Is Quantum Gravity a Linear Theory?

Version for publishing in
http://www.arxiv.org

Andrey Simonian, *RUSSIA*
Peter Svirov, *RUSSIA*

Abstract

The main idea of our quantum gravity model is following: in usual quantum mechanics we can represent any state of the many-particle system **as sum over products of one-particle functions, generally, a path integral**

$$\Psi = \sum_{u\ \dots\ v} \Psi_u^{(1)}(tx_1) \cdot \dots \cdot \Psi_v^{(N)}(tx_N) \cdot I_{u\ \dots v}$$

In quantum gravity we also represent each state as such superposition. In the last part we try to make numerical calculations using our quantum gravity theory.

In the case of interaction with arbitrary electromagnetic and gravitational field the wave function is similar:

$$\Psi = \sum_{u\ \dots\ v,\, a, b} \Psi_u^{(1)}(tx_1) \cdot \dots \cdot \Psi_v^{(N)}(tx_N) \cdot$$
$$\cdot \Psi_a^{em}\left(t, \{\vec{q}(\vec{x})\}\right) \Psi_b^{G}\left(t, \{g_{\alpha\beta}(\vec{x})\}\right) \cdot I_{u\ \dots v,\, a, b}$$

Contents

1.1 Quantization of Einstein equation: case of free gravitational field.

Suppose that we have arbitrary frame with time and coordinate parameters. We postulate that the wave function of arbitrary gravitational field is function of the metric matrix functions $g_{\alpha\beta}$:

$$\Psi = \Psi(t, \{g_{\alpha\beta}(\vec{x})\}) \tag{1.1.0}$$

The change of metric by time is in the time parameter which is in wave the function. How this wave function changes from one frame to another - see Part 5. We will receive in the end the invariant equation

$$i\hbar \frac{\partial \Psi}{\partial w} = g_{\gamma\eta} w^{\gamma} \hat{p}^{\eta} \Psi \tag{1.1.0`}$$

The final result will not contain the Fourier transformation of fields which will greatly help in making the theory of quantum fields simple, relativistic invariant, without divergences which usually appear. Now let's try to calculate action, Lagrangian and Hamiltonian in the arbitrary metric in classic case:

$$S = \frac{c^3}{16\pi G} \int d\Omega R = \frac{c^3}{16\pi G} \int dt d\vec{x} \sqrt{-\det g}\, R = -\int dt L, \tag{1.1.1}$$

where R is a scalar curvature,

$$L = -\int d\vec{x} \frac{c^3}{16\pi G} \sqrt{-\det g}\, R = \int d\vec{x}\, \Lambda , \quad \Lambda = -\frac{c^3}{16\pi G} \sqrt{-\det g}\, R \tag{1.1.2}$$

The momentum density

$$p_{\mu\nu} = \frac{\partial \Lambda}{\partial \dot{g}_{\mu\nu}} \tag{1.1.3}$$

The Hamiltonian density

$$\Lambda_H = \dot{g}_{\mu\nu} p_{\mu\nu} - \Lambda \tag{1.1.4}$$

The Hamiltonian in classical and quantum is calculated using easy rule

$$\hat{H} = \int d\vec{x}\, \hat{\Lambda}_H \tag{1.1.5}$$

We want to make the quantum equation relativistic invariant:

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi , \tag{1.1.6}$$

We see here that time stands out in contradiction to the relativistic equality of time and space. Let's fix it: let's count the derivative not only by time but by the arbitrary 4-vector v^{α} :

$$i\hbar \frac{\partial \Psi}{\partial w} = i\hbar w^{\gamma} \frac{\partial \Psi}{\partial x^{\gamma}} = w^0 \hat{H} \Psi - \sum_{\bar{\gamma}=1...3} w^{\bar{\gamma}} \hat{p}^{\bar{\gamma}} \Psi = g^L_{\gamma\eta} w^{\gamma} \hat{p}^{\eta} \Psi , \tag{1.1.7}$$

where $g^{L}_{\gamma\eta}$ is a Lorenz metric. The generalization to the arbitrary metric of the equation for one particle movement will be

$$i\hbar\frac{\partial\Psi}{\partial w}=g_{\gamma\eta}w^{\gamma}\hat{p}^{\eta}\Psi \tag{1.1.8}$$

The question remains – how the wave function changes from one frame to another – see Part 5.

1.2 Quantization of many particle system + fields. How to make many particle state out of one-particle states.

We postulate that the wave function in the case arbitrary metric is function of the particle coordinates, electromagnetic field and the metric functions $g_{\alpha\beta}$.

Now let's make many-particle state out of one-particle states: if we know how the one-particle states change from one frame to another, then we know how the entire state changes from one frame to another.

In usual quantum mechanics in the path integral approach we make the sum over the sets of Feynman paths $y_1,...,y_N$. Let's formulate usual quantum mechanics in a different way: let's make the sum not over all Feynman paths, but over all possible basis states of the particles (at each moment of time those states are basis states):

$$\Psi=\int D\Psi_1...D\Psi_N\cdot\Psi_1(tx_1)\cdot...\cdot\Psi_N(tx_N)\cdot I\left[\Psi_1...\Psi_N\right] \tag{1.2.1}$$

where $I\left[\Psi_1...\Psi_N\right]=\left(\Psi_1...\Psi_N,\Psi\right)$ can be calculated by continuous von Neumann projection of the state Ψ on $\Psi_1...\Psi_N$, physically it is the probability amplitude of the system characterized by the wave function Ψ to be in state $\Psi_1...\Psi_N$ during the period of time starting from t_0 to t - it is similar to a usual scalar product in quantum mechanics.

Proof of (1.1.1). Using the Dirac bra- and cket- notion, we can rewrite (1.1.1):

$$\left|\Psi\right\rangle=\int D\Psi_1...D\Psi_N\cdot\left|\Psi_1\right\rangle\cdot...\cdot\left|\Psi_N\right\rangle\cdot\left\langle\Psi_1...\Psi_N\right|\Psi\rangle \tag{1.2.2}$$

If we postulate the normalization condition of our functional integral

$$1=\int D\Psi_1...D\Psi_N\cdot\left|\Psi_1\right\rangle\cdot...\cdot\left|\Psi_N\right\rangle\cdot\left\langle\Psi_1...\Psi_N\right|, \tag{1.2.3}$$

then the equation to prove (1.1.1) and the equivalent equation (1.1.2) starts to be obvious:

$$\left|\Psi\right\rangle=\left|\Psi\right\rangle \tag{1.2.4}$$

Equation (1.1.1) proven.

1.3 Quantization of particle in the arbitrary metric.

Let's make quantization of the particle in arbitrary metric. Knowing the classical action, let's calculate Lagrangian and then let's calculate the Hamiltonian in classics and then let's make standard quantization procedure.

$$S = mc^2 \int d\tau + q/c \cdot \int A_\alpha dx^\alpha = mc \int \sqrt{g_{\alpha\beta} v_\alpha v_\beta}\, dt + q/c \cdot \int A_\alpha v_\alpha dt \,,$$

where $v_\alpha = dx_\alpha / dt$ (1.3.1)

So the action of a classic particle in the arbitrary metric and the arbitrary electromagnetic field

$$S = \int \left(mc \sqrt{g_{\alpha\beta} v_\alpha v_\beta} + q/c \cdot A_\alpha v_\alpha \right) dt \tag{1.3.2}$$

From this equation we introduce the Lagrangian:

$$L = mc \sqrt{g_{\alpha\beta} v_\alpha v_\beta} + q/c \cdot A_\alpha v_\alpha \tag{1.3.3}$$

Now we calculate the momentum

$$p_\gamma = \frac{\partial L}{\partial v_\gamma} = mc \frac{g_{\gamma\mu} v_\mu}{\sqrt{g_{\alpha\beta} v_\alpha v_\beta}} + q/c \cdot A_\gamma \tag{1.3.4}$$

Note that the real amount of independent momentum components is not 4, but 3. From (1.3.4) it easily follows that there is a relation between the momentum components:

$$g^{\gamma\nu} \left(p_\gamma - q/c \cdot A_\gamma \right) \left(p_\nu - q/c \cdot A_\nu \right) = m^2 c^2 \tag{1.3.5}$$

Now let's calculate the Hamiltonian of the particle in the arbitrary metric and electromagnetic field from the Lagrangian using the well known equation:

$$\begin{aligned} H &= \sum_{\bar\gamma = 1...3} p_{\bar\gamma} v_{\bar\gamma} - L = mc \frac{g_{\bar\gamma\mu} v_\mu v_{\bar\gamma}}{\sqrt{g_{\alpha\beta} v_\alpha v_\beta}} + q/c \cdot A_{\bar\gamma} v_{\bar\gamma} - mc \sqrt{g_{\alpha\beta} v_\alpha v_\beta} - q/c \cdot A_\alpha v_\alpha = \\ &= mc \frac{g_{\bar\gamma\mu} v_\mu v_{\bar\gamma} - g_{\alpha\beta} v_\alpha v_\beta}{\sqrt{g_{\alpha\beta} v_\alpha v_\beta}} - q/c \cdot A_0 v_0 = mc \frac{-g_{0\mu} v_\mu v_0}{\sqrt{g_{\alpha\beta} v_\alpha v_\beta}} - q/c \cdot A_0 v_0 = -v_0 p_0 = -c p_0 \end{aligned} \tag{1.3.6}$$

Then p_0 has to be calculated from the remaining momentum components using (1.3.5). Note that the reason of the Hamiltonian = energy of the particle having the minus sign is following: we have chosen not the right sign of the Lagrangian (1.3.3) and the action (1.3.1) – usually the minus sign is written is there – see, for example, Landau, Lifshitz, volume 2 - see [1]. Let's rewrite (1.3.5):

$$g^{\gamma\nu} P_\gamma P_\nu = m^2 c^2 \tag{1.3.7}$$

Or, the same,

$$g^{00}P_0P_0+2g^{\bar{\gamma}0}P_{\bar{\gamma}}P_0+g^{\bar{\gamma}\bar{\nu}}P_{\bar{\gamma}}P_{\bar{\nu}}=m^2c^2, \tag{1.3.8}$$

where the bars of the summation indexes means that that this summation index changes not from 0 to 3 as usual, but from 1 to 3. This is a quadratic equation on p_0. If we make the condition of the energy p_0 to be nonnegative, then

$$P_0=\frac{g^{\bar{\gamma}0}P_{\bar{\gamma}}+\sqrt{\left(g^{\bar{\gamma}0}P_{\bar{\gamma}}\right)^2+g^{00}\left(m^2c^2-g^{\bar{\gamma}\bar{\nu}}P_{\bar{\gamma}}P_{\bar{\nu}}\right)}}{g^{00}}, \tag{1.3.9}$$

Or, equivalently,

$$p_0=\frac{g^{\bar{\gamma}0}\left(p_{\bar{\gamma}}-q/c\cdot A_{\bar{\gamma}}\right)+\sqrt{\left(g^{\bar{\gamma}0}\left(p_{\bar{\gamma}}-q/c\cdot A_{\bar{\gamma}}\right)\right)^2+g^{00}\left(m^2c^2-g^{\bar{\gamma}\bar{\nu}}\left(p_{\bar{\gamma}}-q/c\cdot A_{\bar{\gamma}}\right)\left(p_{\bar{\nu}}-q/c\cdot A_{\bar{\nu}}\right)\right)}}{g^{00}}+q/c\cdot A_0 \tag{1.3.10}$$

The Hamiltonian can be easily found from this equation in analogy to (1.3.12), we also will omit the minus sign so that the energy would have it's usual physical meaning:

$$H=\frac{g^{\bar{\gamma}0}\left(cp_{\bar{\gamma}}-qA_{\bar{\gamma}}\right)+\sqrt{\left(g^{\bar{\gamma}0}\left(cp_{\bar{\gamma}}-qA_{\bar{\gamma}}\right)\right)^2+g^{00}\left(m^2c^4-g^{\bar{\gamma}\bar{\nu}}\left(cp_{\bar{\gamma}}-qA_{\bar{\gamma}}\right)\left(cp_{\bar{\nu}}-qA_{\bar{\nu}}\right)\right)}}{g^{00}}+qA_0 \tag{1.3.11}$$

When we make the quantization, we suppose that all physical quantities participating here all operators. Also we substitute all the products of the operators as following:

$$AB\to 1/2\cdot\left(\hat{A}\hat{B}+\hat{B}\hat{A}\right), \tag{1.3.12}$$

$$ABC\to 1/6\cdot\left(\hat{A}\hat{B}\hat{C}+\hat{A}\hat{C}\hat{B}+\hat{B}\hat{A}\hat{C}+\hat{B}\hat{C}\hat{A}+\hat{C}\hat{A}\hat{B}+\hat{C}\hat{B}\hat{A}\right), \tag{1.3.13}$$

…

1.4 The quantization of electromagnetic field in Lorenz metric.

We will show how the electromagnetic field can be quantized, the final result will not contain the Fourier transformation of fields which will greatly help in making the theory of quantum fields simple, relativistic invariant, without divergences which usually appear.

As well known, the action **in Lorenz frame**

$$S = \frac{1}{16\pi c} \int d\Omega F_{\alpha\beta} F^{\alpha\beta} = -\int dt d\vec{x} \frac{\vec{E}^2 - \vec{H}^2}{8\pi} = -\int dt L, \quad (1.4.1)$$

where $F^{\alpha\beta} = \frac{\partial A^{\beta}}{\partial x_{\alpha}} - \frac{\partial A^{\alpha}}{\partial x_{\beta}} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -H_3 & H_2 \\ E_2 & H_3 & 0 & -H_1 \\ E_3 & -H_2 & H_1 & 0 \end{pmatrix}$, $L = \int d\vec{x} \frac{\vec{E}^2 - \vec{H}^2}{8\pi} = \int d\vec{x}\,\Lambda$, $\Lambda = \frac{\vec{E}^2 - \vec{H}^2}{8\pi}$

One of the ways to produce right Hamiltonian = energy $H = \int d\vec{x} \frac{\vec{E}^2 + \vec{H}^2}{8\pi} = \int d\vec{x}\,\Lambda_H$,

$\Lambda_H = \frac{\vec{E}^2 + \vec{H}^2}{8\pi}$ is to introduce the 3-vector field obeying the equation

$$\dot{\vec{q}} = \vec{E} \quad (1.4.2)$$

Then if magnetic field $\vec{H}$ does not depend on $\dot{\vec{q}}$ (as we will see below), then

$$\Lambda_H = \dot{\vec{q}} \frac{\partial \Lambda}{\partial \dot{\vec{q}}} - \Lambda = \frac{\vec{E}^2 + \vec{H}^2}{8\pi} \quad (1.4.3)$$

Obviously, the Hamiltonian will not change if we multiply the coordinate by a constant:

$$\dot{\vec{q}} = -c\vec{E} = \frac{\partial \vec{A}}{\partial t} + c\,grad\,\varphi \quad (1.4.4)$$

Now let's take an integral by time:

$$\vec{q} = \vec{A} + \int dt \cdot c\,grad\,\varphi \quad (1.4.5)$$

Taking into account that $rot\,grad\,\varphi = 0$, we can find the magnetic field

$$\vec{H} = rot\,\vec{q} = rot\,\vec{A} \quad (1.4.6)$$

We have calculated the magnetic and electric field using our new coordinate $\vec{q}$ - see (1.4.2) and (1.4.6). Now we can calculate the action, Lagrangian and Hamiltonian:

$$S = \frac{1}{16\pi c} \int d\Omega F_{\alpha\beta} F^{\alpha\beta} = -\int dt d\vec{x} \frac{(\dot{\vec{q}}/c)^2 - (rot\,\vec{q})^2}{8\pi} = -\int dt L, \quad (1.4.7)$$

where $F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} = \begin{pmatrix} 0 & -\dot{q}_1/c & -\dot{q}_2/c & -\dot{q}_3/c \\ \dot{q}_1/c & 0 & -(rot\,\vec{q})_3 & (rot\,\vec{q})_2 \\ \dot{q}_2/c & (rot\,\vec{q})_3 & 0 & -(rot\,\vec{q})_1 \\ \dot{q}_3/c & -(rot\,\vec{q})_2 & (rot\,\vec{q})_1 & 0 \end{pmatrix}$,

$$L = \int d\vec{x}\, \frac{(\dot{\vec{q}}/c)^2 - (rot\vec{q})^2}{8\pi} = \int d\vec{x}\, \Lambda,\ \Lambda = \frac{(\dot{\vec{q}}/c)^2 - (rot\vec{q})^2}{8\pi}$$

Hamiltonian = energy $H = \int d\vec{x}\, \frac{(\dot{\vec{q}}/c)^2 + (rot\vec{q})^2}{8\pi} = \int d\vec{x}\, \Lambda_H$,

$$\Lambda_H = \frac{(\dot{\vec{q}}/c)^2 + (rot\vec{q})^2}{8\pi} \qquad (1.4.8)$$

The momentum density

$$\vec{p} = \frac{\partial \Lambda}{\partial \dot{\vec{q}}} = \frac{\dot{\vec{q}}}{4\pi c^2} = -\frac{\vec{E}}{4\pi c} \qquad (1.4.9)$$

It means that Hamiltonian

$$H = \int d\vec{x}\, \frac{(4\pi c\vec{p})^2 + (rot\vec{q})^2}{8\pi}, \qquad (1.4.10)$$

Now we make the quantization: the state if the free field depends on time and the “3-coordinate” $\vec{q}(\vec{x})$ the same way as the state of the free particle depends on time and the 3-coordinate $\vec{x}$:

$$\Psi = \Psi(t, \{\vec{q}(\vec{x})\}) \qquad (1.4.11)$$

The probability for the field to have the value $\vec{q}(\vec{x})$ and change in the “small interval” with volume $D\{\vec{q}(\vec{x})\}$ at moment of time t is equal to

$$dP = D\{\vec{q}(\vec{x})\} |\Psi(t, \{\vec{q}(\vec{x})\})|^2 \qquad (1.4.12)$$

It means that there is a probability normalization condition:

$$1 = \int D\{\vec{q}(\vec{x})\} \Psi^*(t, \{\vec{q}(\vec{x})\}) \Psi(t, \{\vec{q}(\vec{x})\}) \qquad (1.4.13)$$

The mean value of field $\vec{q}(\vec{x})$ at the moment of time t_0 and coordinate $\vec{x}_0$ is equal to

$$< \vec{q}(t_0, \vec{x}_0) > = \int D\{\vec{q}(\vec{x})\} \Psi^*(t_0, \{\vec{q}(\vec{x})\}) \vec{q}(\vec{x}_0) \Psi(t_0, \{\vec{q}(\vec{x})\}) \qquad (1.4.14)$$

The question is to calculate the electric field $< \vec{E}(t_0, \vec{x}_0) > = < -\dot{\vec{q}}(t_0, \vec{x}_0)/c >$ - see (1.4.4). Using the appropriate theorem from quantum mechanics showing how to calculate the derivative, we can write: $< \dot{\vec{q}}(t_0, \vec{x}_0) > = < -\frac{i}{\hbar c} [\hat{H}, \hat{\vec{q}}(t_0, \vec{x}_0)] >$. But the problem is that we don’t know the Hamiltonian $\hat{H}$. Let’s calculate it.

The action, Lagrangian and Hamiltonian are also operators in quantum mechanics:

$$\hat{S}=\frac{1}{16\pi c}\int d\Omega \hat{F}_{\alpha\beta}\hat{F}^{\alpha\beta}=-\int dt d\vec{x}\,\frac{(\dot{\hat{\vec{q}}}/c)^2-\left(rot\,\hat{\vec{q}}\right)^2}{8\pi}=-\int dt\hat{L}\,, \qquad (1.4.15)$$

where $\hat{F}_{\alpha\beta}=\frac{\partial\hat{A}_\beta}{\partial x^\alpha}-\frac{\partial\hat{A}_\alpha}{\partial x^\beta}=\begin{pmatrix} 0 & -\dot{\hat{q}}_1/c & -\dot{\hat{q}}_2/c & -\dot{\hat{q}}_3/c \\ \dot{\hat{q}}_1/c & 0 & -\left(rot\,\hat{\vec{q}}\right)_3 & \left(rot\,\hat{\vec{q}}\right)_2 \\ \dot{\hat{q}}_2/c & \left(rot\,\hat{\vec{q}}\right)_3 & 0 & -\left(rot\,\hat{\vec{q}}\right)_1 \\ \dot{\hat{q}}_3/c & -\left(rot\,\hat{\vec{q}}\right)_2 & \left(rot\,\hat{\vec{q}}\right)_1 & 0 \end{pmatrix}$,

$$\hat{L}=\int d\vec{x}\,\frac{(\dot{\hat{\vec{q}}}/c)^2-\left(rot\hat{\vec{q}}\right)^2}{8\pi}=\int d\vec{x}\,\hat{\Lambda}\,,\ \hat{\Lambda}=\frac{(\dot{\hat{\vec{q}}}/c)^2-\left(rot\hat{\vec{q}}\right)^2}{8\pi}$$

Hamiltonian = energy

$$\hat{H}=\int d\vec{x}\,\frac{(\dot{\hat{\vec{q}}}/c)^2+\left(rot\,\hat{\vec{q}}\right)^2}{8\pi}=\int d\vec{x}\,\hat{\Lambda}_H\,,\ \hat{\Lambda}_H=\frac{(\dot{\hat{\vec{q}}}/c)^2+\left(rot\hat{\vec{q}}\right)^2}{8\pi} \qquad (1.4.16)$$

The momentum

$$\hat{\vec{p}}=\frac{\dot{\hat{\vec{q}}}}{4\pi c^2}=-\frac{\hat{\vec{E}}}{4\pi c} \qquad (1.4.17)$$

It means that Hamiltonian

$$H=\int d\vec{x}\,\frac{(4\pi c\hat{\vec{p}})^2+\left(rot\hat{\vec{q}}\right)^2}{8\pi}\,, \qquad (1.4.18)$$

We still did not postulate the momentum operator. Let’s state the following:

$$\left(\hat{\vec{p}}(\vec{x}_0)\Psi\right)\left(\{\vec{q}(\vec{x})\}\right)=-i\hbar\,\frac{\partial\Psi\left(\{\vec{q}(\vec{x})\}\right)}{\partial\vec{q}(\vec{x}_0)} \qquad (1.4.19)$$

where $-i\hbar\,\frac{\partial}{\partial\vec{q}(\vec{x}_0)}$ is a derivative by coordinate with number $\vec{x}_0$ where $\vec{x}_0$ is the element of our usual 3-dimensional vector space over real numbers, in analogy with usual momentum operator $\hat{p}_\alpha=-i\hbar\,\frac{\partial}{\partial q^\alpha}$ with derivative by coordinate number α where α takes values $1, 2, 3$. In usual quantum mechanics we have commutation relations $\frac{i}{\hbar}[\hat{p}_\alpha,\hat{x}_\beta]=\delta_{\alpha\beta}$.

Let’s prove that here we have similar commutation relations

$$\frac{i}{\hbar}[\hat{p}_\alpha(\vec{x}_0),\hat{q}_\beta(\vec{x}_1)]=\delta_{\alpha\beta}\delta(\vec{x}_0-\vec{x}_1) \qquad (1.4.20)$$

Indeed,

$$\frac{i}{\hbar}\left[\hat{p}_\alpha(\vec{x}_0),\hat{q}_\beta(\vec{x}_1)\right]\Psi\left(\{\vec{q}(\vec{x})\}\right)=\frac{\partial\left(q_\beta(\vec{x}_1)\Psi\left(\{\vec{q}(\vec{x})\}\right)\right)}{\partial q_\alpha(\vec{x}_0)}-q_\beta(\vec{x}_1)\,\frac{\partial\Psi\left(\{\vec{q}(\vec{x})\}\right)}{\partial q_\alpha(\vec{x}_0)}=$$
$$=\frac{\partial q_\beta(\vec{x}_1)}{\partial q_\alpha(\vec{x}_0)}\Psi\left(\{\vec{q}(\vec{x})\}\right)=\delta_{\alpha\beta}\,\delta(\vec{x}_0-\vec{x}_1)\Psi\left(\{\vec{q}(\vec{x})\}\right) \tag{1.4.21}$$

We use the fact

$$\frac{\partial q_\beta(\vec{x}_1)}{\partial q_\alpha(\vec{x}_0)}=\delta_{\alpha\beta}\,\delta(\vec{x}_0-\vec{x}_1) \tag{1.4.22}$$

which is similar to usual relations $\frac{\partial x_\beta}{\partial x^\alpha}=\delta_{\alpha\beta}$.

From (1.4.21) the equation which we were proving immediately follows:

$$\frac{i}{\hbar}\left[\hat{p}_\alpha(\vec{x}_0),\hat{q}_\beta(\vec{x}_1)\right]=\delta_{\alpha\beta}\,\delta(\vec{x}_0-\vec{x}_1) \tag{1.4.23}$$

Now we can write the Hamiltonian of free electromagnetic field in the Lorenz frame

$$H=\int d\vec{x}\,\frac{1}{8\pi}\left(-\hbar^2\frac{4\pi c\partial^2}{\partial\,\vec{q}(\vec{x})^2}+\left(rot\hat{\vec{q}}(\vec{x})\right)^2\right), \tag{1.4.24}$$

1.5 The quantization of electromagnetic field in arbitrary metric.

Let's postulate that the wave function in the case arbitrary metric is function of the particle coordinates, function of electromagnetic field coordinates and the metric components $g_{\alpha\beta}$.

The final result also as in previous part will not contain the Fourier transformation of fields which will greatly help in making the theory of quantum fields simple, relativistic invariant, without divergences which usually appear.

Now let's try to calculate action, Lagrangian and Hamiltonian in the arbitrary metric in classic case:

$$S = \frac{1}{16\pi c}\int d\Omega F_{\alpha\beta}F^{\alpha\beta} = \frac{1}{16\pi c}\int d\Omega F_{\alpha\beta}F_{\mu\nu}\, g^{\alpha\mu}g^{\beta\nu} = \\ = \frac{1}{16\pi}\int dt d\vec{x}\sqrt{-\det g}\, F_{\alpha\beta}F_{\mu\nu}\, g^{\alpha\mu}g^{\beta\nu} = -\int dtL \quad , \tag{1.5.1}$$

where $F_{\alpha\beta} = \dfrac{\partial A_\beta}{\partial x^\alpha} - \dfrac{\partial A_\alpha}{\partial x^\beta} = \begin{pmatrix} 0 & -\dot{q}_1/c & -\dot{q}_2/c & -\dot{q}_3/c \\ \dot{q}_1/c & 0 & -(rot\,\vec{q})_3 & (rot\,\vec{q})_2 \\ \dot{q}_2/c & (rot\,\vec{q})_3 & 0 & -(rot\,\vec{q})_1 \\ \dot{q}_3/c & -(rot\,\vec{q})_2 & (rot\,\vec{q})_1 & 0 \end{pmatrix}$,

$$L = -\int d\vec{x}\,\frac{\sqrt{-\det g}\,F_{\alpha\beta}F_{\mu\nu}\,g^{\alpha\mu}g^{\beta\nu}}{16\pi} = \int d\vec{x}\,\Lambda,\ \Lambda = -\frac{\sqrt{-\det g}\,F_{\alpha\beta}F_{\mu\nu}\,g^{\alpha\mu}g^{\beta\nu}}{16\pi}$$

The momentum density

$$p_\gamma = \frac{\partial\Lambda}{\partial\dot{\vec{q}}_\gamma} = -\frac{\partial}{\partial\dot{\vec{q}}_\gamma}\frac{\sqrt{-\det g}\,F_{\alpha\beta}F_{\mu\nu}\,g^{\alpha\mu}g^{\beta\nu}}{16\pi} = -\frac{\partial}{\partial\dot{\vec{q}}_\gamma}\frac{\sqrt{-\det g}\,F_{\alpha\beta}F_{\mu\nu}\,g^{\alpha\mu}g^{\beta\nu}}{16\pi} = \\ = -\frac{\partial}{\partial\dot{\vec{q}}_\gamma}\frac{\sqrt{-\det g}\,F_{\alpha\beta}F_{\mu\nu}\,g^{\alpha\mu}g^{\beta\nu}}{16\pi} = -\frac{\sqrt{-\det g}}{16\pi}\frac{\partial}{\partial\dot{\vec{q}}_\gamma}\left(F_{\alpha\beta}F_{\mu\nu}\right)g^{\alpha\mu}g^{\beta\nu} = \\ = -\frac{\sqrt{-\det g}}{16\pi}\frac{\partial}{\partial\dot{\vec{q}}_\gamma}\left(\frac{\partial F_{\alpha\beta}}{\partial\dot{\vec{q}}_\gamma}F_{\mu\nu} + F_{\alpha\beta}\frac{\partial F_{\mu\nu}}{\partial\dot{\vec{q}}_\gamma}\right)g^{\alpha\mu}g^{\beta\nu} \tag{1.5.2}$$

Note that according to (1.5.25) $\dfrac{\partial F_{\alpha\beta}}{\partial\dot{\vec{q}}_\gamma}$ may have no-zero value only if $\alpha \neq 0 \;\; \beta \neq 0$. It means that

$$p_\gamma = -\frac{\sqrt{-\det g}}{16\pi}\frac{\partial F_{0\beta}}{\partial \dot{\vec{q}}_\gamma} F_{\mu\nu} g^{0\mu} g^{\beta\nu} - \frac{\sqrt{-\det g}}{16\pi}\frac{\partial F_{\alpha 0}}{\partial \dot{\vec{q}}_\gamma} F_{\mu\nu} g^{\alpha\mu} g^{0\nu} - \\ - \frac{\sqrt{-\det g}}{16\pi} F_{\alpha\beta}\frac{\partial F_{0\nu}}{\partial \dot{\vec{q}}_\gamma} g^{\alpha 0} g^{\beta\nu} - \frac{\sqrt{-\det g}}{16\pi} F_{\alpha\beta}\frac{\partial F_{\mu 0}}{\partial \dot{\vec{q}}_\gamma} g^{\alpha\mu} g^{\beta 0} \tag{1.5.3}$$

Again using (1.5.25) we have:

$$p_\gamma = \frac{\sqrt{-\det g}}{16\pi}\delta_{\gamma\beta}/c\cdot F_{\mu\nu}\, g^{0\mu} g^{\beta\nu} - \frac{\sqrt{-\det g}}{16\pi}\delta_{\alpha\gamma}/c\cdot F_{\mu\nu}\, g^{\alpha\mu} g^{0\nu} - \\ + \frac{\sqrt{-\det g}}{16\pi} F_{\alpha\beta}\,\delta_{\gamma\nu}/c\cdot g^{\alpha 0} g^{\beta\nu} - \frac{\sqrt{-\det g}}{16\pi} F_{\alpha\beta}\,\delta_{\mu\gamma}/c\cdot g^{\alpha\mu} g^{\beta 0} = \\ = \frac{\sqrt{-\det g}}{16\pi}\delta_{\gamma\beta}/c\cdot F^{0\beta} - \frac{\sqrt{-\det g}}{16\pi}\delta_{\alpha\gamma}/c\cdot F^{\alpha 0} - \\ + \frac{\sqrt{-\det g}}{16\pi} F^{0\nu}\cdot\delta_{\gamma\nu}/c - \frac{\sqrt{-\det g}}{16\pi} F_{\mu 0}\cdot\,\delta_{\mu\gamma}/c = \frac{\sqrt{-\det g}}{8\pi c}\left(F^{0\gamma} - F^{\gamma 0}\right) \tag{1.5.4}$$

We have found the momentum density of electromagnetic field in the arbitrary metric

$$p_\gamma = \frac{\sqrt{-\det g}}{8\pi c}\left(F^{0\gamma} - F^{\gamma 0}\right) \tag{1.5.5}$$

Let’s note that if $F_{\alpha\beta} = -F_{\beta\alpha}$, then

$$F^{\alpha\beta} = -F^{\beta\alpha} \tag{1.5.6}$$

It means that we can introduce notations.

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -H^3 & H^2 \\ E^2 & H^3 & 0 & -H^1 \\ E^3 & -H^2 & H^1 & 0 \end{pmatrix} \tag{1.5.7}$$

Note that E^γ and H^γ are not tensors. It follows from (1.5.5) and (1.5.7)

$$p_\gamma = -\frac{\sqrt{-\det g}}{4\pi c} E^\gamma \tag{1.5.8}$$

This equation is similar to (1.5.17).

Hamiltonian density obeys equation $H = \int d\vec{x}\,\Lambda_H$,

$$\Lambda_H = p_\gamma \dot{\vec{q}}_\gamma - \Lambda = \frac{\sqrt{-\det g}}{4\pi c} E^\gamma\cdot cE_\gamma - \sqrt{-\det g}\,\frac{E_\gamma E^\gamma - H_\gamma H^\gamma}{8\pi} \tag{1.5.9}$$

Or, the same

$$\Lambda_H = \sqrt{-\det g}\,\frac{E_\gamma E^\gamma + H_\gamma H^\gamma}{8\pi} \tag{1.5.10}$$

The Hamiltonian of the electromagnetic field in the arbitrary metric

$$H = \int d\vec{x}\sqrt{-\det g}\,\frac{E_\gamma E^\gamma + H_\gamma H^\gamma}{8\pi}, \tag{1.5.11}$$

where we take the integral over arbitrary hypersurface.

In quantum case everything is very similar:

$$\hat{F}^{\alpha\beta} = \begin{pmatrix} 0 & -\hat{E}^1 & -\hat{E}^2 & -\hat{E}^3 \\ \hat{E}^1 & 0 & -\hat{H}^3 & \hat{H}^2 \\ \hat{E}^2 & \hat{H}^3 & 0 & -\hat{H}^1 \\ \hat{E}^3 & -\hat{H}^2 & \hat{H}^1 & 0 \end{pmatrix} \tag{1.5.12}$$

Also

$$\hat{p}_\gamma = -\frac{\sqrt{-\det \hat{g}}}{4\pi c}\hat{E}^\gamma, \tag{1.5.13}$$

where $\hat{g}_{\alpha\beta}$ are the operators which multiply the wave function by the metric matrix functions similar to Schrödinger case, where the coordinate operator is the operator, which multiplies the wave function by a coordinate. The equation (1.5.13) is very similar to (1.5.17).

Hamiltonian density

$$\hat{\Lambda}_H = \sqrt{-\det\hat{g}}\,\frac{\hat{E}_\gamma \hat{E}^\gamma + \hat{H}_\gamma \hat{H}^\gamma}{8\pi} \tag{1.5.14}$$

The Hamiltonian of the electromagnetic field in the arbitrary metric

$$\hat{H} = \int d\vec{x}\sqrt{-\det\hat{g}}\,\frac{\hat{E}_\gamma \hat{E}^\gamma + \hat{H}_\gamma \hat{H}^\gamma}{8\pi}, \tag{1.5.15}$$

where we take the integral over arbitrary hypersurface.

1.6 **About the relativistic invariance.**

Suppose that we have one particle, it's quantum equation, and we want to make it relativistic invariant:

$$i\hbar\frac{\partial\Psi}{\partial t}=\hat{H}\Psi\,, \tag{1.6.1}$$

We see here that time stands out in contradiction to the relativistic equality of time and space. Let's fix it: let's count the derivative not only by time but by the arbitrary 4-vector v^{α}:

$$i\hbar\frac{\partial\Psi}{\partial w}=i\hbar w^{\gamma}\frac{\partial\Psi}{\partial x^{\gamma}}=w^{0}\hat{H}\Psi-\sum_{\bar{\gamma}=1...3}w^{\bar{\gamma}}\hat{p}^{\bar{\gamma}}\Psi=g^{L}_{\gamma\eta}w^{\gamma}\hat{p}^{\eta}\Psi\,, \tag{1.6.2}$$

where $g^{L}_{\gamma\eta}$ is a Lorenz metric.

The generalization to the arbitrary metric of the equation for one particle movement will be

$$i\hbar\frac{\partial\Psi}{\partial w}=g_{\gamma\eta}w^{\gamma}\hat{p}^{\eta}\Psi \tag{1.6.3}$$

The question remains – how the wave function changes from one frame to another.

Let's note that the wave function is not invariant. One of the reasons is that the probability density $\rho(t_{\alpha},\vec{x}_{\alpha})=\left|\Psi(t_{\alpha},\vec{x}_{\alpha})\right|^{2}$ is not the invariant. In classical mechanics it transits from one frame to another like the 0-component of the 4-density vector. But the classical proof of vector transformation law of this value does not hold in quantum mechanics. So we don't know if 4-density transforms like a 4-vector or not in quantum mechanics.

Let's start with a simple case and consider one free particle. Let's suppose that the wave evolution law is following:

$$i\hbar\frac{\partial\Psi}{\partial t}=\hat{H}\Psi\,, \tag{1.6.4}$$

where $\hat{H}=\sqrt{\hat{\vec{p}}^{2}c^{2}+m^{2}c^{4}}$, $\hat{\vec{p}}=-i\hbar\partial/\partial\vec{x}$.

As well known solution of this equation is a harmonic wave:

$$\Psi=e^{i/\hbar\cdot\left(\vec{p}(\vec{x}-\vec{x}_{0})-E(t-t_{0})\right)}, \tag{1.6.5}$$

where $E=\sqrt{\vec{p}^{2}c^{2}+m^{2}c^{4}}$.

If we set the initial condition as the delta-function,

$$\Psi(t_{0})=\delta(\vec{x}-\vec{x}_{0})=\frac{1}{(2\pi)^{3}\hbar}\int d\vec{p}\,e^{i/\hbar\cdot\vec{p}(\vec{x}-\vec{x}_{0})}, \tag{1.6.6}$$

then the evolution law of this wave function is called propagator which shows how the singular delta-function change as time changes:

$$K(t,\vec{x},t_0,\vec{x}_0)=\frac{1}{(2\pi)^3\hbar}\int d\vec{p}\,e^{i/\hbar\left(\vec{p}(\vec{x}-\vec{x}_0)-E(t-t_0)\right)} \qquad (1.6.7)$$

If we try to write it in Lorenz invariant form, then

$$K(t,\vec{x},t_0,\vec{x}_0)=\int dp^1dp^2dp^3\,e^{-i/\hbar\cdot g^L_{\alpha\beta}p^\alpha\Delta x^\beta}, \qquad (1.6.8)$$

$$\begin{cases} g^L_{\alpha\beta}p^\alpha p^\beta=m^2c^4 \\ p^0=E/c>0 \end{cases}$$

where the Lorenz metric is $g^L_{\alpha\beta}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$.

We see that the propagator $K(t,\vec{x},t_0,\vec{x}_0)$ is seen not to be invariant. It can be understood, the reason is that the wave function is not invariant – see beginning of this Part 1.3.

Let's try to find how one-particle wave function transits from one frame to another. Let's postulate the following axiom concerning the coordinate:

Axiom M1. If in one frame the coordinates are 100% known (the wave function is a delta-function), then in the other frame the coordinates are 100% known (in the new frame the wave function is a delta-function multiplied by a coefficient).

$$F\left[\Psi=\delta(\tilde{x}^\gamma-x^\gamma)\right]\quad \text{is equal to}\quad \Psi'=a(\tilde{x}'^\gamma)\delta(\tilde{x}'^\gamma-x'^\gamma) \qquad (1.6.9)$$

Let's also postulate the following statement consulting general momentum:

Axiom M2. If in one frame the general momentum was with 100% known (the wave function was a harmonic function in case of linear transformations), then in the other frame the general momentum is also with 100% known (the wave function is a harmonic function multiplied by a coefficient in the new frame).

$$F\left[\Psi=e^{i\vec{p}_\gamma\vec{x}^\gamma/\hbar}\right]\ \text{in case of linear trasform is equal to}\ \Psi'=b(\vec{p}'_\gamma)e^{i\vec{p}'_\gamma\vec{x}'^\gamma/\hbar} \qquad (1.6.10)$$

Let's postulate the superposition principle:

Axiom M3. If in one frame wave function was a sum (superposition) of two wave functions, then in the new frame the wave function will be superposition of the same states but taken in the new frame.

If in one frame wave function was a product of a wave function and a constant, then in the new frame the wave function will be also the product of the same state but taken in the new frame and the same constant.

If in one frame wave function was an integral over a wave function and a coefficient, then in the new frame the wave function will be also the integral of the same state but taken in the new frame and the same coefficient.

$$F[\Psi_1 + \Psi_2] = F[\Psi_1] + F[\Psi_2] \qquad (1.6.11)$$

$$F[\gamma\Psi] = \gamma F[\Psi] \qquad (1.6.12)$$

$$F\left[\int dx^\gamma f(x^\gamma)\Psi(x^\gamma)\right] = \int dx^\gamma f(x^\gamma) F[\Psi(x^\gamma)] \qquad (1.6.13)$$

Axiom M4. The probability element remains the same if we transit from one frame to another

$$dV|\Psi|^2 = dV'|\Psi'|^2 \qquad (1.6.14)$$

dV and dV' are the volume elements in the coordinate space (excluding time). The relation between them can be calculated as the relation between the volumes formed by the infinitesimal coordinate basis vectors:

$$\frac{dV'}{dV} = \frac{dV'(e'_1, e'_2, e'_3)}{dV(e_1, e_2, e_3)} \qquad (1.6.15)$$

The wave function transition law from one frame to another may be calculated from those axioms. From Axiom M1 and M4 we conclude:

$$F[\Psi(x^\gamma)] = F\left[\int d\tilde{x}^\gamma \delta(\tilde{x}^\gamma - x^\gamma)\Psi(\tilde{x}^\gamma)\right] = \int d\tilde{x}^\gamma \Psi(\tilde{x}^\gamma) F[\delta(\tilde{x}^\gamma - x^\gamma)] =$$
$$= \int d\tilde{x}^\gamma\, \Psi(\tilde{x}^\gamma) a(\tilde{x}'^\gamma)\delta(\tilde{x}'^\gamma - x'^\gamma) = \int d\tilde{x}'^\gamma J(\tilde{x}'^\gamma)\Psi(\tilde{x}'^\gamma) a(\tilde{x}'^\gamma)\delta(\tilde{x}'^\gamma - x'^\gamma) = \qquad (1.6.16)$$
$$= \ J(x'^\gamma)\Psi(x'^\gamma)a(x'^\gamma) = W(x'^\gamma)\Psi(x'^\gamma)$$

We see that the wave function transition law from one frame to another is always a multiplication by a function:

$$F[\Psi(x^\gamma)] = W(x'^\gamma)\Psi(x'^\gamma) \qquad (1.6.17)$$

Now let's consider the case of linear transformations between the Lorenz frame and the new frame. According to the Axiom M2 if in the Lorenz frame the general momentum was with 100% known (the wave function was a harmonic), then in the new frame the momentum will also be 100% known (the wave function will be a harmonic up to multiplication by coefficient):

$$F\left[e^{i\vec{p}_\gamma \vec{x}^\gamma/\hbar}\right] = W(x'^\gamma)e^{i\vec{p}'_\gamma \vec{x}'^\gamma/\hbar} = b(\vec{p}'_\gamma)e^{i\vec{p}'_\gamma \vec{x}'^\gamma/\hbar} \qquad (1.6.18)$$

It can be only in one case: when $W(x'^\gamma) = b(\vec{p}'_\gamma)$ so that $W = const$. We see that transition law of the wave function from one frame to another is case of linear transition from Lorenz frame to another frame is just simple multiplication by a constant:

$$F[\Psi(x^\gamma)] = W \cdot \Psi(x'^\gamma) \qquad (1.6.19)$$

where the modulus of W can be calculated using Axiom 4, the phase can be arbitrary. Let's state that the phase is zero, then $W = \left(\frac{dV}{dV'}\right)^{0.5}$. For example, in the case Lorenz transformation $W = \left(\frac{1}{\sqrt{1 - v^2/c^2}}\right)^{0.5}$.

Now let's consider not the wave function of one particle, but the wave function of gravitational field

$$\Psi = \Psi(t, \{g_{\alpha\beta}(\vec{x})\}) \qquad (1.6.20)$$

and let's try to calculate how it changes from one frame to another. The question may rise: the metric functions are the functions of space coordinates, but not the functions of time? The answer is following: the time evolution of metric is in the wave function which is a function of time.

Axiom G1. If in one frame the metric is 100% known (the wave function is a delta-function), then in the other frame the metric is 100% known (in the new frame the wave function is a delta-function multiplied by a coefficient).

$F[\Psi_0 = \delta(\{g_{\alpha\beta}(x^\gamma)\} - \{g0_{\alpha\beta}(x^\gamma)\})]$ is equal to $\Psi'_0 = a(\{g'0_{\alpha\beta}(x'^\gamma)\})\delta(\{g'_{\alpha\beta}(x'^\gamma)\} - \{g'0_{\alpha\beta}(x'^\gamma)\})$

(1.6.21)

Let's also postulate the following statement consulting general momentum:

Axiom G2. If in one frame the general momentum was with 100% known (the wave function was a harmonic functional in case of linear transformations), then in the other frame the general momentum is also with 100% known (the wave function is a harmonic function multiplied by a coefficient in the new frame).

$$F\left[\Psi_{p^{\gamma\mu}(t\vec{x})}: g_{\gamma\mu}(\vec{x}) \to e^{-i\int d\vec{x}\, p^{\gamma\mu}(t\vec{x}) g_{\gamma\mu}(\vec{x})/\hbar}\right] = \Psi'_{p'^{\gamma\mu}(t'\vec{x}')}: g'_{\gamma\mu}(\vec{x}') \to b\{p'^{\gamma\mu}(t'\vec{x}')\} e^{-i\int d\vec{x}' p'^{\gamma\mu}(t'\vec{x}') g_{\gamma\mu}(\vec{x}')/\hbar} \quad (1.6.22)$$

Let's postulate the superposition principle:

Axiom G3. If in one frame wave function was a sum (superposition) of two wave functions, then in the new frame the wave function will be superposition of the same states but taken in the new frame.

If in one frame wave function was a product of a wave function and a constant, then in the new frame the wave function will be also the product of the same state but taken in the new frame and the same constant.

If in one frame wave function was an integral over a wave function and a coefficient, then in the new frame the wave function will be also the integral of the same state but taken in the new frame and the same coefficient.

$$F[\Psi_1 + \Psi_2] = F[\Psi_1] + F[\Psi_2] \quad (1.6.23)$$

$$F[\gamma\Psi] = \gamma F[\Psi] \quad (1.6.24)$$

$$F\left[\int D[g_{\gamma\mu}(x^\gamma)] f[g_{\gamma\mu}(x^\gamma)] \Psi[g_{\gamma\mu}(x^\gamma)]\right] = \int D[g_{\gamma\mu}(x^\gamma)] f[g_{\gamma\mu}(x^\gamma)] F[\Psi[g_{\gamma\mu}(x^\gamma)]] \quad (1.6.25)$$

Axiom G4. The probability element remains the same if we transit from one frame to another

$$D[g_{\gamma\mu}(x)] \left|\Psi_0[g_{\gamma\mu}(x)]\right|^2 = D[g'_{\gamma\mu}(x')] \left|\Psi'_0[g'_{\gamma\mu}(x')]\right|^2 \quad (1.6.26)$$

where $D[g_{\gamma\mu}(x)]\left|\Psi[g_{\gamma\mu}(x)]\right|^2$ in analogy with usual $dV\left|\Psi\right|^2$ is approximately the probability for the metric components to change between $g_{\gamma\mu}(x)$ and $g_{\gamma\mu}(x)+\Delta g_{\gamma\mu}(x)$ with small $\Delta g_{\gamma\mu}(x)$, Ψ_0 and Ψ'_0 are written in Axiom G1.

<u>The wave function transition law from one frame to another</u> may be calculated from those axioms. Let's us think of the state as the vector which has it's value at each moment of parameter time. It means that the metric, functional of which the state is, also is function of time:

From Axiom 1 and 4 we see:

$$
\begin{aligned}
&\Psi' = F[\Psi(t,\{g_{\alpha\beta}(t,\vec{x})\})] = F\left[\int D\tilde{g}_{\alpha\beta}(t,\vec{x})\,\delta\left(\{\tilde{g}_{\alpha\beta}(t,\vec{x})\}-\{g_{\alpha\beta}(t,\vec{x})\}\right)\Psi(t,\{\tilde{g}_{\alpha\beta}(t,\vec{x})\}\right]= \\
&=\int D\tilde{g}_{\alpha\beta}(t,\vec{x})\Psi(t,\{\tilde{g}_{\alpha\beta}(t,\vec{x})\})F\left[\delta(\{\tilde{g}_{\alpha\beta}(t,\vec{x})\}-\{g_{\alpha\beta}(t,\vec{x})\})\right]= \\
&=\int D\tilde{g}'J(D\tilde{g}/D\tilde{g}')\Psi(t,\{\tilde{g}_{\alpha\beta}(t,\vec{x})\})\,a\left(\{\tilde{g}'_{\alpha\beta}(t',\vec{x}')\}\right)\delta\left(\{\tilde{g}'_{\alpha\beta}(t',\vec{x}')\}-\{g'_{\alpha\beta}(t',\vec{x}')\}\right)= \\
&=\int D\tilde{g}'J(D\tilde{g}/D\tilde{g}')\Psi\left(t(t',\vec{x}'),\left\{\frac{\partial x'^{\eta}}{\partial x^{\alpha}}\frac{\partial x'^{\mu}}{\partial x^{\beta}}\tilde{g}'_{\eta\mu}(t',\vec{x}')\right\}\right)a\left(\{\tilde{g}'_{\alpha\beta}(t',\vec{x}')\}\right)\delta\left(\{\tilde{g}'_{\alpha\beta}(t',\vec{x}')\}-\{g'_{\alpha\beta}(t',\vec{x}')\}\right) = \\
&=J(Dg/Dg')\Psi\left(t(t',\vec{x}'),\left\{\frac{\partial x'^{\eta}}{\partial x^{\alpha}}\frac{\partial x'^{\mu}}{\partial x^{\beta}}g'_{\eta\mu}(t',\vec{x}')\right\}\right)a\left(\{g'_{\alpha\beta}(t',\vec{x}')\}\right)=W\cdot\Psi
\end{aligned}
$$

(1.6.27)

We see that the wave function transition law from one frame to another is always a multiplication by a function:

$$\Psi' = F[\Psi] = W\cdot\Psi \qquad (1.6.28)$$

Now let's consider the case of linear transformations between the Lorenz frame and the new frame. According to the Axiom 2 if in the Lorenz frame the general momentum was with 100% known (the wave function was a harmonic), then in the new frame the momentum will also be 100% known (the wave function will be a harmonic up to multiplication by coefficient):

$$
\begin{aligned}
&F\left[e^{-i\int d\vec{x}\,p^{\gamma\mu}(t\vec{x})g_{\gamma\mu}(\vec{x})/\hbar}\right]=b\{p'^{\gamma\mu}(t'\vec{x}')\}e^{-i\int d\vec{x}'p'^{\gamma\mu}(t'\vec{x}')g_{\gamma\mu}(\vec{x}')/\hbar}= \\
&=W\cdot\Psi = J(Dg/Dg')\,a\left(\{g'_{\alpha\beta}(t',\vec{x}')\}\right)e^{-i\int d\vec{x}'p'^{\gamma\mu}(t'\vec{x}')g_{\gamma\mu}(\vec{x}')/\hbar}
\end{aligned}
\qquad (1.6.29)
$$

It can be only in one case: when $W=b=const$ so that $W=const$. We see that transition law of the wave function from one frame to another is case of linear transition from Lorenz frame to another frame is just simple multiplication by a constant:

$$F[\Psi] = W\cdot\Psi \qquad (1.6.30)$$

where the modulus of W can be calculated using Axiom G4, the phase can be arbitrary. Let's state that the phase is zero.

Now let's consider not the wave function of one particle, but the wave function of free electromagnetic field

$$\Psi = \Psi(t, \{A^{\alpha}(\vec{x})\}) \tag{1.6.31}$$

and let's try to calculate how it changes from one frame to another. The question may rise: the electromagnetic field vector functions are the functions of space coordinates, but not the functions of time? The answer is following: the time evolution of electromagnetic field is in the wave function which is a function of time.

Axiom E1. If in one frame the metric is 100% known (the wave function is a delta-function), then in the other frame the metric is 100% known (in the new frame the wave function is a delta-function multiplied by a coefficient).

$$F\left[\Psi = \delta(\{A^{\alpha}(x^{\gamma})\} - \{A_0^{\alpha}(x^{\gamma})\})\right] \quad \text{is equal to} \quad \Psi' = a(\{A'^{\alpha}_0(x'^{\gamma})\})\delta(\{A^{\alpha}(x'^{\gamma})\} - \{A'^{\alpha}_0(x'^{\gamma})\}) \tag{1.6.32}$$

Let's also postulate the following statement consulting general momentum:

Axiom E2. If in one frame the general momentum was with 100% known (the wave function was a harmonic function in case of linear transformations), then in the other frame the general momentum is also with 100% known (the wave function is a harmonic function multiplied by a coefficient in the new frame).

$$F\left[\Psi_{p_{\gamma}(t\vec{x})}: A^{\gamma}(\vec{x}) \rightarrow e^{-i\int d\vec{x}\, p_{\gamma}(t\vec{x})A^{\gamma}(\vec{x})/\hbar}\right] = \Psi'_{p'_{\gamma}(t'\vec{x}')}: A'^{\gamma}(\vec{x}') \rightarrow b(\{p'_{\gamma}(t'\vec{x}')\})e^{-i\int d\vec{x}'\, p'_{\gamma}(t'\vec{x}')\, A'^{\gamma}(\vec{x}')/\hbar} \tag{1.6.33}$$

Let's postulate the superposition principle:

Axiom E3. If in one frame wave function was a sum (superposition) of two wave functions, then in the new frame the wave function will be superposition of the same states but taken in the new frame.

If in one frame wave function was a product of a wave function and a constant, then in the new frame the wave function will be also the product of the same state but taken in the new frame and the same constant.

If in one frame wave function was an integral over a wave function and a coefficient, then in the new frame the wave function will be also the integral of the same state but taken in the new frame and the same coefficient.

$$F\left[\Psi_1 + \Psi_2\right] = F\left[\Psi_1\right] + F\left[\Psi_2\right] \tag{1.6.34}$$

$$F[\gamma\Psi] = \gamma F[\Psi] \tag{1.6.35}$$

$$F\left[\int DA^{\alpha} f(A^{\alpha})\Psi(A^{\alpha})\right] = \int DA^{\alpha} f(A^{\alpha}) F\left[\Psi(A^{\alpha})\right] \tag{1.6.36}$$

<u>Axiom E4.</u> The probability element remains the same if we transit from one frame to another

$$D\left[A^{\gamma}(x)\right]\left\|\Psi_0\left[A^{\gamma}(x)\right]\right\|^2 = D\left[A'^{\gamma}(x')\right]\left\|\Psi'_0\left[A'^{\gamma}(x')\right]\right\|^2 \tag{1.6.37}$$

where $D\left[A^{\gamma}(x)\right]\left\|\Psi_0\left[A^{\gamma}(x)\right]\right\|^2$ in analogy with usual $dV\left|\Psi\right|^2$ is approximately the probability for the field components to change between $A^{\gamma}(\vec{x})$ and $A^{\gamma}(x) + \Delta A^{\gamma}(x)$ with small $\Delta g_{\gamma\mu}(x)$, Ψ_0 and Ψ'_0 are written in Axiom E1.

<u>The wave function transition law from one frame to another</u> may be calculated from those axioms. From Axiom E1 and E4 we see:

$$\begin{aligned}
\Psi' &= F\left[\Psi(t, \{A^{\gamma}(t,\vec{x})\})\right] = F\left[\int D\tilde{A}^{\gamma}(t,\vec{x})\, \delta\left(\{\tilde{A}^{\gamma}(t,\vec{x})\} - \{A^{\gamma}(t,\vec{x})\}\right)\Psi(t, \{\tilde{A}^{\gamma}(t,\vec{x})\})\right] = \\
&= \int D\tilde{A}^{\gamma}(t,\vec{x})\Psi(t, \{\tilde{A}^{\gamma}(t,\vec{x})\}) F\left[\delta\left(\{\tilde{A}^{\gamma}(t,\vec{x})\} - \{A^{\gamma}(t,\vec{x})\}\right)\right] = \\
&= \int D\tilde{A}'^{\gamma} J(D\tilde{A}^{\gamma} / D\tilde{A}'^{\gamma})\Psi(t, \{\tilde{A}^{\gamma}(t,\vec{x})\})\, a\left(\{\tilde{A}'^{\gamma}(t',\vec{x}')\}\right) \delta\left(\{\tilde{A}'^{\gamma}(t',\vec{x}')\} - \{A'^{\gamma}(t',\vec{x}')\}\right) = \\
&= \int D\tilde{A}'^{\gamma} J(D\tilde{A}^{\gamma} / D\tilde{A}'^{\gamma})\Psi\left(t(t',\vec{x}'), \left\{\frac{\partial x'^{\gamma}}{\partial x^{\eta}} \tilde{A}'^{\eta}(t',\vec{x}')\right\}\right) a\left(\{\tilde{A}'^{\gamma}(t',\vec{x}')\}\right) \delta\left(\{\tilde{A}'^{\gamma}(t',\vec{x}')\} - \{A'^{\gamma}(t',\vec{x}')\}\right) = \\
&= J(DA / DA')\Psi\left(t(t',\vec{x}'), \left\{\frac{\partial x'^{\gamma}}{\partial x^{\eta}} A'^{\eta}(t',\vec{x}')\right\}\right) a\left(\{A'^{\gamma}(t',\vec{x}')\}\right) = W \cdot \Psi
\end{aligned}$$

(1.6.38)

We see that the wave function transition law from one frame to another is always a multiplication by a function:

$$\Psi' = F[\Psi] = W \cdot \Psi \tag{1.6.39}$$

Now let's consider the case of linear transformations between the Lorenz frame and the new frame. According to the Axiom 2 if in the Lorenz frame the general momentum was with 100% known (the wave function was a harmonic), then in the new frame the momentum will also be 100% known (the wave function will be a harmonic up to multiplication by coefficient):

$$\begin{aligned} & F\left[e^{-i\int d\vec{x}\, p_\gamma(t\vec{x})\, A^\gamma(\vec{x})/\hbar} \right] = b\{p'_\gamma(t'\vec{x}')\} e^{-i\int d\vec{x}'\, p'_\gamma(t'\vec{x}')\, A'^\gamma(\vec{x}')/\hbar} = \\ & = W \cdot \Psi = J(DA/DA')\, a\left(\{A'^\gamma(t', \vec{x}')\}\right) e^{-i\int d\vec{x}'\, p'_\gamma(t'\vec{x}')\, A'^\gamma(\vec{x}')/\hbar} \end{aligned} \tag{1.6.40}$$

It can be only in one case: when $W = b = const$ so that $W = const$. We see that transition law of the wave function from one frame to another is case of linear transition from Lorenz frame to another frame is just simple multiplication by a constant:

$$F[\Psi] = W \cdot \Psi \tag{1.6.41}$$

where the modulus of W can be calculated using Axiom G4, the phase can be arbitrary. Let's state that the phase is zero.

For many particle movement we have according to the Abstract:

$$\Psi = \sum_{u\,...\,v,a,b} \Psi_u^{(1)}(tx_1)\cdot ...\cdot \Psi_v^{(N)}(tx_N)\cdot \Psi_a^{em}(t,\{\vec{q}(\vec{x})\})\Psi_b^G(t,\{g_{\alpha\beta}(\vec{x})\})\cdot I_{u\,...v,\,a,b} \tag{1.6.42}$$

If we know the transition law from one frame to another of each component, then we know the transition law of the whole state.

1.7 Semiclassical case of low fields and velocities: how to calculate a low field metric?

In order to find the low gravity metric, we compare the classical and gravitational actions. Also we suppose in the sake of simplicity that classical gravitation is classical electrodynamics where the charge is equal to the mass: $q = m$.

Then we compare the action of a particle in the classical electromagnetic field $S_e = mc^2\int d\tau + q/c\cdot\int A_\alpha dx^\alpha = mc^2\int d\tau + m/c\cdot\int A_\alpha v^\alpha d\tau = mc^2\int\left(1+1/c^3\cdot A_\alpha v^\alpha\right)d\tau$, where $v^\alpha = dx^\alpha/d\tau$ and the action of a classic particle in the arbitrary metric $S_g = mc^2\int d\tau_g = mc\cdot\int\sqrt{g_{\alpha\beta}dx^\alpha dx^\beta} = mc\cdot\int\sqrt{g_{\alpha\beta}v^\alpha v^\beta}d\tau$, where $v^\alpha = dx^\alpha/d\tau$ supposing that the charge is equal to the mass

$$\int\left(1+1/c^3\cdot A_\alpha v^\alpha\right)d\tau = 1/c\cdot\int\sqrt{g_{\alpha\beta}v^\alpha v^\beta}d\tau \qquad (1.7.1)$$

Suppose that the parts under integrals are also equal:

$$1+1/c^3\cdot A_\alpha v^\alpha = 1/c\cdot\sqrt{g_{\alpha\beta}v^\alpha v^\beta} \qquad (1.7.2)$$

Then take a square from both parts:

$$1+\left(1/c^3\cdot A_\alpha v^\alpha\right)^2 + 2/c^3\cdot A_\alpha v_\alpha = 1/c^2\cdot g_{\alpha\beta}v^\alpha v^\beta \qquad (1.7.3)$$

Suppose that the classic field is small and the square of it $A_\alpha\cdot A_\beta$ can be neglected. Also make the notation: $1/c^2\cdot g_{\alpha\beta}v^\alpha v^\beta = 1/c^2\cdot\left(g^L_{\alpha\beta}+\Delta g_{\alpha\beta}\right)v^\alpha v^\beta = 1+1/c^2\cdot\Delta g_{\alpha\beta}v^\alpha v^\beta$, where $g^L_{\alpha\beta}$ is a Lorenz metric:

$$1+2/c^3\cdot A_\alpha v^\alpha = 1+1/c^2\cdot\Delta g_{\alpha\beta}v^\alpha v^\beta \qquad (1.7.4)$$

Or,

$$2/c\cdot A_\alpha v^\alpha = \Delta g_{\alpha\beta}v^\alpha v^\beta \qquad (1.7.5)$$

Note that the following metric is the solution of the last equation (note that is depends on velocity, below we show how to overcome this problem):

$$\Delta g_{\alpha\beta} = 1/c^3\cdot\left(A_\alpha v_\beta + A_\beta v_\alpha\right) \qquad (1.7.6)$$

I suppose that if we suppose the symmetry of metric, then the solution (1.7.6) of (1.7.5) is unique, but I haven’t proven it yet. The dependence of metric on velocity leads to supposition that (1.7.5) is an “equation valid at low velocities”, it is valid up to the second order by velocity

Obviously,

$$\Delta g^{\alpha\beta} = -1/c^3 \cdot \left(A^{\alpha} v^{\beta} + A^{\beta} v^{\alpha} \right) \qquad (1.7.7)$$

If we introduce usual Newtonian velocity $V^{\alpha} = dx^{\alpha} / dt$, and use the formula $v^{\alpha} = V^{\alpha} / \sqrt{1 - \vec{V}^2 / c^2}$, then we can rewrite (1.7.5):

$$2/c \cdot A_{\alpha} V^{\alpha} \sqrt{1 - \vec{V}^2 / c^2} = \Delta g_{\alpha\beta} V^{\alpha} V^{\beta} \qquad (1.7.8)$$

Let's consider 3 cases:

0. Zero approximation by $\vec{V}$: $\Delta g_{00} = 2A_0 / c^2$
1. First approximation by $\vec{V}$: $\Delta g_{\alpha 0} = \Delta g_{0\alpha} = A_{\alpha} / c^2$ (1.7.9)
2. Second approximation by $\vec{V}$: $\Delta g_{\alpha\alpha} = -1 - A_0 / c^2$ if $\alpha \geq 1$, otherwise $\Delta g_{\alpha\beta} = 0$.

No we have a metric in the approximation of low gravitational field and low velocities:

$$g_{\alpha\beta} = \begin{pmatrix} 1 + 2A_0/c^2 & A_1/c^2 & A_2/c^2 & A_3/c^2 \\ A_1/c^2 & -1 - A_0/c^2 & 0 & 0 \\ A_2/c^2 & 0 & -1 - A_0/c^2 & 0 \\ A_3/c^2 & 0 & 0 & -1 - A_0/c^2 \end{pmatrix}, \qquad (1.7.10)$$

where $A_{\alpha} = g^{L}_{\alpha\beta} A^{\beta}$, $\Box A^{\alpha} = 4\pi G \rho^{\alpha}$.

It means that

$$g_{\alpha\beta} = \begin{pmatrix} 1 + 2A^0/c^2 & -A^1/c^2 & -A^2/c^2 & -A^3/c^2 \\ -A^1/c^2 & -1 - A^0/c^2 & 0 & 0 \\ -A^2/c^2 & 0 & -1 - A^0/c^2 & 0 \\ -A^3/c^2 & 0 & 0 & -1 - A^0/c^2 \end{pmatrix} \qquad (1.7.11)$$

1.8 The numerical predictions.

We will show how the numerical predictions can be done in our quantum gravity theory. In the arbitrary case the wave function is function of metric matrix elements. Suppose that we have the semiclassical approximation **(with simple generalization in the case of arbitrary fields and metric fields also where the wave function is a function of metric matrix components):**

$$\hat{g}_{\alpha\beta} = \begin{pmatrix} 1+2\hat{\varphi}g/c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.8.1}$$

where we introduce $\hat{\varphi}g$ in the following way. If the wave function was a function of metric matrix elements, than now the wave function is a function of 4-vector of gravitational potential obeying the wave equation. If $\hat{A}^{\mu}_{g} = (\hat{\varphi}_g, \hat{\vec{A}}_g)$, then

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \vec{x}^2}\right)\hat{A}^{\mu}_{g} = 4\pi G\hat{\rho}^{\mu}_{g} \tag{1.8.2}$$

$\hat{\rho}^{\mu}_{g}$ is the operator of mass 4-density.

We see from (1.8.1) that contrary to the case electromagnetic fields the physical result may depend on the zero component of the semiclassical gravitational field 4-vector $\hat{\varphi}g$. So in analogy with (1.6.5) we introduce the 4-coordinate

$$\hat{q}^{\mu}_{g} = \left(\hat{q}^{g}_{0}\ \hat{\vec{q}}_{g}\right) = \left(\hat{\varphi}_g, \hat{\vec{A}}_g + \int dt \cdot c\,grad\,\hat{\varphi}_g\right) \tag{1.8.3}$$

Note that the operator value of the Hamiltonian will not change, the reason is that adding 0-component to the coordinate gives the 0-component of momentum equal to 0:

$$p_0 = \frac{\partial \Lambda}{\partial \dot{\vec{q}}_0} = 0 \tag{1.8.4}$$

So quantization gives:

$$\hat{p}^{g}_{0} = 0 \tag{1.8.5}$$

Now let's calculate $\hat{F}^{\alpha\beta}$:

$$\begin{aligned} \hat{F}^{\alpha\beta} &= \hat{F}_{\mu\nu}\hat{g}^{\alpha\mu}\hat{g}^{\beta\nu} = \left(\hat{g}^{\circ}\,\hat{F}_{\circ}\,\hat{g}^{\circ}\right)^{\alpha\beta} = \left((\hat{g}_L + \Delta\hat{g})^{\circ}\,\hat{F}_{\circ}\,(\hat{g}_L + \Delta\hat{g})^{\circ}\right)^{\alpha\beta} = \\ &= \left(\hat{g}_L{}^{\circ}\,\hat{F}_{\circ}\,\hat{g}_L{}^{\circ}\right)^{\alpha\beta} + \left(\Delta\hat{g}^{\circ}\hat{F}_{\circ}\,\hat{g}_L{}^{\circ}\right)^{\alpha\beta} + \left(\hat{g}^{\circ}_{L}\,\hat{F}_{\circ}\,\Delta\hat{g}^{\circ}\right)^{\alpha\beta} + o(\Delta\hat{g}) \end{aligned} \tag{1.8.6}$$

In particular,

$$\hat{E}^{\alpha} = \hat{F}^{\alpha 0} = (1 - 2\hat{\varphi}g/c^2)\hat{E}_{\alpha} + o(\Delta\hat{g}) \tag{1.8.7}$$

$$\hat{H}^{\alpha} = \hat{H}_{\alpha} + o(\Delta\hat{g}) \qquad (1.8.8)$$

Also,

$$\sqrt{-\det\hat{g}} = 1 + \hat{\varphi}g/c^2 + o(\Delta\hat{g}) \qquad (1.8.9)$$

It means that the Hamiltonian of electromagnetic field is following - see (1.6.15)

$$\begin{aligned}\hat{H}_{em} &= \int d\vec{x}\sqrt{-\det\hat{g}}\,\frac{\hat{E}_{\gamma}\hat{E}^{\gamma} + \hat{H}_{\gamma}\hat{H}^{\gamma}}{8\pi} = \int d\vec{x}\,\frac{(1+3\hat{\varphi}g/c^2)\hat{E}^{\gamma}\hat{E}^{\gamma} + (1+\hat{\varphi}g/c^2)\hat{H}^{\gamma}\hat{H}^{\gamma}}{8\pi} + \\ &+ o(\Delta\hat{g}) = \int d\vec{x}\,\frac{(1+\hat{\varphi}g/c^2)\left(4\pi c\hat{\hat{p}}(\vec{x})\right)^2 + (1+\hat{\varphi}g/c^2)\left(rot\,\hat{\hat{q}}(\vec{x})\right)^2}{8\pi} + o(\Delta\hat{g})\end{aligned} \qquad (1.8.10)$$

Let’s suppose that the Hamiltonian of the 2 quantum particles interacting with gravitational and electromagnetic field in the approximation of weak fields is following

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_{em} + \hat{H}_g \qquad (1.8.11)$$

where $\hat{H}_1$ and $\hat{H}_2$ are the Hamiltonians of the first and second particles which can be found from (1.3.17) - (1.3.19), they can be very simplified using the weak field approximation, $\hat{H}_{em}$ can be found from (1.6.10), also our gravity hear in approximation of weak fields **(in case of arbitrary fields the wave function is a function of metric matrix components and the particle coordinates)** in our semiclassical case is:

$$\hat{H}_g = \int d\vec{x}\,\frac{\hat{E}_g^{\gamma}\hat{E}_g^{\gamma} + \hat{H}_g^{\gamma}\hat{H}_g^{\gamma}}{8\pi} + o(\Delta\hat{g}) = \int d\vec{x}\,\frac{\left(4\pi c\hat{\hat{p}}_g(\vec{x})\right)^2 + \left(rot\,\hat{\hat{q}}_g(\vec{x})\right)^2}{8\pi} + o(\Delta\hat{g}) \qquad (1.8.12)$$

The acceleration of the first particle, for example, be found from simple well known rules

$$\left\langle \frac{d^2\vec{x}_1}{dt^2} \right\rangle = \left\langle \frac{i}{\hbar}\left[\hat{H}, \frac{i}{\hbar}\left[\hat{H}, \vec{x}_1\right]\right] \right\rangle \qquad (1.8.13)$$

One of the terms of this equation which can give quantum gravity effects is

$$\left\langle \frac{d^2\vec{x}_1}{dt^2} \right\rangle_{QG1} = \left\langle \frac{q_1\hat{\hat{E}}_1}{m_1c^2}\hat{\varphi}g \right\rangle \qquad (1.8.14)$$

2.1 Conclusion

Following simple quantization procedures we have done the quantization of gravitational field, electromagnetic field and particle quantization, using the canonical variables coordinate and momentum. We have constructed the many-particle wave function in usual way.

The quantization of fields was little bit different from the one which is done by followers of Fourier method, may be this was the reason why we have no divergences, ultraviolet divergences, etc. which usually happen, and did not appear within our method. Our quantum gravity model is the model with the ability to give numerical predictions – see (1.8.14).

References

[1] L. D. Landau, E. M. Lifshitz, Field Theory, Moscow, “Nauka”, 1988.
[2] S. Weinberg, Gravitation and Cosmology, Moscow, “Nauka”, 1975.
[3] http://avortementetmoi.webcindario.com/Христианство болезни исцеление.
[4] http://avortementetmoi.webcindario.com/Christianity illness and healing.
[5] C. Rovelli, L. Smolin, Nucl. Phys. B331 (1990) 80.
[6] C. Rovelli, L. Smolin, Nucl. Phys. B442 (1995) 593-622.
[7] A. Ashtekar, Phys. Rev. Lett. 57 (1986) 2244-2247.
[8] A. Ashtekar, Phys. Rev. D36 (1987) 1587-1602.
[9] M. Maggiore, Phys. Lett. B 319 (1993) 83.
[10] M. Maggiore, Phys. Lett. B 304 (1993) 65.
[11] M. Maggiore, Phys. Rev. D 49 (1994) 5182.
[12] A. Kempf, G. Mangano, R. B. Mann, Phys. Rev. D52 (1995) 1108-1118.
[13] H. Hinrichsen and A. Kempf, J. Math. Phys. 37 (1996) 2121.
[14] A. Kempf, J. Math. Phys. 38 (1997) 1347.
[15] A. Kempf and G. Mangano, Phys. Rev. D 55 (1997) 7909.
[16] T. Matsuo, Y. Shibusa, Mod. Phys. Lett. A21 (1006) 1285-1296.
[17] F. Scardigli, Phys. Lett. B 452 (1999) 39.
[18] S. Capozziello, G. Lambiase and G. Scarpetta, Int. J. Theor. Phys. 39 (1000) 15.
[19] L. B. Crowell, Found. Phys. Lett. 12 (1999) 585.
[20] L. N. Chang, D. Minic, N. Okamura and T. Takeuchi, Phys. Rev. D 65 (1002) 125028.
[21] W. Kim, Y. W. Kim and Y. J. Park, Phys. Rev. D 74 (1006) 104001.
[22] M. I. Park, Phys. Lett. B 659 (1008) 698.
[23] F. Scardigli and R. Casadio, Int. J. Mod. Phys. D 18 (1009) 319.
[24] W. Kim, E. J. Son and M. Yoon, JHEP 0801 (1008) 035.
[25] A. Bina, K. Atazadeh and S. Jalalzadeh, Int. J. Theor. Phys. 47 (1008) 1354.
[26] M. V. Battisti and G. Montani, Phys. Rev. D 77 (1008) 023518.
[27] T. Zhu, J. R. Ren and M. F. Li, Phys. Lett. B 674 (1009) 204.

+++ IHM