The angular momentum in the hydrogen atom

Sangwha-Yi
Department of Math, Taejon University 300-716

ABSTRACT
The article treats that the angular momentum in the hydrogen atom.
If calculates the electron motion in the hydrogen atom, can do the quantization in the
Bohr’s theory about the hydrogen atom. In this time, the electron’s orbit velocity $v$ is
non-relativity velocity.

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e-mail address:sangwha1@nate.com
Tel:051-624-3953
I. Introduction

The article treats that the angular momentum in the hydrogen atom.

If calculates the electron motion in the hydrogen atom by Coulomb’s law in the classical mechanic [4].

\[
F = ma = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r^2} = m \frac{v^2}{r} \quad (1)
\]

\(m\) is the electron’s mass.

The electron’s kinetic Energy is

\[
K = \frac{1}{2}mv^2 = \frac{e^2}{8\pi\varepsilon_0 r} \quad (2)
\]

The potential Energy in the hydrogen atom is

\[U = V(-e) = -\frac{e^2}{4\pi\varepsilon_0 r} \quad , \quad V = \frac{e}{4\pi\varepsilon_0 r} \quad (3)\]

The total Energy in the hydrogen atom is

\[E = K + U = -\frac{e^2}{8\pi\varepsilon_0 r} \quad (4)\]

The Bohr’s frequency condition is

\[h\nu = E_k - E_j \quad (5)\]

By Eq(2), the electron’s the orbit velocity \(v\), the momentum \(p\), the angular momentum \(L\) are

\[v = \sqrt{\frac{e^2}{4\pi\varepsilon_0 mr}} , \quad p = mv = \sqrt{\frac{me^2}{4\pi\varepsilon_0 r}} , \quad L = pr = \sqrt{\frac{me^2 r}{4\pi\varepsilon_0}} \quad (6)\]

The Bohr’s hypothesis is

\[L = n \frac{h}{2\pi} = nh \quad (7) \quad n = 1,2,3,\ldots\]

By Eq(6), Eq(7), the radius \(r\) is

\[r = r_n = n^2 \frac{h^2 \varepsilon_0}{\pi me^2} \quad (8) \quad n = 1,2,3,\ldots\]

By Eq(4), Eq(8), the total energy \(E\) is

\[E = E_n = -\frac{me^4}{8\varepsilon_0^2 h^2 n^2} \quad (9) \quad n = 1,2,3,\ldots\]

By Eq(5), Eq(9), the hydrogen line spectra’s frequency \(\nu\) is
\[ \nu = \frac{me^4}{8\varepsilon_0^2 h^3} \left( \frac{1}{j^2} - \frac{1}{k^2} \right) \quad (10) \quad j, k \text{ is number.} \]

By Eq(6), Eq(8), the quantization of the electron's the orbit velocity \( \nu \), the momentum \( p \) is in the hydrogen atom

\[ \nu = \nu_n = \sqrt{\frac{e^2}{4\pi\varepsilon_0 m r}} = \frac{e^2}{2n\hbar \varepsilon_0}, \quad \nu_1 = \frac{e^2}{2\hbar \varepsilon_0} << c \quad (11) \quad n = 1, 2, 3, \ldots \]

\[ p = p_n = mv = \sqrt{\frac{me^2}{4\pi\varepsilon_0 r}} = \frac{me^2}{2n\hbar \varepsilon_0} \quad (12) \quad n = 1, 2, 3, \ldots \]

In this time, the electron's orbit velocity \( \nu \) in the hydrogen atom is not continuous.

Bohr’s orbit that it’s radius is \( r \) is

\[ n\lambda = 2\pi r \quad (13) \quad n = 1, 2, 3, \ldots \]

Therefore, the quantization of the electron's wavelength \( \lambda \) is in the hydrogen atom

\[ r = \frac{n\lambda}{2\pi} = \frac{n^2 \hbar^2 \varepsilon_0}{\pi m e^2} \]

\[ \lambda = \lambda_n = \frac{2n \hbar \varepsilon_0}{me^2} \quad (14) \quad n = 1, 2, 3, \ldots \]

By Eq(2), Eq(8), the quantization of the electron’s kinetic Energy is

\[ K = \frac{1}{2} m v^2 = K_n = \frac{e^2}{8\pi\varepsilon_0 r} = \frac{e^4 m}{8n^2 \hbar^2 \varepsilon_0^2} \quad (15) \quad n = 1, 2, 3, \ldots \]

By Eq(3), Eq(8), the quantization of the potential Energy in the hydrogen atom is

\[ U_n = V_n (-e) = -\frac{e^2}{4\pi\varepsilon_0 r} = -\frac{e^4 m}{4n^2 \hbar^2 \varepsilon_0^2}, \quad V_n = \frac{e}{4\pi\varepsilon_0 r} = \frac{e^3 m}{4n^2 \hbar^2 \varepsilon_0^2} \quad (16) \quad n = 1, 2, 3, \ldots \]

By Eq(1), Eq(8), the quantization of the electric force is in the hydrogen atom is

\[ F = F_n = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r^2} = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{n^2 \hbar^2 \varepsilon_0} = \frac{e^4 m}{4n^2 \hbar^2 \varepsilon_0^2}, \quad n = 1, 2, 3, \ldots \quad (17) \]

\[ F = F_n = \frac{mv^2}{r} = \frac{e^4 m}{4n^2 \hbar^2 \varepsilon_0^2} \]

In this time, the electric force in the hydrogen atom is quantized.

In this time, de Broglie wavelength is in the hydrogen atom
\[ \lambda = \lambda_n = \frac{2n\hbar^2 \varepsilon_0}{me^2} = \frac{\hbar}{p} = \frac{\hbar}{p_n} \quad (18) \quad n = 1,2,3,\ldots \]

II. Additional chapter

Generally, the electron's angular momentum \( L \) is in the quantum mechanics,

\[ L = \frac{\hbar}{2\pi} \sqrt{l(l+1)} = \frac{\hbar}{r} \sqrt{l(l+1)} \quad (19) \quad l \text{ is the orbital quantum number} \]

\[ l = 0,1,2,\ldots,(n-1) \]

Eq(19) is different from Eq(7) about the electron's angular momentum \( L \).

Therefore, Eq(19) has to be include the principal quantum number \( n \) likely Eq(7).

In the Schrödinger wave equation, the radius function \( R = R(r) \) is

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} R \right) + \left[ \frac{2m}{\hbar} \left( \frac{e^2}{4\pi \varepsilon_0 r} + E \right) - \frac{l(l+1)}{r^2} \right] R = 0 \]

\[ E = E_n + V = E_n - \frac{e^2}{4\pi \varepsilon_0 r}, \quad E_n = -\frac{me^4}{8\varepsilon_0^2 \hbar^2 n^2} \]

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} R \right) + \frac{2m}{\hbar} \left[ E_n - \frac{\hbar^2 l(l+1)}{2mr^2} \right] R = 0 \quad (20) \]

Therefore, the electron's orbital kinetic energy \( K_{\text{orbital}} \) is \( K \) is by Eq(2)

\[ K_{\text{orbital}} = \frac{\hbar^2 l(l+1)}{2mr^2} + K = \frac{\hbar^2 l(l+1)}{2mr^2} + \frac{e^2}{8\pi \varepsilon_0 r} \quad (21) \]

\[ K_{\text{orbital}} = \frac{1}{2} mv_{\text{orbital}}^2 \quad (22) \]

And, the electron's angular momentum \( L \) is

\[ L = mv_{\text{orbital}} r \quad (23) \]

Therefore, the electron's orbital kinetic energy \( K_{\text{orbital}} \) is

\[ K_{\text{orbital}} = \frac{L^2}{2mr^2} \quad (24) \]

Therefore, according to Eq(21),

\[ \frac{L^2}{2mr^2} = \frac{\hbar^2 l(l+1)}{2mr^2} + \frac{e^2}{8\pi \varepsilon_0 r} \quad (25) \]

The electron's angular momentum \( L \) is

\[ L = \hbar \sqrt{l(l+1) + \frac{4\pi^2}{\hbar^2} \frac{me^2 r}{4\pi \varepsilon_0}} \]
\[ L = \hbar \sqrt{l(l + 1) + \frac{m \pi e^2 r}{\varepsilon_0 \hbar^2}} \quad (26) \]

In this time, hypothesizes that the \( r \) in Eq(26) is same the \( r \) in Eq(8)

\[ r = n^2 \frac{\hbar^2 \varepsilon_0}{\pi n e^2} \quad (27) \]

Therefore, the electron’s angular momentum \( L \) is

\[ L = \hbar \sqrt{l(l + 1) + \frac{m \pi e^2}{\varepsilon_0 \hbar^2}, n^2 \frac{\hbar^2 \varepsilon_0}{\pi n e^2}} = \hbar \sqrt{l(l + 1) + n^2} \quad (28) \]

\( n \) is the principal quantum number, \( l \) is the orbital quantum number
\[ n = 1, 2, 3, \ldots \quad l = 0, 1, 2, \ldots, (n - 1) \]

**III. Conclusion**

If \( l = 0 \),

\[ L = n \frac{\hbar}{2\pi} = nh \]

\[ v_{\text{orbital}} = v = \sqrt{\frac{e^2}{4\pi \varepsilon_0 mr}} = \frac{e^2}{2\hbar \varepsilon_0}, \quad r = n^2 \frac{\hbar^2 \varepsilon_0}{\pi n e^2} \]

\[ n = 1, 2, 3, \ldots \]

\[ K_{\text{orbital}} = \frac{L^2}{2mr^2} = \frac{1}{2} mv_{\text{orbital}}^2 = K = \frac{e^2}{8 \pi \varepsilon_0 r} = \frac{1}{2} mv^2 \quad (29) \]

Therefore, the electron’s angular momentum \( L \) is

\[ L = \hbar \sqrt{l(l + 1) + n^2} \quad (30) \]

\( n \) is the principal quantum number, \( l \) is the orbital quantum number
\[ n = 1, 2, 3, \ldots \quad l = 0, 1, 2, \ldots, (n - 1) \]

In this time, the \( r \) of the radius function \( R = R(r) \) has to be continuous in the Schrödinger wave equation but the \( r \) of the electron’s angular momentum \( L \) need not to be continuous in the quantum mechanic

**Reference**