More properties in Goldbach's Conjecture

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Summary

This paper reveals that the reference function $G(2n)=2n/(\ln(n))^2$ plays a significant role in the distribution of the total number of pairs (p, q) of primes that fulfill the condition ($p + q = 2n$), which constitutes Goldbach's conjecture. Numerical experiments up to $2n$=500,000 show that, in the plot of the number of pairs versus $2n$, the ratio of the lowest points over $G(2n)$ tends asymptotically to the value 2/3. The latter fact dictates that the lower bound concerning the minimum number of pairs that fulfill Goldbach's conjecture is equal to $4n/[3(\ln(n))^2]$. Moreover, smoothed sequences by treatment of the aforementioned pairs are revealed.

1. Introduction

Goldbach's conjecture [1] states that "every even natural number > 4 can be written as a sum of two primes", namely:

$$2n = p + q \quad \text{where } n>2, \text{ and } p, q \text{ are prime numbers,} \tag{1}$$

where the set of primes is $\mathbb{P} = \{2,3,5,7,11,\ldots\}$. This is also called the 'strong formulation' of the conjecture.

In 1900, Hilbert said that Goldbach's conjecture was one of the 23 most difficult problems for mathematicians of the 20th century [2], while Landau sorted four main problems for the first few numbers including Goldbach's conjecture [3,4].

The ***weak formulation*** of the conjecture has not been yet proven, but there have been some useful although somewhat failed attempts. The first of these works was in 1923 when, using the 'circle method' and assuming the validity of the hypothesis of a generalized Riemann, Hardy and Littlewood [5] proved that every sufficiently large odd integer is sum of three odd primes and almost all the even number is the sum of two primes. In 1919, Brun [6], using the method of his sieve proved that every large even number is the sum of two numbers each of whom has at least nine factors of primes. Then in 1930, using the Brun's method along with his own idea of "density" of a sequence of integers, Schnirelman [7] proved that every sufficiently large integer is the sum of maximum c primes for a given number c. Then in 1937, Vinogradov [8], using the circle method and his own method to estimate the exponential sum in a variable prime number, was able to overcome the dependence of the great Riemann hypothesis and thus provide the evidence of the findings of Hardy and Littlewood now without conditions. In other words, he directly proved (theorem of Vinogradov's theorem) that all sufficiently large odd number can be expressed as the sum of three primes. The original proof of Vinogradov, based on inefficient theorem of Siegel-Walfisz, did not put a limit for the term "sufficiently large", while his student K. Borozdkin [9] showed in 1956 that ότι $n_0 = 3^{3^{15}} = 3^{14348907}$ is sufficiently large (has 6,846,169 digits). Later, after improvements in the method of Brun, in 1966 Chen Jing-Run [10] managed to prove that every large integer is the sum of a prime and a product of at most two primes. In 2002, Liu and Wang [11] lowered the threshold around $n > e^{3100} \approx 2\times 10^{1346}$. The exponent is too large to allow control of all smaller numbers with the assistance of a digital computer. According to Internet reports [12,13], the computer assisted search arrived for the strong Goldbach conjecture up to order 10^{18} (http://www.ieeta.pt/~tos/goldbach.html) and, for the weak Goldbach conjecture not much more. In 1997, Deshouillers et al. [14] showed that the generalized Riemann hypothesis implies the weak Goldbach's conjecture for all numbers. Also, Kaniecki [15] showed that every odd number is the sum of at most five primes, provided the validity of Riemann Hypothesis.

Most of these classic works have been included in a collective volume by Wang [16]. Specifically, in this volume the first section includes the representation of an odd number as a sum of three primes with six papers (Hardy and Littlewood; Vinogradov; Linnik; Pan; Vaughan; Deshouillers, Effinger, Riele & Zinoviev), the second section includes the representation of an even number as a sum of two nearly primes in six other works of (Brun; Buchstab; Kuhn; Selberg; Wang; Selberg) and finally the third section includes the representation of an even number as a sum of a prime and an almost prime in nine works (Renyi; Wang; Pan; Barban Til; Buchstab; Vinogradov; Bombieri; Chen Jing-Run; Pan). Finally, apart from the individual reports of certain articles, the collective volume includes 234 additional citations arranged by author, referring to the period 1901-2001.

The ***strong formulation*** of Goldbach conjecture, which is the subject of this paper, is much more difficult than the above weak one. Using the above method of Vinogradov [8], in separate works Chudakov [17], van der Corput [18] and Estermann [19] showed that almost all even number can be written as a sum of two primes (in the sense that the fraction of even number tends to the unit). As mentioned above, in 1930, Schnirelman [7] showed that every even number $n \geq 4$ can be written as a sum of at most 20 primes. This result in turn enriched by other authors; the most well-known result due to Ramaré [20] who in 1995 showed that every even number $n \geq 4$ is indeed a maximum sum of 6 primes. Indeed, resolving the weak Goldbach conjecture will come through that every even number $n \geq 4$ is the sum of at most 4 primes [21]. In 1973, using sieve theory methods (sieve theory) Chen Jing-run showed that every (sufficiently large even number can be written as a sum either of two primes or of one prime and one semiprime (i.e. a product of two primes) [22], e.g. $100 = 23 + 7{\cdot}11$. In 1975, Montgomery and Vaughan [23] showed that "most" even number is a sum of two primes. In fact, they showed that there was a positive constants c and C such that for all sufficiently large numbers N, every even number less than N is the sum of two primes with CN^{1-c} exceptions at the most. In particular, all the even integers that are not sum of two primes have zero density. Linnet [24] proved, in 1951, the existence of a constant K such that every sufficiently large even number is the sum of two primes and a maximum of K powers of 2. Heath-Brown and Puchta [25] in 2002 found that the value $K = 13$ works well. The latter improved to $K = 8$ by Pintz and Ruzsa [26] in 2003.

It is noteworthy that in 2000 Eq(1) was verified using computers for even numbers up to $4{\times}10^{16}$ [27], and the attempt was repeated by T. Oliveira e Silva with the help of distributed computing network to $n \leq 1.609{\times}10^{18}$ and in selected areas up to $4{\times}10^{18}$ [13]. However, mathematically these checks do not constitute conclusive evidence of validity of Eq(1), and the effort continues today [28,29]. Initial attempts were [30-32].

In this paper we present experimental results on all even numbers between 6 and 500,000.

2. The reference function *G*(2*n*)

In 1793 Gauss gave the approximate formula

$$\pi\left(n\right) = \frac{n}{\ln n}, \tag{2}$$

for the estimation of prime numbers less or equal to n. It can be used to produce something like the 'probability' that a randomly chosen odd number is prime.

Moreover, Sheldon [33] showed that the probability that the number $2\lambda+1$ is prime, is approximated by:

$$\frac{2}{\ln(2\lambda)} \tag{3}$$

The proof is based on the fact that, for large λ, by virtue of Eq(2) the quantity $\pi(2\lambda+1)$-$\pi(2\lambda-1)$ is almost exactly $\frac{2}{\ln(2\lambda)}$, which can be interpreted as the expected number of primes in the set $\{2\lambda, 2\lambda+1\}$. Since there is only one candidate for being a prime, namely $2\lambda+1$, the proof of Eq(2) is complete■.

Definition-1. Let us define the function $p_{2n}(N)$ as the number of ways that an even number ($N = 2n$, $n = 3, 4, \ldots$) can be written as the sum of two primes, p and q that fulfill Eq(1). In other words, it is the total true number of pairs (p, q) that fulfill Eq(1).

Definition-2. Let us assume independency of events and define the function $G(N)$ as a probabilistic approximation of $p_{2n}(N)$.

Since $p+q=2n$, if we take that one of the two primes is smaller or equal to the other prime, for example $p \le q$, then $p \le n$ and $n \le q \le 2n$. To evaluate the function $G(N)$ we have to find each prime p and establish whether or not the corresponding q is prime. We can apply Eq(2) and find an estimate of the number of primes p, particularly the value $\pi(n)$, which corresponds to the maximum possible value of p, i.e. $p=n$. Each of the aforementioned $\pi(n)$ values of p generates a q, equal to $2n$-p, which may or may not be prime.

Concerning q, which may take either of the extreme values n or $2n$, according to Eq(3) the corresponding number of primes almost exactly varies between $\frac{2}{\ln(2n)}$ and $\frac{2}{\ln(n)}$. However, since $\ln(2n) = \ln 2 + \ln(n) \approx 0.6931 + \ln(n)$, for large n, the differences are rather of minor importance. Therefore, since Eq(2) underestimates the true prime-counting function, we consider the greater value, i.e. $\frac{2}{\ln(n)}$.

As a result, we can say that the expected number of pairs of primes (p,q) is given by the multiplication of $\pi(n)$ times the maximum probability, i.e. $\frac{n}{\ln n}$ by $\frac{2}{\ln n}$, that is:

$$G(2n) = \frac{2n}{(\ln n)^2}. \tag{4}$$

The *reference* function $G(2n)$ given by Eq(4) appears for the first time.

It is the aim of this paper to determine a lower bound for p_{2n} as a function of $2n$, and particularly in terms of $G(2n)$.

3. Processing the numerical data

A MATLAB® code was developed to determine functions $G(2n)$ and p_{2n} for all even numbers $2n$ lying between 6 and 500,000. The results are presented in **Figure 1**, where the ratio R_{2n} is determined as:

$$R_{2n} = \frac{p_{2n}}{G(2n)} \tag{5}$$

It is noticed that the ratio R_{2n} fluctuates near to the value 2/3 (see details in **Table 1**). By definition, the reference function $G(2n) = 2n/(\ln n)^2$ corresponds to $R_{2n} = 1$; although the aforementioned line $R_{2n} = 1$ is closer to the bottom than top, there are 95035 points beyond (38 percent) and 154963 ones below (62 percent) it.

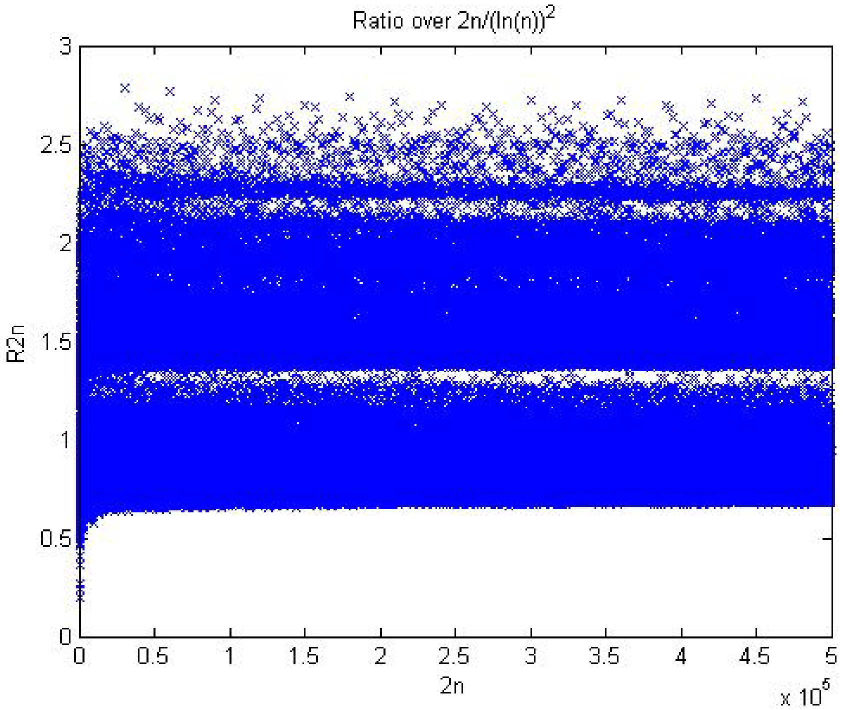

Figure 1: Convergence of ratio R_{2n} versus increasing even numbers $2n$.

Table 1: Details for the minimum ratio R_{2n} defined by Eq(5).

Interval of even numbers **($2n$)**	**Minimum ratio** **(R_{2n})**
280,000 – 500,000	0.6751
460,000 – 280,000	0.6680
440,000 – 460,000	0.6742
420,000 – 440,000	0.6736
400,000 – 420,000	0.6719
380,000 – 400,000	0.6690
360,000 – 380,000	0.6768
...	...
20,000 – 34,000	0.6316
12,000 – 30,000	0.6223
8,000 – 24,000	0.5762

3.1 Lower bound

Since the limit of the ratio R_{2n} is a little higher than 2/3, a lower bound for p_{2n}^L is approximated by:

$$p_{2n}^L = \frac{4n}{3(\ln n)^2}. \tag{6}$$

The quality of Eq(6) is excellent, as shown in **Figure 2** for all even numbers between 440,000 and 500,000.

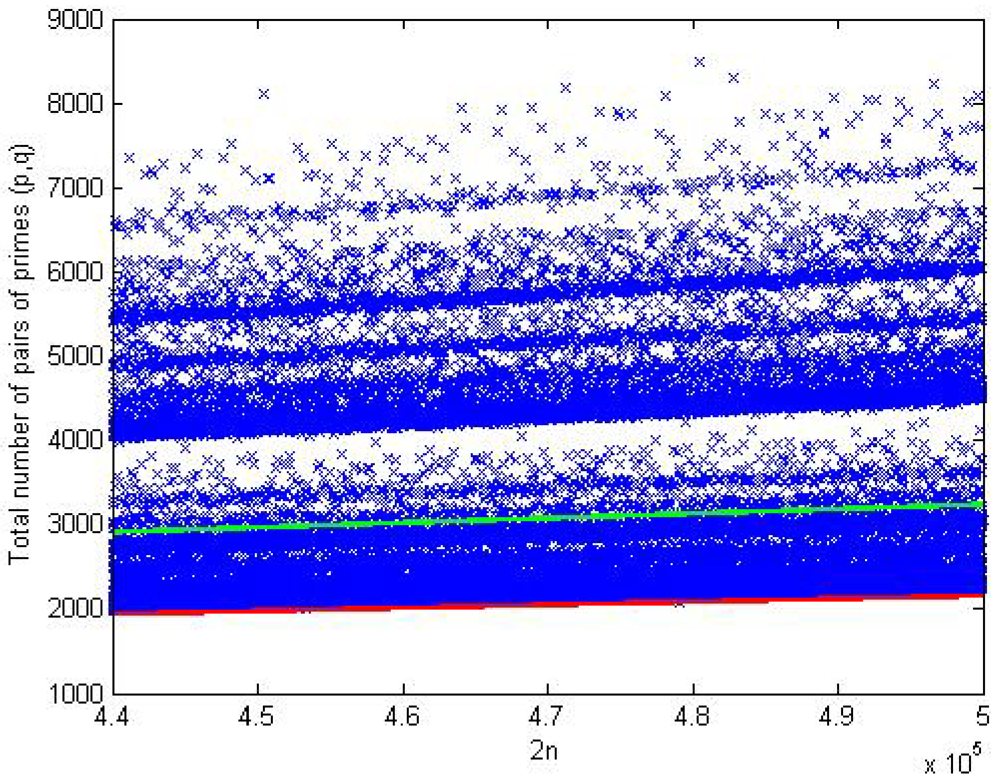

Figure 2: The lower bound (in red color) given by Eq(6) and the reference function (in green color).

3.2 Basic cell

The lowest white-colored horizontal gap that appears in **Figure 1** is due to the fact that, when $2n$=6λ (λ=2,3,…), the value p_{2n} may approximately be twice larger than that obtained either for $2n=6\lambda-2$ or $2n=6\lambda+2$. A theoretical explanation is given in the 'Discussion' section.

Within this context, for a better insight, we split the interval [6, 500000] into 166,665 consecutive cells of the form (6λ-2, 6λ, 6λ+2), λ=2,3,4 and so on. We start with the cell (10,12,14), i.e. λ=2 and continue with (16,18,20) , i.e. λ=3, (22,24,26), i.e. λ=4, etc.

For every cell we calculate the following three ratios:

$$R_{6\lambda-2} = \frac{p_{6\lambda-2}}{G(6\lambda-2)}, \quad R_{6\lambda} = \frac{p_{6\lambda}}{G(6\lambda)} \quad \text{and} \quad R_{6\lambda+2} = \frac{p_{6\lambda+2}}{G(6\lambda+2)}, \ \lambda = 2,3,\ldots \tag{7}$$

for which the corresponding results are shown in **Figure 3**.

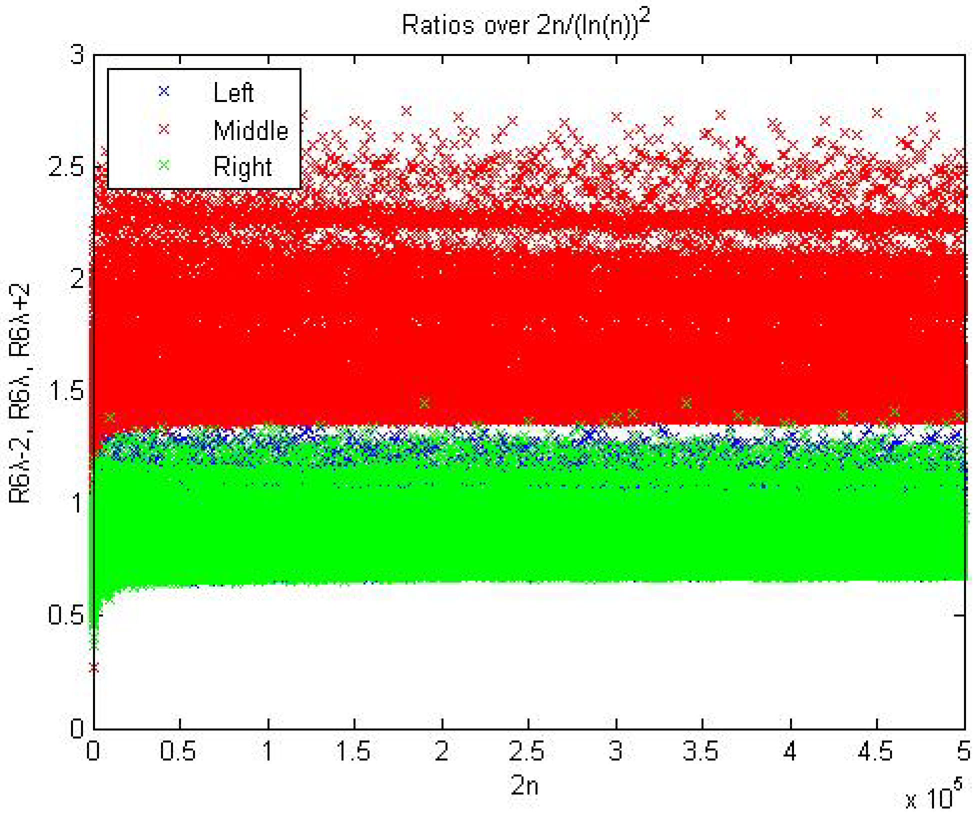

Figure 3: The three ratios for the basic cell (6λ-2, 6λ, 6λ+2) are shown in blue, red and green color, respectively.

It is clear that, both the left and right ends of cells occupy a very similar area (however the left end is somehow higher, in the average sense), whereas the middle point is about twice higher (again, in the average sense). These facts explain the appearance of the above-mentioned lowest white-colored horizontal gap.

3.3 Smoothing procedures

The above findings have revealed that the points within the cloud are interrelated at least on the level of the basic cell. Moreover, **in Figure 4** we have again divided the set of natural numbers in consecutive cells of the form $2n = (6\lambda\text{-}2,\ 6\lambda,\ 6\lambda\text{+}2)$, the first being (10, 12, 14). Then, based on every cell, we construct triangles (in blue lines) of which the centroids are connected by straight lines (in magenta color). The green line represents Eq(4), whereas the red line in the bottom refers to p_{2n}^{L} of Eq(6). We notice that most of the centroids are above the green line.

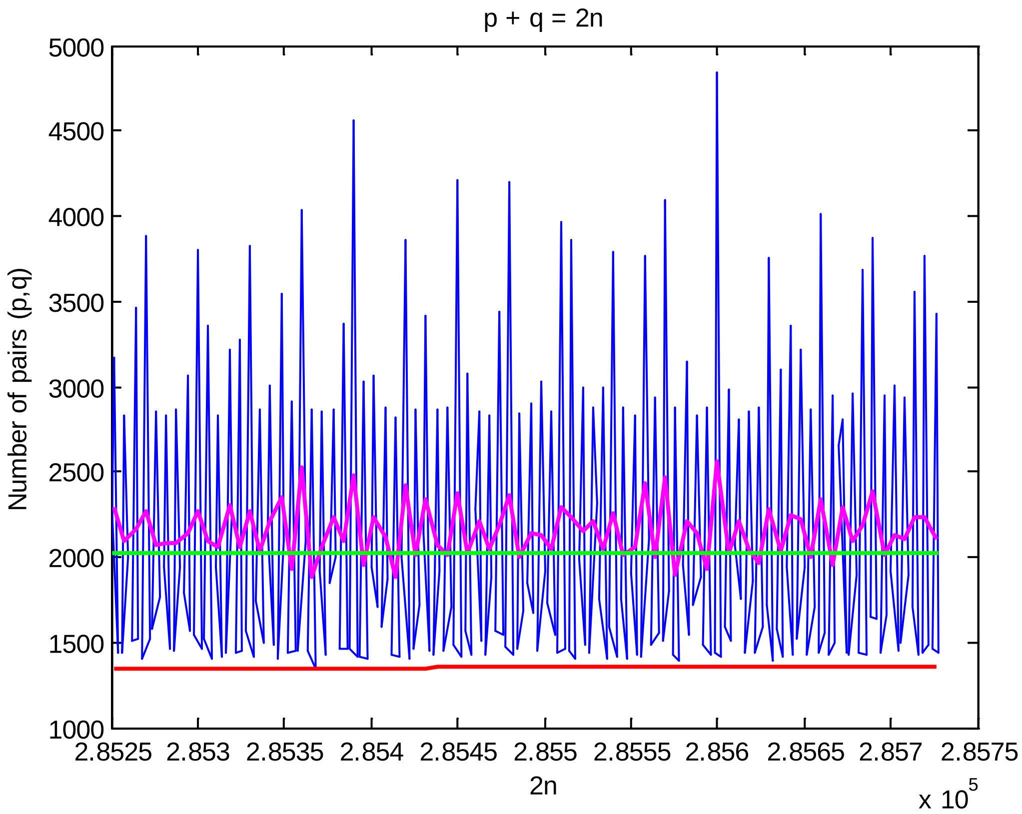

Figure 4: Number of pairs (p,q) that fulfill Goldbach's conjecture and B_0-smoothing in the neighborhood of the even number $2n$=285368.

As shown in **Figure 5**, let us now denote by $\mathbf{B_0}$ the family of the basic cells [(10, **12**, 14), (16,**18**,20), (22,**24**,26) and so on] in which the set $\mathbb{N}$ is divided; its step is $s_0 = 2$.

In the sequence, we denote by $\mathbf{B_1}$ the family of second-level cells in which we take the midpoint of every $\mathbf{B_0}$-cell as elements of another cell sequence with step equal to $s_1 = 6$; the first of these cells is (12,**18**,24), the second is (30,**36**,42), the third is (48,**54**,60) and so on.

Moreover, we denote by $\mathbf{B_2}$ the family of third-level cells in which we take the midpoint of every $\mathbf{B_1}$-cell as elements of another cell sequence with step equal to $s_2 = 18$; the first of these cells is (18,**36**,54), the second is (72,**90**,108) and so on.

In analogy, we can define $\mathbf{B_3}$-family with step equal to $s_3 = 54$ and so on.

Obviously, whenever we pass from family $\mathbf{B_n}$ to family $\mathbf{B_{n+1}}$ we have to multiply its previous s_n step by 3 (i.e., $s_{n+1}=3s_n$).

From a physical point of view, the abovementioned $\mathbf{B_i}$-families constitute the procedure for determining the 'linearly distributed' center of mass of the cloud of points that refer to the function $\mathbb{F}: 2n \rightarrow p_{2n}$, where all points of the cloud possess the same mass. Actually, by definition, the basic $\mathbf{B_0}$-family refers to the centers of masses of the triangles that are formed by the ends and the middle of the $\mathbf{B_0}$-cells (cf. Fig.4). Also, by definition, the $\mathbf{B_1}$-family consists of the centroids of the already found centroids of $\mathbf{B_0}$-family. Then, again by definition, the $\mathbf{B_2}$-family consists of the centroids of the already found centroids of $\mathbf{B_1}$-family, and so on. By continuously shifting the starting point to the right and extending the step (successively multiplying by 3) we obtain a smoother and smoother average curve.

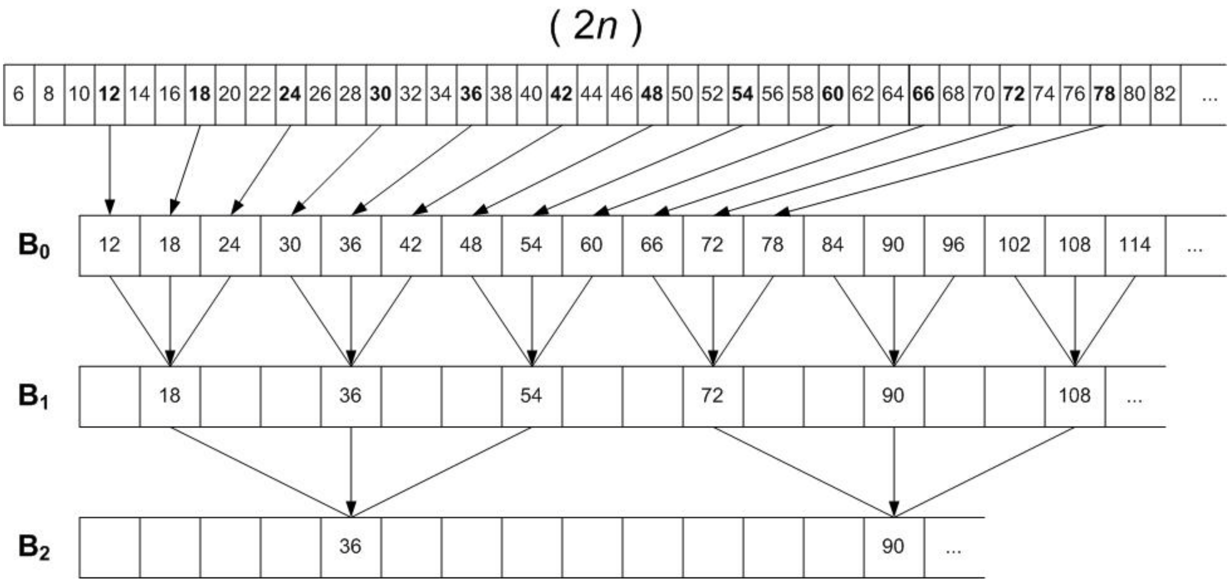

Figure 5: Construction of smoothing families: B_0, B_1 and B_2.

As shown in **Figure 6**, the abovementioned successive smoothings lead to decreasing amplitudes.

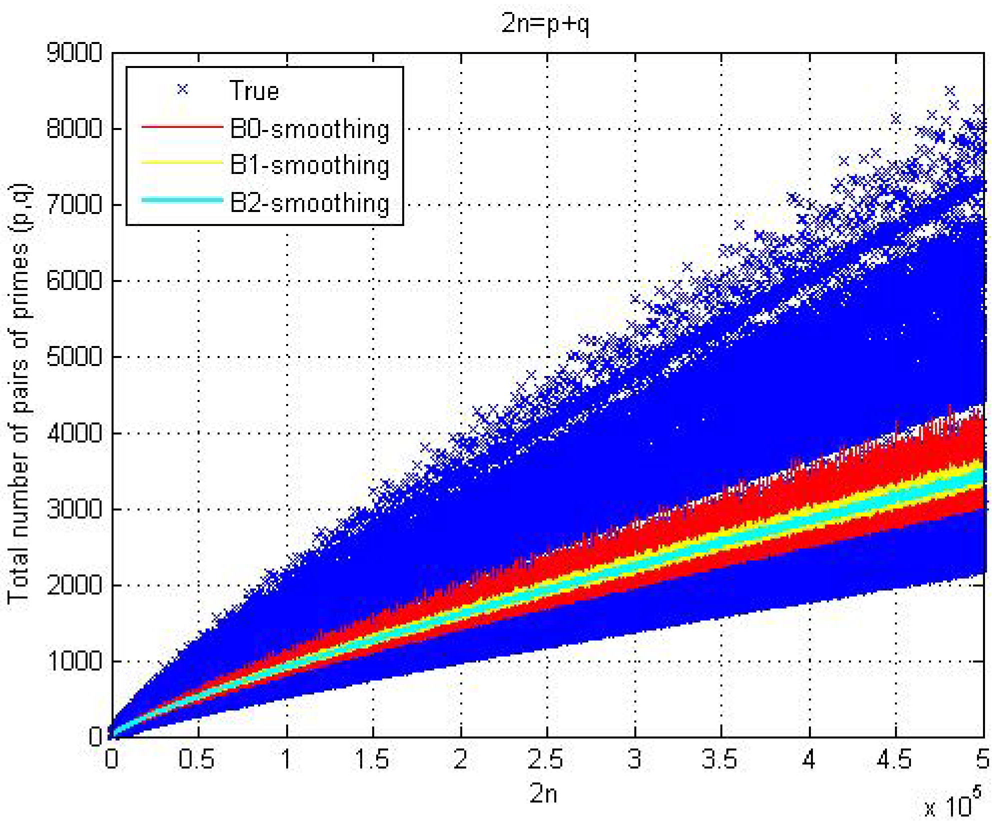

Figure 6: Smoothed distribution of total number of pairs (p,q), at the midpoints of successively enlarged cells, whereas the smoothing procedure is shown in Figure 5.

4. Discussion

Due to the high difficulty of the topic under investigation, as a first step for further close examination, a non-strictly mathematical approach was followed in order to reveal some of the laws or properties hidden in the conjecture. The results so far (until $2n$ = 500,000) show an excellent agreement between reality and the proposed Equation (6). The latter formula is based on the assumption that the logarithmic-like curve (Eq(4)) can be considered as a *reference* function in conjunction with the finding that the ratio of the total number of pairs over the reference function is very close to 2/3.

Setting in the reference function a factor exactly equal to 2/3, for even numbers less or equal to 285,368 the proposed formula slightly deviates at 463 positions (by only 1 to 17 units); see also Table 1. For even numbers greater or equal to 285,370 no violation has been observed up to the value $2n$ = 500,000. To fix this drawback there are two easy ways. The first way is to decrease to factor from 2/3 to a slightly smaller one (according to Table 1), whereas the second way is to split the interval [6, 500000] into a few broad subintervals and use a multiply defined function p_{2n}.

A remaining issue is the fact of an approximately double, or similar, ratio for the mid-points of every cell (6λ-2, 6λ, 6λ+2). Below we try to justify the reason for this fact, in a qualitative way. First of all, we have to mention the procedure of determining the number p_{2n} of pairs (p,q) that fulfill the conjecture, i.e. $p+q=2n$. The procedure of determining the value of p_{2n} is quite deterministic. One standard way to do it is described by the following *Theorem-1*.

4.1 Determination of the total number of pairs (*p*,*q*) that fulfill the conjecture

Theorem-1: Every even natural number $2n$ can be decomposed into a sum of two odd natural numbers (primes or composites) in so many different ways, n_s, as the integer part *(floor)* of the rational number $(n-1)/2$, that is $n_s = \lfloor (n-1)/2 \rfloor$. The index '*s*' results from the word 'sample', thus referring to sample of n_s odd numbers (from which we will later isolate the prime numbers).

Proof

We distinguish two cases (Case-1 and Case-2) as follows.

Case-1: When *n* is odd, we form the sets:

A = {3, 5, …, *n*} and B = {2*n*-3, 2*n*-5, …, *n*}. Since the order of items is not important in the sets, in order to maintain the desired sequence (in the form of rows or columns) we form the vectors $\vec{a} = [3,5,\ldots,n]$ και $\vec{b} = [2n-3, 2n-5,\ldots,n]$. It is obvious that all elements of the vector $\vec{c} = \vec{a} + \vec{b}$ are strictly defined and are equal to $2n$ as opposed to *probabilistic pairs* that can be derived from the sets A and B. Also, it is evident that any enhancement of the vector $\vec{a}$ will give terms contained in the vector $\vec{b}$, displayed from right to left, so it makes no sense. Finally, it is obvious that the cardinality of two sets is the same, ie *cardA* = *cardB* = (*n*-1)/2.

Case-2: When *n* is even, we form the sets:

A = {3, 5, …, *n*-1} and B = {2*n*-3, 2*n*-5, …, *n*+1}. As previously, we consider the new vectors $\vec{a}' = [3,5,\ldots,n-1]$ and $\vec{b}' = [2n-3, 2n-5,\ldots,n+1]$. It is obvious that all elements of the vector $\vec{c}' = \vec{a}' + \vec{b}'$ are again equal to $2n$. As previously, any enhancement of the vector $\vec{a}$ will give terms included into the vector $\vec{b}$, displayed from right to left. Finally, it is obvious that the cardinality of two sets is the same, i.e. *cardA* = *cardB* = *n*/2-1.

Summarizing the results of the two above cases, it is easily concluded that:

$$cardA = cardB = \left\lfloor \frac{(n-1)}{2} \right\rfloor \blacksquare \tag{8}$$

4.2 The unit cell

The reason that (the order of magnitude) of the number of pairs of primes that correspond to the end values (6λ-2, 6λ+2) of every cell is about half of that for the midpoint (6λ) is as follows.

The triple of numbers (6λ-2, 6λ, 6λ+2) are consecutive even numbers. This implies that the subsets of even numbers $\mathbb{N}$ ={x/x 6λ-2, 6λ, 6λ+2} for λ = 1,2,3 ... and $\lambda \in \mathbb{N}$ are disjoint and their union is the set $\mathbb{N}_2$ of all even numbers ≥ 4:

$$\mathbb{N}_2 = \{x/x \ \ 6\lambda\text{-}2, 6\lambda, 6\lambda\text{+}2\} \text{ where } \lambda \geq 1. \tag{9}$$

If we replace λ in Eq(9) with two natural numbers λ_i and λ_j where $(\lambda_i, \lambda_j) \in \mathbb{N}$, such as:

$$\lambda = \lambda_i + \lambda_j \ , \tag{10}$$

the triple of the successive even numbers is transformed to

$$6(\lambda_i + \lambda_j) - 2 \tag{11a}$$
$$6(\lambda_i + \lambda_j) \tag{11b}$$
$$6(\lambda_i + \lambda_j) + 2. \tag{11c}$$

Proposition-1. The set of natural numbers $\mathbb{N}$ is divided into six equivalence classes, those of elements with remainder 0, 1, 2, 3, 4 and 5, namely:

$$K(0) = \{x/x = 6\lambda + 0 \quad \lambda \in \mathbb{N}\} \tag{12a}$$
$$K(1) = \{x/x = 6\lambda + 1, \ \lambda \in \mathbb{N}\} \tag{12b}$$
$$K(2) = \{x/x = 6\lambda + 2, \ \lambda \in \mathbb{N}\} \tag{12c}$$
$$K(3) = \{x/x = 6\lambda + 3, \ \lambda \in \mathbb{N}\} \tag{12d}$$
$$K(4) = \{x/x = 6\lambda + 4, \ \lambda \in \mathbb{N}\} \tag{12e}$$
$$K(5) = \{x/x = 6\lambda + 5, \ \lambda \in \mathbb{N}\}, \tag{12f}$$

which are disjoint each other and their union gives the set $\mathbb{N}$, that is:

$$K(0) \cup K(1) \cup K(2) \cup K(3) \cup K(4) \cup K(5) = \mathbb{N} \tag{13}$$

Proposition-2. The set of natural numbers contained in the equivalence classes K(0), K(2), K(3) and K(4) are *composite numbers* as multiples of 2 and 3.

Proposition-3: The classes K(1) and K(5) include *all primes* (except of 2 and 3) as well as the *multiples of primes* being > 3; these can be combined in the formula:

$$\boxed{6\lambda \pm 1 = \text{Primes} + \text{Multiples of Primes} > 3} \tag{14}$$

Lemma: The mid-point of the basic cell (6λ–2, 6λ, 6λ+2) corresponds to a higher number of pairs $p_{6\lambda}$ than their endpoints.

Proof

Based on Proposition-3, Equations (11) may be further transformed as follows:

$$\text{Left end:} \quad 6(\lambda_i + \lambda_j) - 2 = (6\lambda_i - 1) + (6\lambda_j - 1), \tag{15a}$$
$$\text{Mid-point:} \quad 6(\lambda_i + \lambda_j) = (6\lambda_i - 1) + (6\lambda_j + 1), \tag{15b}$$
$$\text{Mid-point:} \quad 6(\lambda_i + \lambda_j) = (6\lambda_i + 1) + (6\lambda_j - 1), \tag{15b′}$$

Right end: $6(\lambda_i + \lambda_j) + 2 = (6\lambda_i + 1) + (6\lambda_j + 1)$. (15c)

Therefore, we achieved to transform all the even numbers into a sum of two odd numbers of the form $6\lambda_i \pm 1$, which (by virtue of Proposition-3) either are primes or multiples of primes ≥ 5. The fact that even numbers of the form 6λ can be analyzed in two different ways as a sum of odd numbers [cf. by virtue of either Eq(15b) and Eq(15b´)], leads to a higher probability for them compared to those in the form $(6\lambda\text{-}2)$ [Eq(15a)] or $(6\lambda+2)$ [Eq(15c)]. In all cases, the procedure of Theorem 1 has been implicitly considered as a sieve. A trivial application is given in **Appendix A**, where it is shown that the mid-point gives nine pairs while the end points only four.

5. Conclusion

In this study we achieved to determine a closed form expression that calculates a lower bound concerning the total number of pairs of primes that fulfill Goldbach's conjecture. For even numbers greater or equal to 285,370 no violation has been observed up to the value 500,000 when applying a factor 2/3 over a probabilistic reference function. Moreover, using basic cells and their multiples, we revealed hidden smoothing curves within the cloud of points that represent the number of pairs of primes that fulfill Goldbach's conjecture.

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APPENDIX A

Formation of pairs of primes on a basic cell (6λ-2, 6λ, 6λ+2)

We apply the trivial case $p+q=2n$, with n=45, where the even number 2n=90 is the mid-point of the basic cell (6λ–2, 6λ, 6λ+2) = (88, 90, 92).

According to Theorem-1, we form two columns, A and B, and then fill them starting from 3 up to the n_s-th row, where $n_s = \left\lfloor \frac{(n-1)}{2} \right\rfloor$. Setting n=44, 45 and 46, it comes out that n_s = 21, 22 and 22, respectively. The procedure is clearly shown in **Table 2**, where the pairs of primes are included.

Table 2: Example for the decomposition of a triad of numbers (6λ–2, 6λ, 6λ+2), for λ = 15, in sums of two odd numbers (primes and composites).

	6λ-2 = 88		**6λ = 90**		**6λ+2 = 92**	
	COLUMNS		**COLUMNS**		**COLUMNS**	
	A	**B**	**A**	**B**	**A**	**B**
1	3	85	3	87	3	89
2	5	83	5	85	5	87
3	7	81	7	83	7	85
4	9	79	9	81	9	83
5	11	77	11	79	11	81
6	13	75	13	77	13	79
7	15	73	15	75	15	77
8	17	71	17	73	17	75
9	19	69	19	71	19	73
10	21	67	21	69	21	71
11	23	65	23	67	23	69
12	25	63	25	65	25	67
13	27	61	27	63	27	65
14	29	59	29	61	29	63
15	31	57	31	59	31	61
16	33	55	33	57	33	59
17	35	53	35	55	35	57
18	37	51	37	53	37	55
19	39	49	39	51	39	53
20	41	47	41	49	41	51
21	43	45	43	47	43	49
22			45	45	45	47
	Sum of pairs that fulfill Goldbach's conjecture = **4**		Sum of pairs that fulfill Goldbach's conjecture = **9**		Sum of pairs that fulfill Goldbach's conjecture = **4**	
	Sum of Pairs: 21 (Equation 15a)		Sum of Pairs: 22 (Equations 15b and 15b´)		Sum of Pairs: 22 (Equation 15c)	

Composite number
Odd prime in the form (6λ-1)
Odd prime in the form (6λ+1)
Prime number 3