A Special Case of the Lonely Runner Conjecture for any *n*

Patrick A Devlin **22nd October 2012** **devlinpa.pd@gmail.com**

Abstract

In number theory, and especially the study of the diophantine approximation, the *Lonely Runner Conjecture* is a conjecture with important and widespread applications in mathematics.

This paper attempts to prove the conjecture for any *n* runners in the special case of speeds with specially correlated prime factors.

Statement of the Conjecture

Consider *n* runners on a circular track of unit length. At time t = 0, all runners are at the same position *O* and start to run; the runners' speeds are pair-wise distinct. A runner is said to be lonely if at a distance of at least 1/n from each other runner. The *Lonely Runner Conjecture* states that every runner gets lonely at some time.

Proof

The track is of unit length, with all runners running at integer speeds. The runners coincide at the starting point *O*, at all integer units of time. Consequently, the condition that every runner becomes lonely, must occur first for times t < 1.

Now we consider if it is possible for any (*n*-1) runners to coincide at the starting point *O*, at any time t < 1. After careful consideration, it is evident that if each of these runners share a common prime factor *p, then they will coincide at the origin at a time T = 1/p.*

This is best explained by example:

Consider that the speeds of the (n-1) runners from 1*st* to (n-1)*th* are **A***p*, **B***p*, **C***p* , **D***p* ...., **Ʊ**p, where *p* is the common factor and **A < B < C< ... < Ʊ**.

After a time t = 1/*p all (n-1) runners will have coincided at the origin O, with the 1st runner having completed **A** cycles, the 2nd **B** cycles and the (n-1)th,* **Ʊ** cycles.

We now enquire as to the location of the *n*th runner at this time T=1/p. We imagine for the moment that he is not at *O* and examine the consequences.

If the nth runner is not at O at time T = 1/p then he is distance DELTA from O. DELTA, is either greater than or less than 1/n. If DELTA is greater than 1/n then that runner is lonely. If DELTA is less than 1/n then after a time 2 x *T, the other runners will have coincided at O once again, whereas the nth runner will have increased his separation from O to a distance 2* x *DELTA. (This follows from the simple application of modular arithmetic) .*

This process can be repeated M times until after a time t = M x *T, M* x *DELTA > 1/n and that runner becomes lonely.*

*Now we consider what conditions are necessary for the nth runner **not** to be at O at time T = 1/p. Clearly, if the condition for coincidence of the (n-1) runners at a time t < 1 is that they share a common factor p, then the condition for the **non-coincidence** of the nth runner is that his speed does not share this prime factor.*

Statement

(1) When the speeds of any (n-1) runners share a common prime factor p, they will coincide at the origin O, at a time T= 1/p, where T < 1.

(2) When the speeds of any (n-1) runners share a common factor p, the excluded runner (whose speed does not share this prime factor) can be shown to become lonely at some time t = M x T, where M is an integer and T = 1/p.

The Case of Specially Correlated Factors

We now give further consideration to the implications of runners speeds sharing common prime factors:

Imagine that there is a set of any *n* prime numbers. From that set of *n* primes, the speeds of *n* runners is constructed by selecting a set of (*n*-1) of those prime factors. Each runner has a speed which is the product of a different set of (*n*-1) of those primes. That is, the speeds of the runners from s1 to sn are as follows:

s1 = p2 x p3 x p3 x p4 .... x pn

s2 = p1 x p3 x p4 x p5 .... x pn

....

s(n-1) = p1 x p2 x p3 x p4 ... x pn

sn = p1 x p2 x p3 x p4 ..... x p(n-1)

There is something special about this arrangement of speeds with highly correlated prime factors:

Every combination of (n-1) runners is such that they share a common factor with each other, whilst not sharing that common factor with the excluded runner. For example, it is evident from the list above, that the *1st to the (n-1)th runners share the common factor* ***pn****. That is to say, in this arrangement, the speed of the* ***Qth*** *runner does not contain the* ***Qth*** *prime factor, whereas the speeds of all other runners do.*

It was demonstrated earlier and summarized in ***Statement 2*** *that:*

When the speeds of any (n-1) runners share a common factor p, the excluded runner can be shown to become lonely at some time t = M x T, where M is an integer and T = 1/p.

It is clear that we have created a situation in which every runner can be considered to be an excluded runner by virtue of some prime factor being absent in its speed, yet *present in the other (n-1) runners speed. In this situation it is clear that all n runners become lonely at some point and the Lonely Runner Conjecture is true.*

Statement

(3) When the prime factors of the speeds of n runners is correlated such that each possesses a different set of (n-1) prime factors, selected from a set of n, *[described above]* then each runner becomes lonely at some time.

It is possible to simplify this idea further:

Imagine now that we have correlated the prime factors of the runners speeds in the method described above. For this purpose, we have used a set of n arbitrary prime factors. We have clearly shown that, in this special case, each runner becomes lonely for any *n* runners.

Let us call the product of all prime factors in our set of *n* prime factors **P** = p1 x p2 x p3 ....x pn

Hence, it follows from our original argument, that the speeds of the n runners are:

1st **P**/p1;

2nd **P**/p2;

3rd **P**/p3;

......

*n*th **P**/pn;

From this observation the following statement then becomes evident.

Statement

(4) If the speeds of any *n* runners is such that, relative to each other, their speeds are related by the inverse proportions of different arbitrary prime numbers, then every runner will become lonely at some time.

Suggestions for further work

Something not discussed until this point, is that the special correlation which satisfies the lonely runner conjecture for any n, may include the powers of those primes. That is to say, as long as the factors are correlated correctly, it does not appear to matter whether any or all are raised to any arbitrary positive integer powers. This fact is not important for these investigations, but may be helpful for any further enquiries. It might be the case that any n runners with arbitrary, pair-wise distinct, integer speeds, can be sufficiently modeled by reference to specially correlated prime factors as discussed in this paper.

Conclusions

In the specific case of any n runners, having speeds with highly correlated prime factors as described above, it has been shown that the *Lonely Runner Conjecture* is true. Following our investigations the following statements are evident:

(1) When the speeds of any (n-1) runners share a common prime factor p, they will coincide at the origin O, at a time T= 1/p, where T < 1.

(2) When the speeds of any (n-1) runners share a common factor p, the excluded runner (whose speed does not share this prime factor) can be shown to become lonely at some time t = M x T, where M is an integer and T = 1/p.

(3) When the prime factors of the speeds of n runners is correlated such that each possesses a different set of (n-1) prime factors, selected from a set of n, [described above] then each runner becomes lonely at some time.

(4) If the speeds of any n runners is such that, relative to each other, their speeds are related by the inverse proportions of different arbitrary prime numbers, then every runner will become lonely at some time.