Binomial Construction of the Trinomial Triangle

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Abstract

The trinomial triangle can be constructed in a binomial way using unit vectors of geometric algebra of quarks. This sheds some light on the question, how it is possible to transform mathematically entities of two elements into entities of three elements or vice versa.

Contents

1. Introduction

Some years ago I described and analysed the bilateral structure of binomial and multinomial expansions, which can be represented by the three Pascal triangles, the four Pascal pyramids or some more Pascal hyper-pyramids [1]. Having later discovered non-commutative equivalents of these structures and having not found them mentioned in the literature, I called them Pauli Pascal triangles, Pauli Pascal pyramids, or Pauli Pascal hyper-pyramids [2]. They surely have some didactical relevance [3].

This paper was uploaded at the internet preprint arxive of Cornell University (www.arxiv.org) at October 7th, 2012 [12]. Some days later the author was informed by the arxiv administration that the present paper is determined inappropriate for www.arxiv.org and that the submission was removed. It therefore seems urgently necessary that scientists intending to work for www.arxiv.org first read books like [13]. It might at least prevent some of their laborious censorship activities.

Recently Miao showed some interesting connections between Pascal pyramids at layer $\mathcal{N}$ and the number of component fields of a given kind in an $\mathcal{N}$-extended supermultiplet for super-conformal field theories [4]. And Gómez-Muñoz prepared some Mathematica applications of the positive Pauli Pascal triangle and other non-commutative expansions [5]. Therefore it might be interesting to reanalyse this subject from a slightly different viewpoint again.

This viewpoint is the viewpoint of geometric algebra of quarks [6], [7].

2. Philosophical Background

Wolfgang Pauli and Carl Gustav Jung vividly discussed the intrinsic description of nature by the numbers three and four in their letters. How can we make four out of three? They even called the problem, how to reach the modern view of our world based on the number four by starting with the Middle Ages view of our world based on the number three, as the "nearly 2000 year old central problem" of our cultural history [8, p. 124]. And Pauli in length analysed the philosophical conflicts between a ternary world view (represented for him by Kepler) and a quaternary world view (represented by the old Greeks, modern physics and scientists like Fludd) [8], [9]. Are theses problems Pauli and Jung discussed solved now?

Of course these philosophical problems are not solved today. We do not know how to make four out of three, because we do not know how to make three out of two. And we do not know how to make three out of two, because we do not know how to make two out of one. Perhaps all we know today is how to make two out of zero by identifying zero with minus one plus one, resulting in the two basic elements 'plus one' and 'minus one'. And it is even not sure whether we really know today how to do this split in a mathematically correct way.

Nevertheless I will try to make three out of two in this paper. I will show how to construct a trinomial triangle in a binomial way using two basic elements only. These two basic elements or two basic paths to follow will be rearranged in a ternary way, resulting in a structure (the trinomial triangle) usually thought of being based on three basic elements or three different paths to follow.

3. Physical Background

In physics we very often describe phenomena using two opposite elements. There are positive and there are negative electric charges, there are magnetic north and there are magnetic south poles, there is attraction and there is repulsion. There is no third form of electric charge, there is no third form of a magnetic pole, and there is no third form of force which is the opposite of attraction and at the same time the opposite of repulsion, too.

It therefore makes surely sense to describe physical laws by using two opposite basic mathematical elements in these cases. If we assume that some physical laws are represented by not too complicated functions $f = f(k_1 a + k_2 b)$ of linear combinations of these two basic elements a and b, the coefficients of Taylor expansions of these functions can be represented in Pascal-like triangles. These triangles are thus situated in the binomial plane.

But sometimes this strategy of choosing only two basic elements fails in physics. Obviously baryons like neutrons and protons can be described in a mathematically appropriate way only, if we use three distinct and somehow opposing basic elements. There are three quarks, two

–8	1																1	–8
21	–7	1														1	–7	21
–14	15	–6	1												1	–6	15	–14
–20	–5	10	–5	1										1	–5	10	–5	–20
24	–15	0	6	–4	1								1	–4	6	0	–15	24
3	9	–9	2	3	–3	1						1	–3	3	2	–9	9	3
–6	3	2	–4	2	1	–2	1				1	–2	1	2	–4	2	3	–6
0	–1	1	0	–1	1	0	–1	1		1	–1	0	1	–1	0	1	–1	0
									1									
								1	1	1								
							1	2	3	2	1							
						1	3	6	7	6	3	1						
					1	4	10	16	19	16	10	4	1					
				1	5	15	30	45	51	45	30	15	5	1				
			1	6	21	50	90	126	141	126	90	50	21	6	1			
		1	7	28	77	161	266	357	393	357	266	161	77	28	7	1		
	1	8	36	112	266	504	784	1016	1107	1016	784	504	266	112	36	8	1	

Fig. 1: The trinomial plane with the three trinomial triangles.

are not enough. This then automatically results in multinomial coefficients of grade three when raising linear combinations of three basic elements a, b, and c to higher powers. They can then be arranged as binominal Pascal pyramids.

But Euler decided to choose the three basic elements x^0, x^1, and x^2 [10] instead. There might be a common mathematical core x in everything, resulting in Taylor expansion coefficients which can be arranged in the trinomial plane as trinomial triangles (see figure 1).

In Chapter two the core philosophical question was outlined: How can we make three out of two? In physics it might be helpful to think about the opposite direction: There are three quarks. Might it be perhaps possible to find two entities which fully and completely represent these three quarks? In this case the question is: How can we make two out of three? Indeed geometric algebra of quark might be helpful to answer this question.

4. Geometric Algebra of Quarks

Geometric algebra of quarks is the algebra of S_3 permutation matrices, which were identified with geometrical objects in the following way.

unit scalar: $$e_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{1} \tag{1}$$

unit vectors: $$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

unit parallelograms: $$e_{12} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{3}$$

The products of the unit vectors then are:

$$e_1^2 = e_2^2 = e_3^2 = e_0 \tag{4}$$

$$e_1e_2 = e_2e_3 = e_3e_1 = e_{12}\,, \quad e_2e_1 = e_3e_2 = e_1e_3 = e_{21} \tag{5}$$

It can be shown that every multiple of the nihilation or nihilistic matrix[1] (or null matrix) **N**

$$\mathbf{N} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = e_1 + e_2 + e_3 = e_0 + e_{12} + e_{21} \tag{6}$$

[1] If you don't like this name, take it as a joke. But it surely makes sense to avoid names like "democratic matrix" or "unit matrix" which can be found for **N** in the literature. These names totally conceal the geometric meaning of the nihilation matrix **N**. Multiplying a (3 x 3) matrix with **N** has the same effect as multiplying with **O** [6].

can be identified geometrically and algebraically with the zero matrix

$$\mathbf{O} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (7)$$

and therefore for every k there is

$$k\,\mathbf{N} = \mathbf{O} \qquad (8)$$

It follows that we do not need negative signs in geometric algebra of quarks, because the (3 x 3) matrix Θ

$$\Theta = e_{12} + e_{21} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad (9)$$

does all that what is usually done by the minus sign "–". Thus in geometric algebra of quarks we live in a totally positive mathematical world. This two-dimensional world can be extended into a three-dimensional world when the pseudoscalar is interpreted as

unit volume: $$\mathbf{I} = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} \quad \text{with} \quad \mathbf{I}^2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \Theta\, e_0 = \Theta \qquad (10)$$

More about geometric algebra of quarks can be found in [6], [7] and [11].

5. Quarkonian Binomial Coefficients (first try)

To get the three standard Pascal triangles, the Taylor expansion (Mac Laurin form) of

$$(\mathbf{1}\,a + \mathbf{1}\,b)^n = (e_0\,a + e_0\,b)^n \qquad (11)$$

is constructed. The coefficients can then be arranged as the three well-known Pascal triangles (see [1]). To get the three Pauli Pascal triangles, the coefficients of the Taylor expansion of

$$(\sigma_x\,a + \sigma_y\,b)^n = \left(\frac{1}{3}\left(\sqrt{2}e_1 + \sqrt{8}e_3 + \sqrt{3}e_4\right) a + \frac{1}{2}\left(\sqrt{8}e_1 + \sqrt{2}e_2 + \sqrt{3}e_4\right) b\right)^n \qquad (12)$$

are arranged in the usual way (see [2] and please feel free to consider these Pauli matrices as (3 x 3) matrices [7]). But now we start with a different binom which is not a vector of Pauli algebra or of standard geometric algebra, but a vector of geometric algebra of quarks or S_3 permuation algebra:

$$(e_1\,a + e_2\,b)^n \qquad (13)$$

This results in the following expressions for $n \geq 0$:

$$(e_1\ a + e_2\ b)^0 = \mathbf{1} \tag{14}$$
$$(e_1\ a + e_2\ b)^1 = e_1\ a + e_2\ b \tag{15}$$
$$\begin{aligned}(e_1\ a + e_2\ b)^2 &= e_1^2\ a^2 + (e_{12} + e_{21})\ ab + e_2^2\ b^2 \\ &= \mathbf{1}\ a^2 + \Theta\mathbf{1}\ ab + \mathbf{1}\ b^2\end{aligned} \tag{16}$$
$$(e_1\ a + e_2\ b)^3 = e_1\ a^3 + (\Theta e_1 + e_2)\ a^2b + (e_1 + \Theta e_2)\ ab^2 + e_2\ b^3 \tag{17}$$
$$\begin{aligned}(e_1\ a + e_2\ b)^4 &= e_1^2\ a^4 + (e_{12} + \Theta e_1^2 + e_{21}\ )\ a^3b \\ &\quad + (\Theta e_{12} + e_2^2 + e_1^2 + \Theta e_{21})\ a^2b^2 + (e_{21} + \Theta e_2^2 + e_{12}\ )\ ab^3 + e_2^2\ b^4 \\ &= \mathbf{1}\ a^4 + \Theta 2\ a^3b + 3\cdot\mathbf{1}\ a^2b^2 + \Theta 2\ ab^3 + \mathbf{1}\ b^4\end{aligned} \tag{18}$$
$$\begin{aligned}(e_1\ a + e_2\ b)^5 &= e_1\ a^5 + (\Theta 2e_1 + e_2)\ a^4b + (3e_1 + \Theta 2e_2)\ a^3b^2 \\ &\quad + (\Theta 2e_1 + 3e_2)\ a^2b^3 + (e_1 + \Theta 2e_2)\ ab^4 + e_2\ b^5\end{aligned} \tag{19}$$
$$(e_1\ a + e_2\ b)^6 = \mathbf{1}\ a^6 + \Theta 3\ a^5b + 6\cdot\mathbf{1}\ a^4b^2 + \Theta 7\ a^3b^3 + 6\cdot\mathbf{1}\ a^2b^4 + \Theta 3\ ab^5 + \mathbf{1}\ b^6 \tag{20}$$

etc...

The coefficients can be arranged as positive (lower) triangle in a Pascal-like way (see figure 2). But there exist three Pascal-like triangles. The second can be found by arranging the coefficients for $n < 0$ and $|a| > |b|$. These coefficients can be found in the same way the coefficients of the negative Pauli triangles are found (see eq. (7a) & (7b) in [2, p. 10]):

$$\begin{aligned}(e_1\ a + e_2\ b)^{-1} &= \frac{e_1a + e_2b}{(e_1a + e_1b)^2} = \frac{e_1a + e_2b}{a^2e_1^{\ 2} + ab(e_{12} + e_{21}) + b^2e_2^{\ 2}} = \frac{e_1a + e_2b}{a^2e_0 + \theta abe_0 + b^2e_0} \\ &= \frac{e_1a + e_2b}{a^2 - ab + b^2} = \frac{e_1a + e_2b}{a^2}\left(\mathbf{1} + \theta\frac{b}{a} + \mathbf{1}\frac{b^2}{a^2}\right)^{-1} \\ &= (e_1\ a^{-1} + e_2\ a^{-2}b)\ (\mathbf{1} + \Theta\ a^{-1}b + \mathbf{1}\ a^{-2}b^2)^{-1}\end{aligned} \tag{21}$$

$a, b \in \mathbb{R}$ are scalars, and the last bracket on the right hand side of eq. (21) can be expanded using the Taylor expansion

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots + \ldots \tag{22}$$

with

$$\begin{aligned}x &= -\ a^{-1}b + a^{-2}b^2 \\ x^2 &= a^{-2}b^2 - 2\ a^{-3}b^3 + a^{-4}b^4 \\ x^3 &= -\ a^{-3}b^3 + 3\ a^{-4}b^4 - 3\ a^{-5}b^5 + a^{-6}b^6 \\ x^4 &= a^{-4}b^4 - 4\ a^{-5}b^5 + 6\ a^{-6}b^6 - 4\ a^{-7}b^7 + a^{-8}b^8\end{aligned} \tag{23}$$

etc...

resulting in

$$\begin{aligned}&(1 - a^{-1}b + a^{-2}b^2)^{-1} \\ &= 1 + a^{-1}b + 0 - a^{-3}b^3 - a^{-4}b^4 + 0 + a^{-6}b^6 + a^{-7}b^7 + 0 - \ldots - \ldots + 0 + \ldots + \ldots + 0 - \ldots\end{aligned} \tag{24}$$

which can be translated into a (3 x 3) matrix equation:

$$\begin{aligned}&(\mathbf{1} + \Theta\ a^{-1}b + \mathbf{1}\ a^{-2}b^2)^{-1} \\ &= \mathbf{1} + \mathbf{1}\ a^{-1}b + \mathbf{O} + \Theta\ a^{-3}b^3 + \Theta\ a^{-4}b^4 + \mathbf{O} + \mathbf{1}\ a^{-6}b^6 + \mathbf{1}\ a^{-7}b^7 + \mathbf{O} + \Theta \ldots + \Theta \ldots + \mathbf{O} + \ldots\end{aligned} \tag{25}$$

Thus we get the matrix equation

$$
\begin{aligned}
(e_1\, a + e_2\, b)^{-1} &= (e_1\, a^{-1} + e_2\, a^{-2}b)\,(\mathbf{1} + \mathbf{1}\, a^{-1}b + 0 + \Theta\, a^{-3}b^3 + \Theta\, a^{-4}b^4 + 0 + \ldots + \ldots + 0 + \ldots) \\
&= e_1\, a^{-1} + e_1\, a^{-2}b + \mathbf{O}\, a^{-3}b^2 + \Theta\, e_1\, a^{-4}b^3 + \Theta\, e_1\, a^{-5}b^4 + \mathbf{O}\, a^{-6}b^5 + e_1\, a^{-7}b^6 + \ldots \\
&\qquad + e_2\, a^{-2}b + e_2\, a^{-3}b^2 + \mathbf{O}\, a^{-4}b^3 + \Theta\, e_2\, a^{-5}b^4 + \Theta\, e_2\, a^{-6}b^5 + \mathbf{O}\, a^{-7}b^6 + \ldots \\
&= e_1\, a^{-1} + (e_1 + e_2)\, a^{-2}b + e_2\, a^{-3}b^2 \\
&\qquad + \Theta\, e_1\, a^{-4}b^3 + \Theta\, (e_1 + e_2)\, a^{-5}b^4 + \Theta\, e_2\, a^{-6}b^5 \qquad (26) \\
&\qquad + \quad e_1 \ldots \quad + \quad (e_1 + e_2) \ldots \quad + \quad e_2 \ldots \\
&\qquad + \Theta\, e_1 \ldots \quad + \Theta\, (e_1 + e_2) \ldots \quad + \Theta\, e_2 \ldots + \ldots
\end{aligned}
$$

We can check this result by multiplying this equation (26) with equation (15):

$$(e_1\, a + e_2\, b)^1\, (e_1\, a + e_2\, b)^{-1} = (e_1\, a + e_2\, b)^0 = \mathbf{1} \qquad (27)$$

In a similar way the following results can be checked too:

$$
\begin{aligned}
(e_1\, a + e_2\, b)^{-2} &= \mathbf{1}\, a^{-2} + \mathbf{1}\, a^{-3}b + \mathbf{O}\, a^{-4}b^2 + \Theta\mathbf{1}\, a^{-5}b^3 + \Theta\mathbf{1}\, a^{-6}b^4 + \ldots && (28) \\
(e_1\, a + e_2\, b)^{-3} &= e_1\, a^{-3} + (2e_1{+}e_2)\, a^{-4}b + (e_1{+}2e_2)\, a^{-5}b^2 + (\Theta 2e_1{+}e_2)\, a^{-6}b^3 + \ldots && (29) \\
(e_1\, a + e_2\, b)^{-4} &= \mathbf{1}\, a^{-4} + 2\cdot\mathbf{1}\, a^{-5}b + \mathbf{1}\, a^{-6}b^2 + \Theta 2\, a^{-7}b^3 + \Theta 4\, a^{-8}b^4 + \ldots && (30) \\
(e_1\, a + e_2\, b)^{-5} &= e_1\, a^{-5} + (3e_1{+}e_2)\, a^{-6}b + (3e_1{+}3e_2)\, a^{-7}b^2 + (\Theta 2e_1{+}3e_2)\, a^{-8}b^3 + \ldots && (31) \\
(e_1\, a + e_2\, b)^{-6} &= \mathbf{1}\, a^{-6} + 3\cdot\mathbf{1}\, a^{-7}b + 3\cdot\mathbf{1}\, a^{-8}b^2 + \Theta 2\, a^{-9}b^3 + \Theta 9\, a^{-10}b^4 + \ldots && (32)
\end{aligned}
$$

etc…

The coefficients of expansions (26) and (28) to (32) form the first negative triangle on the upper right side of figure 2. The second negative triangle on the upper left side is constructed with the coefficients for $n < 0$ and $|a| < |b|$:

$$
\begin{aligned}
(e_1\, a + e_2\, b)^{-1} &= e_2\, a^0b^{-1} + (e_1{+}e_2)\, ab^{-2} + e_1\, a^2b^{-3} + \Theta e_2\, a^3b^{-4} + \Theta(e_1{+}e_2)\, a^4b^{-5} + \ldots && (33) \\
(e_1\, a + e_2\, b)^{-2} &= \mathbf{1}\, a^0b^{-2} + \mathbf{1}\, ab^{-3} + \mathbf{O}\, a^2b^{-4} + \Theta\mathbf{1}\, a^3b^{-5} + \Theta\mathbf{1}\, a^4b^{-6} + \ldots && (34) \\
(e_1\, a + e_2\, b)^{-3} &= e_2\, a^0b^{-3} + (e_1{+}2e_2)\, ab^{-4} + (2e_1{+}e_2)\, a^2b^{-5} + (e_1{+}\Theta 2e_2)\, a^3b^{-6} + \ldots && (35) \\
(e_1\, a + e_2\, b)^{-4} &= \mathbf{1}\, a^0b^{-4} + 2\cdot\mathbf{1}\, ab^{-5} + \mathbf{1}\, a^2b^{-6} + \Theta 2\, a^3b^{-7} + \Theta 4\, a^4b^{-8} + \ldots && (36) \\
(e_1\, a + e_2\, b)^{-5} &= e_2\, a^0b^{-5} + (e_1{+}3e_2)\, ab^{-6} + (3e_1{+}3e_2)\, a^2b^{-7} + (3e_1{+}\,\Theta 2e_2)\, a^3a^{-8} + \ldots && (37) \\
(e_1\, a + e_2\, b)^{-6} &= \mathbf{1}\, a^0b^{-6} + 3\cdot\mathbf{1}\, ab^{-7} + 3\cdot\mathbf{1}\, a^2b^{-8} + \Theta 2\, a^3b^{-9} + \Theta 9\, a^4b^{-10} + \ldots && (38)
\end{aligned}
$$

etc…

Comparing the trinomial triangle (figure 1) with the wrong trinomial triangle (figure 2), we observe, that the integers of every second diagonal line are identical, while the integers of the other diagonal lines have reversed signs.

This can be repaired after having analysed and understood the construction rule for quarkonian Pascal triangles.

6. The Construction Rule for Quarkonian Pascal Triangles

An analysis of these results clearly show that the construction of quarkonian binomial coefficients (abbreviated by QBC in figure 3) follows a recursive pattern. To find a quarkonian binomial coefficient (QBC) in row (n + 1), three steps have to be made.

$\mathbf{1} \qquad\qquad\qquad\qquad \mathbf{1}$

$e_1+5e_2 \quad e_2 \qquad\qquad\qquad\qquad e_1 \quad 5e_1+e_2$

$\mathbf{4} \quad \mathbf{1} \qquad\qquad\qquad\qquad \mathbf{1} \quad \mathbf{4}$

$4e_1+6e_2 \quad e_1+4e_2 \quad e_2 \qquad\qquad\qquad\qquad e_1 \quad 4e_1+e_2 \quad 6e_1+4e_2$

$\mathbf{3} \quad \mathbf{3} \quad \mathbf{1} \qquad\qquad\qquad\qquad \mathbf{1} \quad \mathbf{3} \quad \mathbf{3}$

$3e_1-2e_2 \quad 3e_1+3e_2 \quad e_1+3e_2 \quad e_2 \qquad\qquad\qquad\qquad e_1 \quad 3e_1+e_2 \quad 3e_1+3e_2 \quad -2e_1+3e_2$

$\mathbf{-2} \quad \mathbf{1} \quad \mathbf{2} \quad \mathbf{1} \qquad\qquad\qquad\qquad \mathbf{1} \quad \mathbf{2} \quad \mathbf{1} \quad \mathbf{-2}$

$-2e_1-4e_2 \quad e_1-2e_2 \quad 2e_1+e_2 \quad e_1+2e_2 \quad e_2 \qquad\qquad\qquad\qquad e_1 \quad 2e_1+e_2 \quad e_1+2e_2 \quad -2e_1+e_2 \quad -4e_1-2e_2$

$\mathbf{-1} \quad \mathbf{-1} \quad \mathbf{0} \quad \mathbf{1} \quad \mathbf{1} \qquad\qquad \mathbf{1} \quad \mathbf{1} \quad \mathbf{0} \quad \mathbf{-1} \quad \mathbf{-1}$

$-e_1 \quad -e_1-e_2 \quad -e_2 \quad e_1 \quad e_1+e_2 \quad e_2 \qquad e_1 \quad e_1+e_2 \quad e_2 \quad -e_1 \quad -e_1-e_2 \quad -e_2$

$\mathbf{1}$

$e_1 \quad e_2$

$\mathbf{1} \quad \mathbf{-1} \quad \mathbf{1}$

$e_1 \quad -e_1+e_2 \quad e_1-e_2 \quad e_2$

$\mathbf{1} \quad \mathbf{-2} \quad \mathbf{3} \quad \mathbf{-2} \quad \mathbf{1}$

$e_1 \quad -2e_1+e_2 \quad 3e_1-2e_2 \quad -2e_1+3e_2 \quad e_1-2e_2 \quad e_2$

$\mathbf{1} \quad \mathbf{-3} \quad \mathbf{6} \quad \mathbf{-7} \quad \mathbf{6} \quad \mathbf{-3} \quad \mathbf{1}$

$e_1 \quad -3e_1+e_2 \quad 6e_1-3e_2 \quad -7e_1+6e_2 \quad 6e_1-7e_2 \quad -3e_1+6e_2 \quad e_1-3e_2 \quad e_2$

$\mathbf{1} \quad \mathbf{-4} \quad \mathbf{10} \quad \mathbf{-16} \quad \mathbf{19} \quad \mathbf{-16} \quad \mathbf{10} \quad \mathbf{-4} \quad \mathbf{1}$

$e_1 \quad -4e_1+e_2 \quad 10e_1-4e_2 \quad -16e_1+10e_2 \quad 19e_1-16e_2 \quad -16e_1+19e_2 \quad 10e_1-16e_2 \quad -4e_1+10e_2 \quad e_1-4e_2 \quad e_2$

$\mathbf{1} \quad \mathbf{-5} \quad \mathbf{15} \quad \mathbf{-30} \quad \mathbf{45} \quad \mathbf{-51} \quad \mathbf{45} \quad \mathbf{-30} \quad \mathbf{15} \quad \mathbf{-5} \quad \mathbf{1}$

$e_1 \quad -5e_1+e_2 \quad 15e_1-5e_2 \quad -30e_1+15e_2 \quad 45e_1-30e_2 \quad -51e_1+45e_2 \quad 45e_1-51e_2 \quad -30e_1+45e_2 \quad 15e_1-30e_2 \quad -5e_1+15e_2 \quad e_1-5e_2 \quad e_2$

Fig. 2: First binomial construction of the three trinomial triangles with wrong signs.

These three steps are:

- Multiply the left quarkonian binomial coefficient in row n just above the position of the wanted quarkonian binomial coefficient by the unit vector e_2.
- Multiply the right quarkonian binomial coefficient in row n just above the position of the wanted quarkonian binomial coefficient by the unit vector e_1.
- Add the two geometric products to get the wanted quarkonian binomial coefficient of row $n+1$.

This procedure is shown in figure 3.

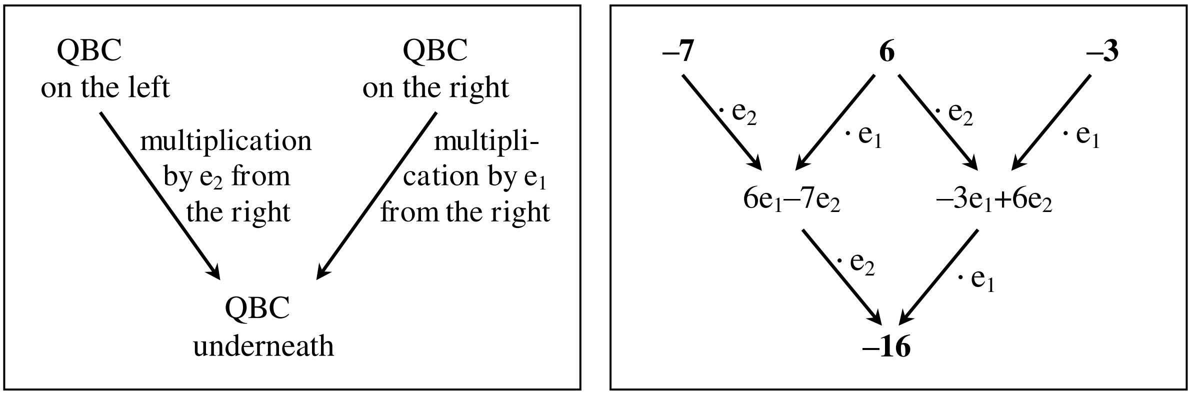

Fig. 3 (a): Construction rule for quarkonian binomial coefficients (QBC),
(b): Example for some quarkonian binomial coefficient constructions.

For example the construction of the “wrong” trinomial coefficient $-\binom{4}{1}_2 = -16$ can be seen as quarkonian binomial coefficient construction by

$$
\begin{aligned}
(\Theta 3e_1 + 6e_2)\, e_1 + (6e_1 + \Theta 7e_2)\, e_2 &= \Theta 3e_0 + 6e_{21} + 6e_{12} + \Theta 7e_0 \\
&= \Theta 3 + 6(e_{21} + e_{12}) + \Theta 7 \\
&= \Theta 3 + \Theta 6 + \Theta 7 \qquad (39) \\
&= \Theta 16 = \begin{pmatrix} 0 & 16 & 16 \\ 16 & 0 & 16 \\ 16 & 16 & 0 \end{pmatrix}
\end{aligned}
$$

This example shows that we get an additional \ominus term of $\Theta 6$ (which can be interpreted as negative integer -6), when the number 6 in figure 3(b) moves down along two different paths. Each number, which reaches the final coefficient on two different routes, is multiplied in two different ways: on the left route by e_{12} and on the right route by e_{21}, resulting in a total multiplication by $(e_{12} + e_{21})$ or Θ.

In figure 3 a multiplication by unit vectors e_1 and e_2 from the right is shown. Of course the final result does not change if instead only multiplications from the left are used.

And another observation is interesting at the “wrong” trinomial triangle of figure 2: The sum of all coefficients of even rows equals **1**, while the sum of all coefficients of odd rows equals $(e_1 + e_2)$. At the “right” trinomial triangle of figure 1, the sums of all coefficients of a row equal powers of three: 1, 3, 9, 27, 81, … To find a binomial equivalent of this correct trinomial triangle, this should be taken into consideration.

7. Quarkonian Binomial Coefficients (second try)

To transform the "wrong" into the correct trinomial triangle, it is necessary to insert an additional \ominus matrix (or minus sign) into the binom of eq. (13). Now we are looking for the binomial expansions of the powers of

$$(e_1\, a + \Theta\, e_2\, b)^n = (e_1\, a + (e_{12} + e_{21})\, e_2\, b)^n = (e_1\, a + (e_1 + e_3)\, b)^n \tag{40}$$

Doing this we change the angle between the two vector parts of our binom. This angle no longer is 0° like in eq. (11) or 90° like in eq. (12) or 120° like in eq. (13). The angle between $e_1 a$ and $\Theta e_2 b = (e_1 + e_3) b$ now equals 60°.

The expansions for $n \geq 0$ are:

$$(e_1\, a + \Theta e_2\, b)^0 = \mathbf{1} \tag{41}$$

$$(e_1\, a + \Theta e_2\, b)^1 = e_1\, a + \Theta e_2\, b \tag{42}$$

$$\begin{aligned}(e_1\, a + \Theta e_2\, b)^2 &= e_1^2\, a^2 + \Theta(e_{12} + e_{21})\, ab + (\Theta e_2)^2\, b^2 \\ &= \mathbf{1}\, a^2 + \mathbf{1}\, ab + \mathbf{1}\, b^2\end{aligned} \tag{43}$$

$$(e_1\, a + \Theta e_2\, b)^3 = e_1\, a^3 + (e_1 + \Theta e_2)\, a^2 b + (e_1 + \Theta e_2)\, ab^2 + \Theta e_2\, b^3 \tag{44}$$

$$\begin{aligned}(e_1\, a + \Theta e_2\, b)^4 &= e_1^2\, a^4 + (\Theta e_{12} + e_1^2 + \Theta e_{21}\,)\, a^3 b \\ &\quad + (\Theta e_{12} + e_2^2 + e_1^2 + \Theta e_{21})\, a^2 b^2 + (\Theta e_{21} + e_2^2 + \Theta e_{12}\,)\, ab^3 + e_2^2\, b^4 \\ &= \mathbf{1}\, a^4 + 2 \cdot \mathbf{1}\, a^3 b + 3 \cdot \mathbf{1}\, a^2 b^2 + 2 \cdot \mathbf{1}\, ab^3 + \mathbf{1}\, b^4\end{aligned} \tag{45}$$

$$\begin{aligned}(e_1\, a + \Theta e_2\, b)^5 &= e_1\, a^5 + (2e_1 + \Theta e_2)\, a^4 b + (3e_1 + \Theta 2e_2)\, a^3 b^2 \\ &\quad + (2e_1 + \Theta 3e_2)\, a^2 b^3 + (e_1 + \Theta 2e_2)\, ab^4 + \Theta e_2\, b^5\end{aligned} \tag{46}$$

$$(e_1\, a + e_2\, b)^6 = \mathbf{1}\, a^6 + 3 \cdot \mathbf{1}\, a^5 b + 6 \cdot \mathbf{1}\, a^4 b^2 + 7 \cdot \mathbf{1}\, a^3 b^3 + 6 \cdot \mathbf{1}\, a^2 b^4 + 3 \cdot \mathbf{1}\, ab^5 + \mathbf{1}\, b^6 \tag{47}$$

etc...

The coefficients can again be arranged as positive (lower) triangle in a Pascal-like way (see figure 4). And again the second Pascal-like triangle can be found by arranging the coefficients for $n < 0$ and $|a| > |b|$ using the following expansion:

$$\begin{aligned}(e_1\, a + \Theta e_2\, b)^{-1} &= \frac{e_1 a + \theta e_2 b}{(e_1 a + \theta e_1 b)^2} = \frac{e_1 a + \theta e_2 b}{a^2 e_1{}^2 + \theta ab(e_{12} + e_{21}) + (\theta b)^2 e_2{}^2} \\ &= \frac{e_1 a + \theta e_2 b}{a^2 e_0 + abe_0 + b^2 e_0} = \frac{e_1 a + \theta e_2 b}{a^2 + ab + b^2} = \frac{e_1 a + \theta e_2 b}{a^2}\left(\mathbf{1} + \mathbf{1}\frac{b}{a} + \mathbf{1}\frac{b^2}{a^2}\right)^{-1} \\ &= (e_1\, a^{-1} + \Theta e_2\, a^{-2} b)\, (\mathbf{1} + a^{-1} b + \mathbf{1}\, a^{-2} b^2)^{-1}\end{aligned} \tag{48}$$

a, b $\in \mathbb{R}$ are scalars, and the last bracket on the right hand side of eq. (48) can be expanded again with the help of the Taylor expansion (22), but this time with

$$x = a^{-1} b + a^{-2} b^2 \tag{49}$$

This results in

$$\begin{aligned}&(1 + a^{-1} b + a^{-2} b^2)^{-1} \\ &\quad = 1 - a^{-1} b + 0 + a^{-3} b^3 - a^{-4} b^4 + 0 + a^{-6} b^6 - a^{-7} b^7 + 0 + \ldots - \ldots + 0 + \ldots\end{aligned} \tag{50}$$

which again can be translated into a (3 x 3) matrix equation:

$$(\mathbf{1} + \mathbf{1}\ a^{-1}b + \mathbf{1}\ a^{-2}b^2)^{-1} \tag{51}$$
$$= \mathbf{1} + \Theta\ a^{-1}b + \mathbf{O} + \mathbf{1}\ a^{-3}b^3 + \Theta\ a^{-4}b^4 + \mathbf{O} + \mathbf{1}\ a^{-6}b^6 + \Theta\ a^{-7}b^7 + \mathbf{O} + \ldots + \Theta \ldots + \mathbf{O} + \ldots$$

Thus we get the matrix equations

$$\begin{aligned}(e_1\ a + \Theta e_2\ b)^{-1} &= (e_1\ a^{-1} + \Theta e_2\ a^{-2}b)\ (\mathbf{1} + \Theta\ a^{-1}b + \mathbf{O} + \mathbf{1}\ a^{-3}b^3 + \Theta\ a^{-4}b^4 + \mathbf{O} + \ldots - \ldots)\\
&= e_1\ a^{-1} + \Theta\ e_1\ a^{-2}b + \mathbf{O}\ a^{-3}b^2 + e_1\ a^{-4}b^3 + \Theta\ e_1\ a^{-5}b^4 + \mathbf{O}\ a^{-6}b^5 + e_1\ a^{-7}b^6 + \Theta \ldots\\
&\qquad + \Theta\ e_2\ a^{-2}b + e_2\ a^{-3}b^2 + \mathbf{O}\ a^{-4}b^3 + \Theta\ e_2\ a^{-5}b^4 + e_2\ a^{-6}b^5 + \mathbf{O}\ a^{-7}b^6 + \Theta \ldots\\
&= e_1\ a^{-1} + \Theta(e_1 + e_2)\ a^{-2}b + e_2\ a^{-3}b^2\\
&\qquad + e_1\ a^{-4}b^3 + \Theta(e_1 + e_2)\ a^{-5}b^4 + e_2\ a^{-6}b^5\\
&\qquad + e_1 \ldots \quad + \Theta(e_1 + e_2) \ldots \quad + e_2 \ldots\end{aligned} \tag{52}$$

$$(e_1\ a + \Theta e_2\ b)^{-2} = \mathbf{1}\ a^{-2} + \Theta\mathbf{1}\ a^{-3}b + \mathbf{O}\ a^{-4}b^2 + \mathbf{1}\ a^{-5}b^3 + \Theta\mathbf{1}\ a^{-6}b^4 + \ldots \tag{53}$$
$$(e_1\ a + \Theta e_2\ b)^{-3} = e_1\ a^{-3} + (\Theta 2e_1 + \Theta e_2)\ a^{-4}b + (e_1 + 2e_2)\ a^{-5}b^2 + (2e_1 + \Theta e_2)\ a^{-6}b^3 + \ldots \tag{54}$$
$$(e_1\ a + \Theta e_2\ b)^{-4} = \mathbf{1}\ a^{-4} + \Theta 2\ a^{-5}b + \mathbf{1}\ a^{-6}b^2 + 2 \cdot \mathbf{1}\ a^{-7}b^3 + \Theta 4\ a^{-8}b^4 + \ldots \tag{55}$$
$$(e_1\ a + \Theta e_2\ b)^{-5} = e_1\ a^{-5} + (\Theta 3e_1 + \Theta e_2)\ a^{-6}b + (3e_1 + 3e_2)\ a^{-7}b^2 + (2e_1 + \Theta 3e_2)\ a^{-8}b^3 + \ldots \tag{56}$$
$$(e_1\ a + \Theta e_2\ b)^{-6} = \mathbf{1}\ a^{-6} + \Theta 3\ a^{-7}b + 3 \cdot \mathbf{1}\ a^{-8}b^2 + 2 \cdot \mathbf{1}\ a^{-9}b^3 + \Theta 9\ a^{-10}b^4 + \ldots \tag{57}$$

etc…

The coefficients of equations (52) to (57) form the first negative triangle on the upper right side of figure 2. The second negative triangle on the upper left side is constructed with the coefficients for $n < 0$ and $|a| < |b|$:

$$(e_1\ a + \Theta e_2\ b)^{-1} = \Theta e_2\ a^0b^{-1} + (e_1 + e_2)\ ab^{-2} + \Theta e_1\ a^2b^{-3} + \Theta e_2\ a^3b^{-4} + (e_1 + e_2)\ a^4b^{-5} + \ldots \tag{58}$$
$$(e_1\ a + \Theta e_2\ b)^{-2} = \mathbf{1}\ a^0b^{-2} + \Theta\mathbf{1}\ ab^{-3} + \mathbf{O}\ a^2b^{-4} + \mathbf{1}\ a^3b^{-5} + \Theta\mathbf{1}\ a^4b^{-6} + \ldots \tag{59}$$
$$(e_1\ a + \Theta e_2\ b)^{-3} = \Theta e_2\ a^0b^{-3} + (e_1 + 2e_2)\ ab^{-4} + (\Theta 2e_1 + \Theta e_2)\ a^2b^{-5} + (e_1 + \Theta 2e_2)\ a^3b^{-6} + \ldots \tag{60}$$
$$(e_1\ a + \Theta e_2\ b)^{-4} = \mathbf{1}\ a^0b^{-4} + \Theta 2\ ab^{-5} + \mathbf{1}\ a^2b^{-6} + 2 \cdot \mathbf{1}\ a^3b^{-7} + \Theta 4\ a^4b^{-8} + \ldots \tag{61}$$
$$(e_1\ a + \Theta e_2\ b)^{-5} = \Theta e_2\ a^0b^{-5} + (e_1 + 3e_2)\ ab^{-6} + (\Theta 3e_1 + \Theta 3e_2)\ a^2b^{-7} + (3e_1 + \Theta 2e_2)\ a^3a^{-8} + \ldots \tag{62}$$
$$(e_1\ a + \Theta e_2\ b)^{-6} = \mathbf{1}\ a^0b^{-6} + \Theta 3\ ab^{-7} + 3 \cdot \mathbf{1}\ a^2b^{-8} + 2 \cdot \mathbf{1}\ a^3b^{-9} + \Theta 9\ a^4b^{-10} + \ldots \tag{63}$$

etc…

What is the meaning of these binomial and trinomial numbers? One explanation is a probabilistic explanation: If we have to decide which one of two possible choices to follow, we can throw a coin.

But if we have to decide which one of three possible choices to follow, there are two possible strategies to make a probabilistic decision: We could throw a three-sided coin. Or we can throw a quarkonian coin two times. Both strategies will work, and perhaps nature works with quarkonian coins.

As we presently have problems in measuring quarkonian probabilities after one, five or seven throws of quarkonian coins, we do not know whether nature actually uses quarkonian coins. But one day we will know, as there are some measurable probabilities of 1/3 after three coin throws (see red probabilities in figure 5). Let's look for them.

$$\mathbf{1} \qquad\qquad \mathbf{1}$$

$$e_1-5e_2 \quad e_2 \qquad\qquad e_1 \quad -5e_1-e_2$$

$$\mathbf{-4} \quad \mathbf{1} \qquad\qquad \mathbf{1} \quad \mathbf{-4}$$

$$-4e_1+6e_2 \quad e_1-4e_2 \quad e_2 \qquad\qquad e_1 \quad -4e_1-e_2 \quad 6e_1+4e_2$$

$$\mathbf{3} \quad \mathbf{-3} \quad \mathbf{1} \qquad\qquad \mathbf{1} \quad \mathbf{-3} \quad \mathbf{3}$$

$$3e_1-2e_2 \quad -3e_1-3e_2 \quad e_1+3e_2 \quad -e_2 \qquad\qquad e_1 \quad -3e_1-e_2 \quad 3e_1+3e_2 \quad 2e_1-3e_2$$

$$\mathbf{2} \quad \mathbf{1} \quad \mathbf{-2} \quad \mathbf{1} \qquad\qquad \mathbf{1} \quad \mathbf{-2} \quad \mathbf{1} \quad \mathbf{2}$$

$$2e_1+4e_2 \quad e_1-2e_2 \quad -2e_1-e_2 \quad e_1+2e_2 \quad -e_2 \qquad\qquad e_1 \quad -2e_1-e_2 \quad e_1+2e_2 \quad 2e_1-e_2 \quad -4e_1-2e_2$$

$$\mathbf{-1} \quad \mathbf{1} \quad \mathbf{0} \quad \mathbf{-1} \quad \mathbf{1} \qquad\qquad \mathbf{1} \quad \mathbf{-1} \quad \mathbf{0} \quad \mathbf{1} \quad \mathbf{-1}$$

$$-e_1 \quad e_1+e_2 \quad -e_2 \quad -e_1 \quad e_1+e_2 \quad -e_2 \quad e_1 \quad -e_1-e_2 \quad e_2 \quad e_1 \quad -e_1-e_2 \quad e_2$$

$$\mathbf{1}$$

$$e_1 \quad -e_2$$

$$\mathbf{1} \quad \mathbf{1} \quad \mathbf{1}$$

$$e_1 \quad e_1-e_2 \quad e_1-e_2 \quad -e_2$$

$$\mathbf{1} \quad \mathbf{2} \quad \mathbf{3} \quad \mathbf{2} \quad \mathbf{1}$$

$$e_1 \quad 2e_1-e_2 \quad 3e_1-2e_2 \quad 2e_1-3e_2 \quad e_1-2e_2 \quad -e_2$$

$$\mathbf{1} \quad \mathbf{3} \quad \mathbf{6} \quad \mathbf{7} \quad \mathbf{6} \quad \mathbf{3} \quad \mathbf{1}$$

$$e_1 \quad 3e_1-e_2 \quad 6e_1-3e_2 \quad 7e_1-6e_2 \quad 6e_1-7e_2 \quad 3e_1-6e_2 \quad e_1-3e_2 \quad -e_2$$

$$\mathbf{1} \quad \mathbf{4} \quad \mathbf{10} \quad \mathbf{16} \quad \mathbf{19} \quad \mathbf{16} \quad \mathbf{10} \quad \mathbf{4} \quad \mathbf{1}$$

$$e_1 \quad 4e_1-e_2 \quad 10e_1-4e_2 \quad 16e_1-10e_2 \quad 19e_1-16e_2 \quad 16e_1-19e_2 \quad 10e_1-16e_2 \quad 4e_1-10e_2 \quad e_1-4e_2 \quad -e_2$$

$$\mathbf{1} \quad \mathbf{5} \quad \mathbf{15} \quad \mathbf{30} \quad \mathbf{45} \quad \mathbf{51} \quad \mathbf{45} \quad \mathbf{30} \quad \mathbf{15} \quad \mathbf{5} \quad \mathbf{1}$$

$$e_1 \quad 5e_1-e_2 \quad 15e_1-5e_2 \quad 30e_1-15e_2 \quad 45e_1-30e_2 \quad 51e_1-45e_2 \quad 45e_1-51e_2 \quad 30e_1-45e_2 \quad 15e_1-30e_2 \quad 5e_1-15e_2 \quad e_1-5e_2 \quad -e_2$$

Fig. 4: Second try of a binomial construction of the three trinomial triangles with correct signs.

No throw:

$$e_0(e_1+\Theta e_2)^0 = \mathbf{100\ \%}$$

First throw:

$$e_1(e_1+\Theta e_2)^{-1} = \frac{1}{3}e_0 + \Theta\frac{1}{3}e_{12} \qquad \Theta e_2(e_1+\Theta e_2)^{-1} = \frac{1}{3}e_0 + \Theta\frac{1}{3}e_{21}$$

Second throw:

$$e_0(e_1+\Theta e_2)^{-2} = \frac{1}{3} \approx \mathbf{33.33\ \%} \qquad e_0(e_1+\Theta e_2)^{-2} = \frac{1}{3} \approx \mathbf{33.33\ \%} \qquad e_0(e_1+\Theta e_2)^{-2} = \frac{1}{3} \approx \mathbf{33.33\ \%}$$

Third throw:

$$e_1(e_1+\Theta e_2)^{-3} = \frac{1}{9}e_0 + \Theta\frac{1}{9}e_{12} \qquad (e_1+\Theta e_2)(e_1+\Theta e_2)^{-3} = \frac{1}{3} \approx \mathbf{33.33\ \%} \qquad (e_1+\Theta e_2)(e_1+\Theta e_2)^{-3} = \frac{1}{3} \approx \mathbf{33.33\ \%} \qquad \Theta e_2(e_1+\Theta e_2)^{-3} = \frac{1}{9}e_0 + \Theta\frac{1}{9}e_{21}$$

Fourth throw:

$$e_0(e_1+\Theta e_2)^{-4} = \frac{1}{9} \approx \mathbf{11.11\ \%} \qquad 2e_0(e_1+\Theta e_2)^{-4} = \frac{2}{9} \approx \mathbf{22.22\ \%} \qquad 3e_0(e_1+\Theta e_2)^{-4} = \frac{1}{3} \approx \mathbf{33.33\ \%} \qquad 2e_0(e_1+\Theta e_2)^{-4} = \frac{2}{9} \approx \mathbf{22.22\ \%} \qquad e_0(e_1+\Theta e_2)^{-4} = \frac{1}{9} \approx \mathbf{11.11\ \%}$$

$$e_1(e_1+\Theta e_2)^{-5} = \frac{1}{27}e_0+\Theta\frac{1}{27}e_{12} \qquad (2e_1+\Theta e_2)(e_1+\Theta e_2)^{-5} = \frac{4}{27}e_0+\Theta\frac{1}{27}e_{12} \qquad (3e_1+\Theta 2e_2)(e_1+\Theta e_2)^{-5} = \frac{7}{27}e_0+\Theta\frac{1}{27}e_{12}$$

$$(2e_1+\Theta 3e_2)(e_1+\Theta e_2)^{-5} = \frac{7}{27}e_0+\Theta\frac{1}{27}e_{21} \qquad (e_1+\Theta 2e_2)(e_1+\Theta e_2)^{-5} = \frac{4}{27}e_0+\Theta\frac{1}{27}e_{21} \qquad \Theta e_2(e_1+\Theta e_2)^{-5} = \frac{1}{27}e_0+\Theta\frac{1}{27}e_{21}$$

$$e_0(e_1+\Theta e_2)^{-6} = \frac{1}{27} \approx \mathbf{3.70\ \%} \qquad 3e_0(e_1+\Theta e_2)^{-6} = \frac{1}{9} \approx \mathbf{11.11\ \%} \qquad 6e_0(e_1+\Theta e_2)^{-6} = \frac{2}{9} \approx \mathbf{22.22\ \%} \qquad 7e_0(e_1+\Theta e_2)^{-6} = \frac{7}{27} \approx \mathbf{25.93\ \%}$$

$$6e_0(e_1+\Theta e_2)^{-6} = \frac{2}{9} \approx \mathbf{22.22\ \%} \qquad 3e_0(e_1+\Theta e_2)^{-6} = \frac{1}{9} \approx \mathbf{11.11\ \%} \qquad e_0(e_1+\Theta e_2)^{-6} = \frac{1}{27} \approx \mathbf{3.70\ \%}$$

Fig. 5: Probabilities when throwing a fair quarkonian coin several times.

8. Outlook

In theoretical physics we can never be sure whether the reality we find outside us is something which comes directly from nature or whether it is something we have invented when inventing mathematics. There might be some conceptual consequences of Taylor expanding formulae in physics. We all live in Pascal space because human theoretical physicists use the Taylor expansion.

The same thing might apply to quarkonian Taylor expansions. We will start living in a quarkonian Pascal space when human theoretical physicists start Taylor expanding expressions of geometric algebra of quarks.

And we can use this quarkonian Taylor expansion in more situations. The next natural step will be to construct a trinomial pyramid. Instead of expanding the binom of eq. (40) we then have to expand the trinom

$$\left(e_1\, a + (e_1 + e_3)\, b + \frac{1}{3}(2\,e_1 + e_3 + \sqrt{6}\, e_4)\, c\right)^n \tag{64}$$

But this is another story.

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10. Attachment: Quarkonian Fibonacci Numbers

Fibonacci numbers are found by adding all the numbers of shallow diagonals of a Pascal triangle. The shallow diagonals of the quarkonian Pascal triangle and their corresponding quarkonian Fibonacci numbers are indicated in figure 6 by different colours. The quarkonian Fibonacci numbers thus are:

$$\left.\begin{array}{ll}
F_{-9} = \Theta 2 + 2e_1 + \Theta e_2 & F_1 = \mathbf{O} + e_1 \\
F_{-8} = \mathbf{O} + \Theta 2e_1 + \Theta 2e_2 & F_2 = \mathbf{1} + \Theta e_2 \\
F_{-7} = \mathbf{1} + 2e_2 & F_3 = \mathbf{1} + e_1 \\
F_{-6} = \mathbf{O} + e_1 & F_4 = 2 \cdot \mathbf{1} + e_1 + \Theta e_2 \\
F_{-5} = \Theta\mathbf{1} + \Theta e_2 & F_5 = 2 \cdot \mathbf{1} + 2e_1 + \Theta e_2 \\
F_{-4} = \mathbf{1} + \Theta e_1 & F_6 = 4 \cdot \mathbf{1} + 2e_1 + \Theta 2e_2 \\
F_{-3} = \mathbf{O} + e_1 + e_2 & F_7 = 5 \cdot \mathbf{1} + 4e_1 + \Theta 2e_2 \\
F_{-2} = \mathbf{O} + \Theta e_2 & F_8 = 8 \cdot \mathbf{1} + 5e_1 + \Theta 4e_2 \\
F_{-1} = \mathbf{O} & F_9 = 11 \cdot \mathbf{1} + 8e_1 + \Theta 5e_2 \\
F_0 = \mathbf{1} & F_{10} = 17 \cdot \mathbf{1} + 11e_1 + \Theta 8e_2
\end{array}\right\} \quad (65)$$

Of course these quarkonian Fibonacci numbers are (3 x 3) matrices. Geometrically they are linear combinations of the unit scalar **1** and the unit vectors e_1 and e_2.

While ordinary Fibonacci numbers satisfy the recurrence relations

$$F_n = F_{n-1} + F_{n-2} \qquad \text{(for increasing n)} \qquad (66)$$

$$F_n = -F_{n+1} + F_{n+2} \qquad \text{(for decreasing n)} \qquad (67)$$

quarkonian Fibonacci numbers satisfy the following recurrence relations:

$$F_n = e_1\, F_{n-1} + \Theta e_2\, F_{n-2} \qquad \text{(for increasing n)} \qquad (68)$$

$$F_n = e_{21}\, F_{n+1} + \Theta e_2\, F_{n+2} \qquad \text{(for decreasing n)} \qquad (69)$$

or

$$F_n = F_{n-1}\, e_1 + F_{n-2}\, \Theta e_2 \qquad \text{(for increasing n)} \qquad (70)$$

$$F_n = F_{n+1}\, e_{12} + F_{n+2}\, \Theta e_2 \qquad \text{(for decreasing n)} \qquad (71)$$

For example, F_{-10} should be:

$$\begin{aligned}
F_{-10} &= e_{21}\,(\Theta 2 + 2e_1 + \Theta e_2) + \Theta e_2\,(\Theta 2e_1 + \Theta 2e_2) \\
&= \Theta 2e_{21} + 2e_2 + \Theta e_3 + 2e_{21} + 2 \cdot \mathbf{1} \\
&= 2 \cdot \mathbf{1} + e_1 + 3e_2
\end{aligned} \qquad (72)$$

Unfortunately a Binet-like formula is still missing.

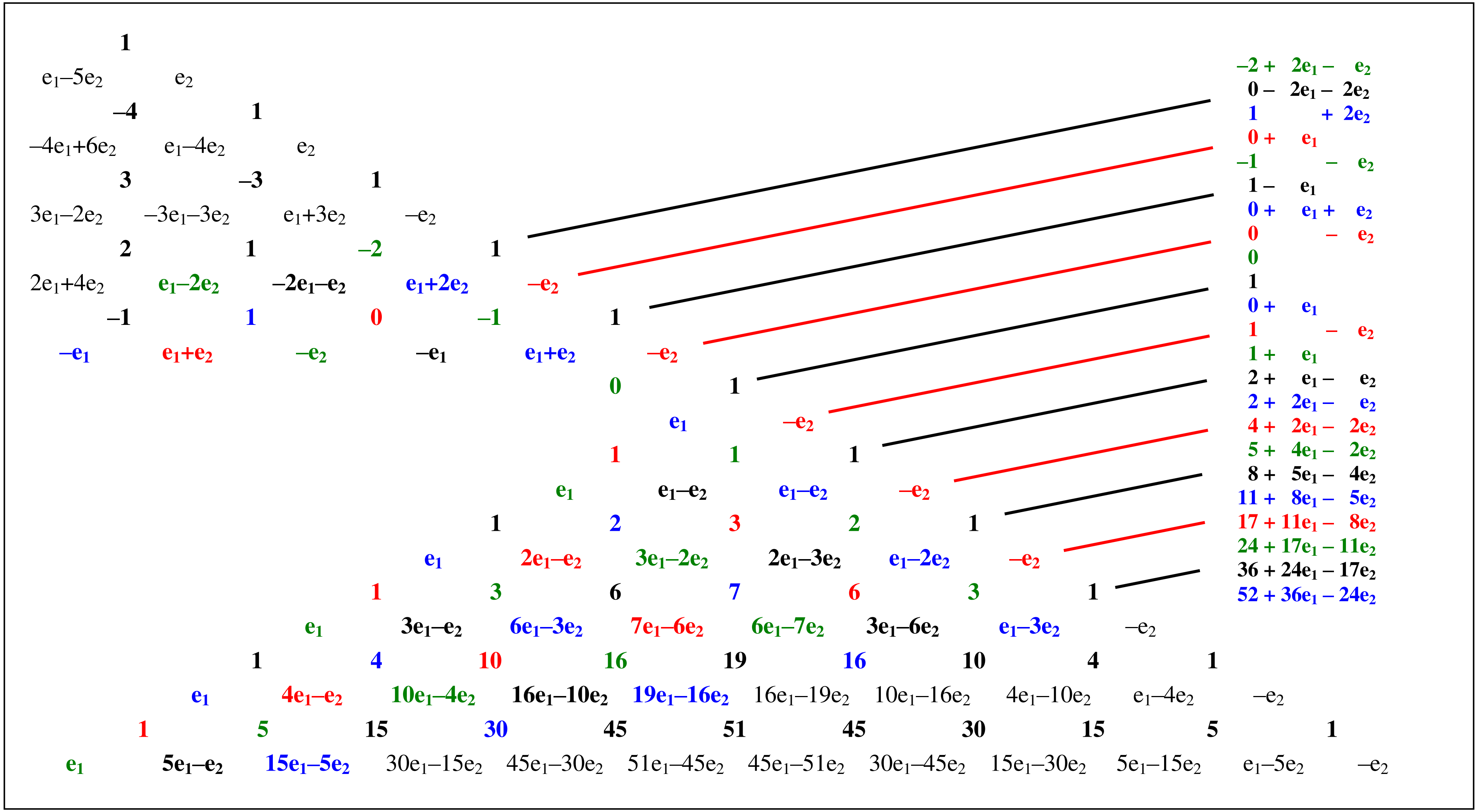

Fig. 6: Construction of quarkonian Fibonacci numbers.