Inconsistency of magnetic monopole

Ali R. Hadjesfandiari

Department of Mechanical and Aerospace Engineering

State University of New York at Buffalo

Buffalo, NY 14260 USA

ah@buffalo.edu

October 23, 2012

It is simply proved that the Hamiltonian for an electric point charge interacting with a fixed magnetic monopole does not exist. This shows that the concept of magnetic monopole is inconsistent within the theory of electrodynamics.

1. Introduction

The inconsistency of magnetic monopole has already been demonstrated in the literature from several different aspects. For example, Zwanziger[1)] and Weinberg[2)] demonstrate that the existence of a magnetic monopole is inconsistent with the *S* matrix theory, and Hagen[3)] shows that the inclusion of a magnetic monopole in electrodynamics is inconsistent with relativistic covariance. However, in the present article, this inconsistency is demonstrated at a more classical level by using the correct potential representing the field of the magnetic monopole.

It has already been shown that the magnetic field of a magnetic monopole, if it is ever found in nature, must be represented by a scalar potential.[4,5)] This is simply the result of the celebrated Helmholtz decomposition theorem. Attempting to utilize a vector potential for this representation violates many fundamental aspects of mathematics. In the course of this work, one appreciates the importance of distribution theory to prevent such mathematical errors. Therefore, considering a non-Euclidean geometry or using fiber bundle theory to justify the use of vector potential is in vein. Dirac has derived his magnetic charge quantization for magnetic monopole by using the Hamiltonian for an electric charge interacting with the field of a fixed magnetic monopole.[6] However, he has used an invalid vector potential instead of a scalar potential to represent the magnetic monopole field.

Interestingly, our argument regarding to the scalar potential can be used to show that the concept of magnetic monopole is inconsistent. This result can be demonstrated by showing that the Hamiltonian for the system used by Dirac cannot be found. This contradiction simply shows that magnetic monopoles do not exist. It has to be appreciated that the magnetic field is only the result of moving electric charges and the vector potential is the only potential representing the magnetic field.

In Section 2 we demonstrate the impossibility of the Hamiltonian for the system of an electric charge in the field of a fixed magnetic monopole. Afterwards, in Section 3, we show that Dirac has actually used a semi-infinite long thin solenoid (magnet) to obtain his magnetic charge quantization, which has nothing to do with a pure magnetic charge. Finally, Section 4 contains a summary and conclusion.

2. Impossibility of Hamiltonian for the system of magnetic and electric charges

Suppose, at the origin, there is a fixed point magnetic monopole of strength q_m. Therefore, in SI units

$$\nabla \bullet \mathbf{B} = \mu_0 q_m \delta^{(3)}(\mathbf{x}) \tag{1}$$

and the static magnetic field is then given by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\hat{\mathbf{r}} \tag{2}$$

Based on the Helmholtz decomposition theorem,[4,5] this field can only be represented by the scalar potential

$$\phi_m(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{q_m}{r} \tag{3}$$

where

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\hat{\mathbf{r}} = -\nabla\phi_m \tag{4}$$

analogous to the theory of electrostatics. In the theory of stationary magnetic charges, it is assumed that the stationary monopoles interact with Coulomb-like forces. This means that two monopoles with magnetic charges q_{m1} and q_{m2} at positions $\mathbf{x}_1$ and $\mathbf{x}_2$ interact with forces

$$\mathbf{F}_{12} = -\mathbf{F}_{21} = \frac{\mu_0}{4\pi}\frac{q_{m1}q_{m2}}{\left|\mathbf{x}_2 - \mathbf{x}_1\right|^3}\left(\mathbf{x}_2 - \mathbf{x}_1\right) \tag{5}$$

Therefore, in general a distribution of stationary magnetic charges generates a magnetic field $\mathbf{B}$ which can be represented by the scalar potential ϕ_m such that the force on a test monopole q_m at $\mathbf{x}$ is represented by

$$\mathbf{F} = q_m\mathbf{B}(\mathbf{x}) = -q_m\nabla\phi_m \tag{6}$$

It is obvious that the quantity

$$V_m(\mathbf{x}) = q_m\phi_m(\mathbf{x}) \tag{7}$$

must be considered as the magnetic potential energy for this monopole q_m in the magnetic field generated by the other fixed magnetic charges. Therefore obtaining a Hamiltonian for this particle is straightforward. It can be simply written as

$$H = \frac{1}{2m_m}\mathbf{p}_m{}^2 + q_m\phi_m \tag{8}$$

where m_m and $\mathbf{p}_m$ are the mass and momentum of the magnetic monopole, respectively. It is seen that we have a copy of interacting electric charges for our interacting magnetic charges. However, one may ask if nature needs to have electric-like monopoles. To answer this question we show that nature only allows one of them to exist, the electric charge. This is established by the inability to describe properly the interaction of a fixed magnetic charge with a moving electric charge. Interestingly, this is the same system used by Dirac to derive his magnetic charge quantization by postulating a Hamiltonian. It is simply demonstrated that the Hamiltonian describing the interacting system cannot exist.

Consider an electric point charge q interacting with a stationary magnetic monopole q_m. By stationary we mean that we somehow constrain the monopole from moving at the origin. The force on the electric charge is the Lorentz force

$$\mathbf{F}_q = q\mathbf{v} \times \mathbf{B} \tag{9}$$

where $\mathbf{v}$ is the velocity of the electric charge and $\mathbf{B}$ is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{q_m}{r^2} \hat{\mathbf{r}} \tag{2}$$

Let us look for the Hamiltonian describing this interaction by assuming its existence. We know that the Hamiltonian of this electric charge q with mass m and momentum $\mathbf{p}$ in an electromagnetic field is generally represented by[7)]

$$H = \frac{1}{2m}(\mathbf{p} - \frac{q}{c}\mathbf{A})^2 + q\phi \tag{10}$$

where ϕ and $\mathbf{A}$ are the scalar and vector potentials representing the field, respectively. If there are also other fields generating additional potential energy V for the electric charge, the Hamiltonian becomes

$$H = \frac{1}{2m}(\mathbf{p} - \frac{q}{c}\mathbf{A})^2 + q\phi + V \tag{11}$$

However, the field of fixed magnetic monopole can only be represented by the scalar potential ϕ_m. Thus, there are no potentials $\mathbf{A}$ and ϕ generated by the fixed magnetic monopole. As a result, we are left with

$$H = \frac{1}{2m}\mathbf{p}^2 + V \tag{12}$$

Nevertheless, we cannot define a potential energy V for the electric charge because the force

$$\mathbf{F}_q = q\mathbf{v} \times \mathbf{B} = -q\mathbf{v} \times \nabla\phi_m \tag{13}$$

acting on electric charge is not a potential force and the quantity $q\phi_m$ is not a potential energy. Therefore, it is impossible to define the general Hamiltonian for the system. This is a contradiction, which shows that the system of a fixed magnetic charge interacting with a moving electric charge does not exist. Therefore, either electric charge or magnetic charge can exist, but not both. We know that there exist particles with electric charge. Thus, the magnetic monopole cannot exist.

Furthermore, it has to be admitted that Maxwell's theory of electrodynamics is complete with electric charges. There is no place for the magnetic charges in electrodynamics. Insisting on the possible existence of magnetic monopoles violates fundamentals of electrodynamics.

3. Origin of magnetic monopole concept

It has been shown previously that the concept of magnetic monopole is the result of the pecular magnetic field of a thin solenoid (magnet).[4,5] Consider a very thin solenoid with length L and uniform magnetic moment per unit length M, which is placed along the z-axis as shown in Figure 1. The magnetic field is given by[4,5]

$$\mathbf{B(x)} = \frac{\mu_0}{4\pi}\frac{M}{r^2}\hat{\mathbf{r}} - \frac{\mu_0}{4\pi}\frac{M}{r_2^2}\hat{\mathbf{r}}_2 \qquad \mathbf{x} \notin OA \tag{14}$$

This may appear as if there were two point magnetic poles with charges q_m, and $-q_m$ at points O and A, where

$$q_m = M \tag{15}$$

However, we should realize that this is only an interesting mathematical result governing the physical phenomenon. The similarity to an electric dipole should not be misleading. We just have a nice mathematical result and concluding the possible existence of a magnetic monopole is not correct because the magnetic field **B(x)** is not defined on the axis of the solenoid by (14). This field can be represented only by the vector potential

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{q_m}{r\sin\theta}(\cos\theta_2 - \cos\theta)\hat{\boldsymbol{\varphi}} \qquad \mathbf{x} \notin OA \tag{16}$$

where

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}) \qquad \mathbf{x} \notin OA \tag{17}$$

We have emphasized that the vector potential **A** and the magnetic field **B** are not defined on the axis of the solenoid by the expressions (16) and (17).

Interestingly, it is seen that for the case of a semi-infinite solenoid (magnet) where $L \to \infty$ ($\theta_2 \to 0$), we have

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{q_m (1-\cos\theta)}{r\sin\theta} \hat{\boldsymbol{\varphi}} \qquad \theta \neq \pi \tag{18}$$

and

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{q_m}{r^2} \hat{\mathbf{r}} \qquad \theta \neq \pi \tag{19}$$

As before $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are not defined along the negative z-axis. This is because the magnet or solenoid is laid on this axis, which represents the distribution of the source current. Strangely enough, these are exactly the same vector fields used by Dirac to represent a monopole field. However, a real monopole should generate an isotropic spherical field

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{q_m}{r^2} \hat{\mathbf{r}} \tag{20}$$

which is not equivalent to (19). As we mentioned this magnetic field can only be represented by the scalar potential

$$\phi_m(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{q_m}{r} \tag{21}$$

Attempting to use the results (18-19) for a semi-infinite solenoid to represent the field of a magnetic monopole obviously is not mathematically valid. These vector fields are based on current generating magnetostatics for the semi-infinite solenoid, which has nothing to do with a pure monopole. The line of singularity has a physical meaning for the solenoid (magnet), but it has been artificially created for monopole by using the vector potential (18). Dirac's derivation[6)] is based on the single valuedness of the wave function of the interacting electric charge around this singular string. However, we have proved in Section 2 that this derivation is not valid because the Hamiltonian for the system does not exist.

For clarification we give a simple example regarding Newtonian gravity. Consider the gravitational field of a homogeneous sphere with mass m and radius R. The gravitational field in spherical coordinates is given by

$$\mathbf{g} = \begin{cases} -\dfrac{Gm}{R^3} r\hat{\mathbf{r}} & r < R \\ -\dfrac{Gm}{r^2}\hat{\mathbf{r}} & r \geq R \end{cases} \tag{22}$$

where the coordinate origin is most conveniently chosen at the center of the sphere. It is seen that for $r \geq R$ the gravitational field is equivalent to the gravitational field of a point particle at the origin with the same mass m. However, it is naïve to think that the mass of the sphere is concentrated at the center. By considering the mass at the center, one might incorrectly conclude the possible existence of point-like elementary particles with huge masses, such as that of the sun. However, we know that (22) is a simple mathematical result. The solenoid case is analogous. It only appears as if there were two point magnetic poles at the tip of the solenoid similar to an electric dipole. However, this similarity is broken on the axis of the magnet. Note this analogy to the gravitational field of the sphere for internal points $r < R$. Searching for magnetic monopoles in the universe is as absurd as looking for elementary particles with mass of billions of kilograms. We have to realize that the magnetic field $\mathbf{B}$ is only generated by moving electric charges.

Why do we need to use a Hamiltonian formulation to demonstrate the impossibility of magnetic monopoles? Why cannot we show non-existence in a more obvious way? This is the subject of a forthcoming article, which shows geometrically why the magnetic field $\mathbf{B}$ can only be generated by moving electric charges.

4. Conclusion

It has been shown that the magnetic monopole is inconsistent within the theory of electrodynamics because a proper Hamiltonian for an electric point charge interacting with a fixed magnetic monopole cannot exist. This is due to the fact that the field of a magnetic monopole cannot be represented by a vector potential. Consequently, the magnetic field is only the result of moving electric charges and magnetic monopoles do not exist.

References

1) D. Zwanziger, Phys. Rev. B137, 647 (1965).

2) S. Weinberg, Phys. Rev. B138, 988 (1965).

3) C.R. Hagen, Phys. Rev., B140, 804 (1965).

4) A. R. Hadjesfandiari, "Character of the magnetic monopole," *International Journal of Materials and Product Technology*, Vol. 34, 37-53, 2009.

5) A.R. Hadjesfandiari, "Field of the Magnetic Monopole", http://arxiv.org/ftp/physics/papers/0701/0701232.pdf.

6) P. A. M. Dirac, Proc. R. Soc. London **A133**, 60 (1931).

7) J. D. Jackson, *Classical Electrodynamics* (John Wiley, New York, 1999).

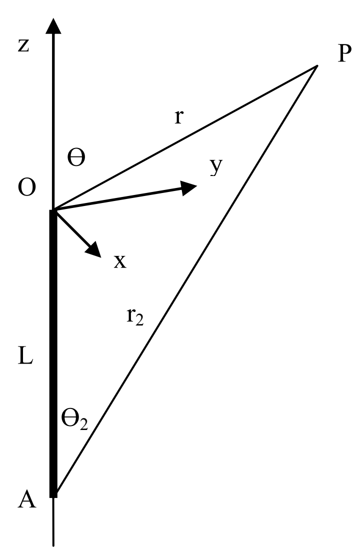

Figure 1