frac{\hat{p}_n^2}{2m}$, expand $\hat{U}(\Delta t)$ to first order, operate to the left, rewrite the first order eigenvalue expansion as an exponential, use $\langle x_n \,|\, p_n\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{x_n p_n}{\hbar}}$, and collect the coefficients in the front

$$\lim_{N\to\infty,\Delta t\to 0} \left(\frac{1}{2\pi\hbar}\right)^N \int_{-\infty}^{\infty} dp_N e^{-i\frac{p_N^2 \Delta t}{2m\hbar}} e^{i\frac{p_N}{\hbar}(x_N - x_{N-1})} \prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dx_n \int_{-\infty}^{\infty} dp_n e^{-i\frac{p_n^2 \Delta t}{2m\hbar}} e^{i\frac{p_n}{\hbar}(x_n - x_{n-1})} \qquad (5)$$

use $e^{u_N} \prod_{n=1}^{N-1} e^{u_n} = \exp\left\{\sum_{n=1}^{N} u_n\right\}$ to collect complex exponentials, then complete the square in the exponents

$$\lim_{N\to\infty,\Delta t\to 0} \left(\frac{1}{2\pi\hbar}\right)^N \left[\prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dx_n\right] \left[\prod_{n=1}^{N} \int_{-\infty}^{\infty} dp_n\right] \exp\left\{\sum_{n=1}^{N} \frac{-i\Delta t}{2m\hbar} \left\langle \left(p_n - \frac{m}{\Delta t}(x_n - x_{n-1})\right)^2 - \left(\frac{m}{\Delta t}(x_n - x_{n-1})\right)^2 \right\rangle\right\} \qquad (6)$$

where the square brackets are meant to indicate that the product goes to N for the momentum integrals but only to $N - 1$ for the position integrals. Using the Gaussian Formula $\int_{-\infty}^{\infty} e^{-au^2} = \sqrt{\frac{\pi}{a}}, a \in \mathbb{C}$, the momentum integrals can be evaluated one by one, and the general formula for N integrals can be found by induction, giving

$$\lim_{N\to\infty,\Delta t\to 0} \left(\frac{1}{2\pi\hbar}\right)^N \sqrt{\pi\frac{2m\hbar}{i\Delta t}}^N \left[\prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dx_n\right] \exp\left\{\sum_{n=1}^{N} \frac{-i\Delta t}{2m\hbar} \left\langle \frac{-m^2}{\Delta t^2}(x_n - x_{n-1})^2 \right\rangle\right\} \qquad (7)$$

which can be rewritten as

$$\lim_{N\to\infty,\Delta t\to 0} \sqrt{\frac{m}{i2\pi\hbar\Delta t}}^N \left[\prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dx_n\right] \exp\left\{\frac{i}{\hbar} \sum_{n=1}^{N} \frac{m}{2} \left(\frac{x_n - x_{n-1}}{\Delta t}\right)^2 \Delta t\right\} \qquad (8)$$

In the limit $N \to \infty, \Delta t \to 0$, the sum in the exponent becomes an integral and its argument a derivative, yielding

$$\lim_{N\to\infty,\Delta t\to 0} \sqrt{\frac{m}{i2\pi\hbar\Delta t}}^N \left[\prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dx_n\right] \exp\left\{\frac{i}{\hbar} \int_{t_0}^{t_f} \frac{mv^2}{2} dt\right\} \equiv \int_{x_0}^{x_f} \mathscr{D}[x(t)] e^{i\frac{S[x(t)]}{\hbar}} \qquad (9)$$

where the last term is the standard symbolic expression for the Feynman path integral in 1+1 dimensions and $S[x(t)] = \int dt\, mv^2/2$ is the classical free particle action.

Conclusion

Superposition and collapse are central features of QM, and while its relation to the path integral is well understood [1], it is so only within the context of poorly 'understood' empirical facts (e.g. the double slit experiment). Here we used theoretical arguments to help elucidate the origin of the Feynman path integral suggesting that the path integral is the spacetime manifestation of objects of lower intrinsic dimensionality than spacetime observers until they give rise to the emergence of actual spacetime objects upon 'Measurement'. A more complete theoretical treatment using the same ideas that does not rely *a priori* on quantum mechanical probability amplitudes has been proposed by this author [2].

References

[1] R. P. Feynman and A. R. Hibbs, "Quantum Mechanics and Path Integrals" (Emended Edition), edited by D. F. Styer, Dover Publications, Mineola, (2010)
[2] A. Nikkhah Shirazi "A Novel Approach For 'Making Sense' Out of the Copenhagen Interpretation" to be published in the AIP proceedings of the *Quantum Theory: Reconsideration of Foundations 6* conference in Växjö, Sweden (2012)