Q-FORMULÆ

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Abstract

This compilation of formula of quaternionic algebra and quaternionic differentials is for a significant part derived from Bo Thidé's book "Electromagnetic Field Theory"; http://www.plasma.uu.se/CED/Book. I have merely converted the vector formula into quaternionic format.

Two types of quaternionic differentiation exist.

- Flat differentiation uses the quaternionic nabla and ignores the curvature of the parameter space.
- Full differentiation uses the distance function $\wp(x)$ that defines the curvature of the parameter space.

The text focuses at applications in quantum mechanics, in electrodynamics and in fluid dynamics.

Contents

1 Introduction

Let x be the position vector (radius vector, coordinate vector) from the origin of the Euclidean space $\mathbb{R}^3$ coordinate system to the coordinate point $(x_1;\ x_2;\ x_3)$ in the same system and let $|x|$ denote the magnitude ('length') of x. Let further $\alpha(x), \beta(x), \gamma(x)$, ..., be arbitrary scalar fields, $\boldsymbol{a}(x), \boldsymbol{b}(x), \boldsymbol{c}(x)$, ..., arbitrary vector fields, and $\boldsymbol{A}(x), \boldsymbol{B}(x), \boldsymbol{C}(x)$, ...,arbitrary rank two tensor fields in this space.

Let q be the position relative to the origin of the space $\mathbb{H}$ that is spanned by the quaternions and that is given by the coordinate point $(q_0;\ q_1;\ q_2;\ q_3))$ and let $|q|$ denote the norm of q.

Let * denote complex or quaternionic conjugate and † denote Hermitian conjugate (transposition and, where applicable, complex or quaternionic conjugation).

1.1 Differentiation in flat space

The differential vector operator $\boldsymbol{\nabla}$ is in Cartesian coordinates given by

$$\boldsymbol{\nabla} = \sum_{i=1}^{3} \boldsymbol{e}_i \frac{\partial}{\partial x_i} \tag{1}$$

The flat quaternionic differential operator ∇ is in Cartesian coordinates given by

$$\nabla = \sum_{i=0}^{3} e_i \nabla_i = \sum_{i=0}^{3} e_i \frac{\partial}{\partial x_i};\quad e = (1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}) \tag{2}$$

$$\nabla f = \sum_{i=0}^{3} \sum_{j=0}^{3} e_i e_j \frac{\partial f_j}{\partial x_i} \tag{3}$$

1.2 Differentiation in curved space

The full quaternionic difference operator $\mathrm{d}\wp$ is given by

$$\mathrm{d}\wp = \sum_{\mu=0}^{3} q^{\mu}\, dx_{\mu} = \sum_{\mu=0}^{3} \frac{\partial \wp}{\partial x_{\mu}} dx_{\mu} = \sum_{\nu=0}^{3} e_{\mu} \sum_{\mu=0}^{3} \frac{\partial \wp_{\nu}}{\partial x_{\mu}} dx_{\mu} \tag{1}$$

Here the coefficients q^{μ} are quaternionic coefficients, which are determined by the quaternionic distance function $\wp(x)$.
$\wp(x)$ has a flat parameter space that is spanned by the quaternions. $\wp(x)$ defines a curved target space. This curved space can act as parameter space to other quaternionic distributions.

$$q^{\mu} = \frac{\partial \wp}{\partial x_{\mu}}; \quad \wp = \sum_{\nu=0}^{3} e_{\nu} \wp_{\nu} \tag{2}$$

The quaternionic infinitesimal interval $d\wp$ defines the quaternionic metric of the curved space that is defined by $\wp(x)$.

In this way, the quaternionic function g(ζ), which has a curved parameter space defined by $\zeta = \wp(x)$ corresponds to a new function h(x)= g($\wp$(x)), which has a flat parameter space. The flattened nabla $\breve{\nabla}$ is defined as:

$$\breve{\nabla} g = \sum_{\nu=0}^{3} e_{\nu} \frac{\partial g(\zeta)}{\partial x_{\nu}} = \sum_{\nu=0}^{3} e_{\nu} \sum_{\lambda=0}^{3} e_{\lambda} \frac{\partial g_{\lambda}}{\partial x_{\nu}} = \sum_{\nu=0}^{3} e_{\nu} \sum_{\lambda=0}^{3} e_{\lambda} \sum_{\mu=0}^{3} \frac{\partial g_{\lambda}}{\partial \zeta_{\mu}} e_{\mu} \frac{\partial \zeta_{\mu}}{\partial x_{\nu}} \tag{3}$$

$$= \sum_{\nu=0}^{3} \sum_{\lambda=0}^{3} \sum_{\mu=0}^{3} e_{\nu} e_{\lambda} e_{\mu} \frac{\partial g_{\lambda}}{\partial \zeta_{\mu}} \frac{\partial \wp_{\mu}}{\partial x_{\nu}}$$

2 Cylindrical circular coordinates

2.1 Base vectors

2.1.1 Cartesian to cylindrical circular

$$\rho = x_1 \, cos(\theta) + x_2 \, sin(\theta) \tag{1}$$

$$\varphi = -x_1 \, sin(\theta) + x_2 \, cos(\theta) \tag{2}$$

$$z = x_3 \tag{3}$$

2.1.2 Cylindrical circular to Cartesian

$$x_1 = \boldsymbol{\rho} \, cos(\theta) - \boldsymbol{\varphi} \, sin(\theta) \tag{1}$$

$$x_2 = \rho \, sin(\theta) + \boldsymbol{\varphi} \, cos(\theta) \tag{2}$$

$$x_3 = z \tag{3}$$

2.1.3 Directed line element

$$dl = dx \frac{\boldsymbol{x}}{|\boldsymbol{x}|} = \boldsymbol{e}_{\boldsymbol{\rho}} d\rho + \boldsymbol{e}_{\boldsymbol{\varphi}} \rho d\varphi + \boldsymbol{e}_{\boldsymbol{z}} dz \tag{1}$$

2.1.4 Solid angle element

$$d\Omega = \sin(\theta)\, d\theta\, d\varphi \tag{1}$$

2.1.5 Directed area element

$$d\boldsymbol{S} = \boldsymbol{e_r}\, r^2\, d\Omega + \boldsymbol{e_\theta}\, r \sin(\theta)\, dr\, d\varphi + \boldsymbol{e_\varphi}\, r\, dr\, d\theta \tag{1}$$

2.1.6 Volume element

$$dV = dx^3 = dr\, r^2\, d\Omega \tag{1}$$

2.1.7 Spatial differential operators

$$\alpha = \alpha(r, \theta, \varphi) \tag{1}$$

$$\boldsymbol{a} = \boldsymbol{a}(r, \theta, \varphi) \tag{2}$$

Gradient

$$\nabla\alpha = \boldsymbol{e_r}\frac{\partial\alpha}{\partial r} + \boldsymbol{e_\theta}\frac{1}{r}\frac{\partial\alpha}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin(\theta)}\frac{\partial\alpha}{\partial\varphi} \tag{3}$$

Divergence

$$\langle\boldsymbol{\nabla}, \boldsymbol{a}\rangle = \frac{1}{r^2}\frac{\partial(r^2\alpha_r)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial(a_\theta\sin(\theta))}{\partial\theta} + \frac{1}{r\sin(\theta)}\frac{\partial a_\varphi}{\partial\varphi} \tag{4}$$

Curl

$$\boldsymbol{\nabla}\times\boldsymbol{a} = \boldsymbol{e_r}\frac{1}{r\sin(\theta)}\left(\frac{\partial\left(a_\varphi\sin(\theta)\right)}{\partial\theta} - \frac{\partial a_\varphi}{\partial\varphi}\right) + \boldsymbol{e_\theta}\frac{1}{r}\left(\frac{1}{\sin(\theta)}\frac{\partial\alpha_r}{\partial\varphi} - \frac{\partial a_\varphi}{\partial r}\right) \tag{5}$$

$$+\boldsymbol{e_\varphi}\frac{1}{r}\left(\frac{\partial r\, a_\varphi}{\partial r} - \frac{\partial a_r}{\partial\theta}\right)$$

The Laplacian

$$\nabla^2\alpha = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\alpha}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\alpha}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2\alpha}{\partial\varphi^2} \tag{6}$$

2.2 Quaternionic algebra

2.2.1 Symmetries

The quaternionic number system exists in sixteen discrete symmetry sets (sign flavors). When the real part is ignored, then eight different symmetry sets result. The values of a continuous distribution all belong to the same symmetry set. The parameter space of the distribution may belong to a different symmetry set.

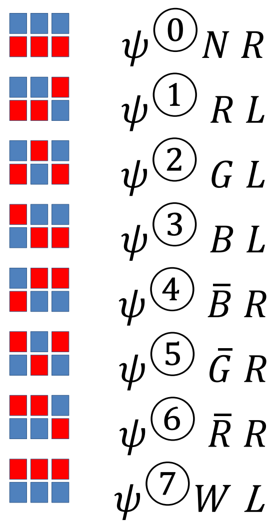


Figure 1: Sign flavors

Eight sign flavors

(discrete symmetries)

Colors N, R, G, B, $\overline{\mathrm{R}}$, $\overline{\mathrm{G}}$, $\overline{\mathrm{B}}$, W

Right or Left handedness R,L

The red block indicates sign up or down with respect to the base sign flavor. For quaternionic distributions the (quaternionic) parameter space acts as base sign flavor.

The 3D Kronecker delta tensor

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \tag{1}$$

The fully antisymmetric Levi-Civita tensor

$$\epsilon_{ijk} = \begin{cases} 1 \text{ if } i,j,k \text{ is an even permutation of } 1,2,3 \\ 0 \text{ if at least two of } i,j,k \text{ are equal} \\ -1 \text{ if } i,j,k \text{ is an even permutation of } 1,2,3 \end{cases} \tag{2}$$

2.2.2 Quaternions

$$a = (a_0, a_1, a_2, a_3) = \sum_{\mu=0}^{3} e_\mu \, \mathrm{a}_\mu = a_0 + \boldsymbol{i}\, a_1 + \boldsymbol{j}\, a_2 + \boldsymbol{k}\, a_3 = \; a_0 + \boldsymbol{a} \tag{1}$$

$$a^* = a_0 - \boldsymbol{a} \tag{2}$$

$$a^* a = a\, a^* = |a|^2 \tag{3}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_{\mu=1}^{3} a_\mu \, b_\mu = \delta_{\mu\nu} a_\mu b_\nu = |\boldsymbol{a}||\boldsymbol{b}| \cos(\theta) \tag{4}$$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} = \pm\left(\epsilon_{\boldsymbol{ijk}} \;\; \boldsymbol{e_i} a_j b_k\right) \tag{5}$$

$$a\, b = \; a_0 \boldsymbol{b} + \; b_0 \, \boldsymbol{a} - \langle \boldsymbol{a}, \boldsymbol{b} \rangle \pm \boldsymbol{a} \times \boldsymbol{b} \tag{6}$$

The colored $\pm$ indicates the handedness of the vector cross product.

$$\boldsymbol{a}\, \boldsymbol{b} = \; -\langle \boldsymbol{a}, \boldsymbol{b} \rangle \pm \boldsymbol{a} \times \boldsymbol{b} \tag{7}$$

$$a\,(b + c) = a\, b + a\, c \tag{8}$$

$$(a \; + \; b)\; c \; = \; a\, c \; + \; b\, c \tag{9}$$

$$(a\, b) c \;\; = \;\; a(b\, c) \tag{10}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \times \boldsymbol{c} \rangle = \; \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c} \rangle \tag{11}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \langle \boldsymbol{a}, \boldsymbol{c} \rangle - \boldsymbol{c} \langle \boldsymbol{a}, \boldsymbol{b} \rangle \tag{12}$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{b} \langle \boldsymbol{a}, \boldsymbol{c} \rangle - \boldsymbol{a} \langle \boldsymbol{b}, \boldsymbol{c} \rangle \tag{13}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = 0 \tag{14}$$

$$\langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c} \times \boldsymbol{d} \rangle = \langle \boldsymbol{a}, \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d}) \rangle = \langle \boldsymbol{a}, \boldsymbol{c} \rangle \langle \boldsymbol{b}. \boldsymbol{d} \rangle - \langle \boldsymbol{a}, \boldsymbol{d} \rangle \langle \boldsymbol{b}. \boldsymbol{c} \rangle \tag{15}$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{d} \rangle \boldsymbol{c} - \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c} \rangle \boldsymbol{d} \qquad 16)$$

3 Quaternionic distributions

3.1.1 Basic properties

A continuous quaternionic distribution contains a scalar field in its real part and a vector field in its imaginary part.

$$f(x) = f_0(x) + \boldsymbol{f}(x) \tag{3}$$

$$a\, f(x) = a_0 \boldsymbol{f}(x) + f_0(x)\, \boldsymbol{a} - \langle \boldsymbol{a}, \boldsymbol{f}(x) \rangle \pm \boldsymbol{a} \times \boldsymbol{f}(x) \tag{2}$$

$$f(x)\, b = f_0(x) \boldsymbol{b} + b_0\, \boldsymbol{f}(x) - \langle \boldsymbol{f}(x), \boldsymbol{b} \rangle \pm \boldsymbol{f}(x) \times \boldsymbol{b} \tag{3}$$

The distributions follow the rules for the quaternion algebra.

$$a\,(f(x) + g(x)) = a\, f(x) + a\, g(x) \tag{4}$$

$$(a + b) f(x) = a\, f(x) + b\, f(x) \tag{5}$$

$$f(x)\, g(x) = f_0(x) \boldsymbol{g}(x) + g_0\,(x) \boldsymbol{f}(x) - \langle \boldsymbol{f}(x), \boldsymbol{g}(x) \rangle \pm \boldsymbol{f}(x) \times \boldsymbol{g}(x) \tag{6}$$

$$\big(f(x) g(x)\big) h(x) = f(x)(g(x)\, h(x)) \tag{7}$$

3.1.2 Symmetries

Continuous quaternionic distributions keep the same discrete symmetries (sign flavor) throughout their domain. The sign flavor of the parameter space acts as reference sign flavor.

3.1.3 Differentials

The quaternionic nabla acts similarly as a normal quaternion

$$\nabla\,(f(x) + g(x)) = \nabla\, f(x) + \nabla\, g(x) \tag{1}$$

$$\nabla\, f(x) = \nabla_0 \boldsymbol{f}(x) + \boldsymbol{\nabla} f_0(x) - \langle \boldsymbol{\nabla}, \boldsymbol{f}(x) \rangle \pm \boldsymbol{\nabla} \times \boldsymbol{f}(x) \tag{2}$$

However

$$\nabla(b\, c) \neq (\nabla\, b) c \tag{3}$$

and

$$\nabla(b\, c) \neq (\nabla\, b) c + b\, \nabla\, c \tag{4}$$

Further

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \alpha \equiv \boldsymbol{\nabla}^{\mathbf{2}} \alpha \tag{5}$$

$$\langle \boldsymbol{\nabla} \times \boldsymbol{\nabla}, \mathbf{a} \rangle = 0 \tag{6}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{a} \rangle = 0 \tag{7}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \mathrm{\alpha} = \mathbf{0} \tag{8}$$

$$\boldsymbol{\nabla}\, \boldsymbol{b} = -\langle \boldsymbol{\nabla}, \boldsymbol{b} \rangle \pm \boldsymbol{\nabla} \times \boldsymbol{b} \qquad (9)$$

$$\boldsymbol{\nabla}\,(\alpha\, \beta) = \alpha \boldsymbol{\nabla}\ \beta + \beta \boldsymbol{\nabla}\, \alpha \qquad (10)$$

$$\boldsymbol{\nabla}\,(\alpha\, \boldsymbol{a}) = \alpha \boldsymbol{\nabla} \times\ \boldsymbol{a} - \alpha \langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle + (\,\boldsymbol{\nabla}\, \alpha) \boldsymbol{a} \qquad (11)$$

$$\langle \boldsymbol{\nabla}, \alpha\, \boldsymbol{a} \rangle = \boldsymbol{a} \boldsymbol{\nabla} \alpha + \alpha \langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle \qquad (12)$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{a} \times \boldsymbol{b} \rangle = \langle \boldsymbol{b}, \boldsymbol{\nabla} \times \boldsymbol{a} \rangle - \langle \boldsymbol{a}, \boldsymbol{\nabla} \times \boldsymbol{b} \rangle \qquad (13)$$

$$\langle \boldsymbol{\nabla}\, \alpha, \boldsymbol{\nabla}\, \beta \rangle = \langle \boldsymbol{\nabla}, \alpha \boldsymbol{\nabla}\, \beta \rangle - \boldsymbol{\alpha} \boldsymbol{\nabla}^{\mathbf{2}} \beta \qquad (14)$$

$$\langle \boldsymbol{\nabla}\, \alpha, \boldsymbol{\nabla} \times \boldsymbol{a} \rangle = -\boldsymbol{\nabla}\,, \boldsymbol{a} \times \boldsymbol{\nabla} \alpha \qquad (15)$$

$$\langle \boldsymbol{\nabla} \times\ \boldsymbol{a}, \boldsymbol{\nabla} \times\ \boldsymbol{b} \rangle = \langle \boldsymbol{b}, \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) \rangle - \langle \boldsymbol{a}, \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{b}) \rangle \qquad (16)$$

$$\boldsymbol{\nabla} \times (\alpha \mathbf{a}) = \alpha \boldsymbol{\nabla} \times \boldsymbol{a} - \boldsymbol{a} \times \boldsymbol{\nabla} \alpha \qquad (17)$$

$$\boldsymbol{\nabla} \times (\alpha \boldsymbol{\nabla} \beta) = (\boldsymbol{\nabla} \alpha) \times \nabla \beta \qquad (18)$$

4 Fourier transform

The Fourier transformation is a linear operator. This transform transfers functions to another parameter space. As a consequence the Fourier transform has no eigenvalues, but the Fourier transform knows functions that are invariant under Fourier transformation.

The Fourier transform cannot cope with functions that have curved parameter spaces. However, it is possible to reduce the parameter space to a domain in which the Fourier transform keeps acceptable accuracy. Another possibility is that the target function is flattened, such that its parameter space becomes flat.

The Fourier transform transfer a orthonormal set of base functions into a new a orthonormal set such that each member of the new set can be written as a linear combination of members of the old set such that none of the coefficients is zero. In fact all coefficients have the same norm.

The Fourier transform converts the nabla operator into an operator that does not differentiate but multiplies the converted function with a factor. That operator will be called a momentum operator.

The Fourier transform has an inverse. It turns the momentum operator into the nabla operator.

The Fourier transform converts convolution of two functions into the multiplication of the two functions and vice versa.

In order to simplify the discussion we restrict it to the case that the parameter spaces of the functions are not curved.

4.1 Fourier transform properties

4.1.1 Linearity

The Fourier transform is a linear operator

$$\mathcal{F}\big(g(q)\big) = \tilde{g}(p) \qquad (1)$$

$$\mathcal{F}(a\, g(q) + b\, h(q)) = a\, \tilde{g}(p) + b\, \tilde{h}(p) \qquad (2)$$

1.1.1 Differentiation

Fourier transformation converts differentiation into multiplication with the canonical conjugated coordinate.

$$\mathrm{g}(q) = \nabla f(q) \qquad (1)$$

$$\tilde{\mathrm{g}}(p) = \mathrm{p}\tilde{f}(p) \tag{2}$$

$$\mathrm{g}(q) = \nabla f(q) = \nabla_0 f_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle \pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q)\right) \tag{3}$$

$$\tilde{\mathrm{g}}(k) = \mathrm{k}\tilde{f}(k) = \mathrm{k}_0 \tilde{f}_0(k) \mp \langle \mathbf{k}, \tilde{\boldsymbol{f}}(k) \rangle \pm \mathrm{k}_0 \tilde{\boldsymbol{f}}(k) + \mathbf{k}\tilde{f}_0(k) \pm \left(\pm \mathbf{k} \times \tilde{\boldsymbol{f}}(k)\right) \tag{4}$$

For the imaginary parts holds:

$$\mathbf{g}(q) = \pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q)\right) \tag{5}$$

$$\tilde{\mathbf{g}}(k) = \pm \mathrm{k}_0 \tilde{\boldsymbol{f}}(k) + \mathbf{k}\tilde{f}_0(k) \pm \left(\pm \mathbf{k} \times \tilde{\boldsymbol{f}}(k)\right) \tag{6}$$

By using

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} f_0(q) = \mathbf{0} \tag{7}$$

and

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{f}(q) \rangle = 0 \tag{8}$$

It can be seen that for the static part ($\nabla_0 f(q) = 0$) holds:

$$\mathbf{g}(q) = \boldsymbol{\nabla} f_0(q) \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q)\right) \tag{9}$$

$$\tilde{\mathbf{g}}(k) = \mathbf{k}\tilde{f}_0(k) \pm \left(\pm\mathbf{k} \times \tilde{\boldsymbol{f}}(k)\right) \tag{10}$$

1.1.2 Parseval's theorem

Parseval's theorem runs:

$$\int f^*(q) \cdot g(q) \cdot dV_q = \int \tilde{f}^*(p) \cdot \tilde{g}(p) \cdot dV_p \tag{1}$$

This leads to

$$\int |f(q)|^2 \cdot dV_q = \int \left|\tilde{f}(p)\right|^2 \cdot dV_p \tag{2}$$

1.1.3 Convolution

Through Fourier transformation a convolution changes into a simple product and vice versa.

$$\mathcal{F}(f(q) \circ g(q)) = \tilde{f}(p) \cdot \tilde{g}(p) \tag{1}$$

4.2 Helmholtz decomposition

The Helmholtz decomposition splits the **static** vector field $\boldsymbol{F}$ in a (transversal) divergence free part $\boldsymbol{F_t}$ and a (one dimensional longitudinal) rotation free part $\boldsymbol{F_l}$.

$$\boldsymbol{F} = \boldsymbol{F_t} + \boldsymbol{F_l} = \boldsymbol{\nabla} \times \boldsymbol{f} - \boldsymbol{\nabla} f_0 \tag{1}$$

Here f_0 is a scalar field and $\boldsymbol{f}$ is a vector field. In quaternionic terms f_0 and $\boldsymbol{f}$ are the real and the imaginary part of a quaternionic field f. $\boldsymbol{F}$ is an imaginary quaternionic distribution.

The significance of the terms "longitudinal" and "transversal" can be understood by computing the local three-dimensional Fourier transform of the vector field $\boldsymbol{F}$, which we call $\widetilde{\boldsymbol{F}}$. Next decompose this field, at each point $\boldsymbol{k}$, into two components, one of which points longitudinally, i.e. parallel to $\boldsymbol{k}$, the other of which points in the transverse direction, i.e. perpendicular to $\boldsymbol{k}$.

$$\widetilde{\boldsymbol{F}}(\boldsymbol{k}) = \widetilde{\boldsymbol{F}}_l(\boldsymbol{k}) + \widetilde{\boldsymbol{F}}_t(\boldsymbol{k}) \tag{2}$$

$$\langle \boldsymbol{k}, \widetilde{\boldsymbol{F}}_t(\boldsymbol{k}) \rangle = 0 \tag{3}$$

$$\boldsymbol{k} \times \widetilde{\boldsymbol{F}}_l(\boldsymbol{k}) = \boldsymbol{0} \tag{4}$$

The Fourier transform converts gradient into multiplication and vice versa. Due to these properties the inverse Fourier transform gives:

$$\boldsymbol{F} = \boldsymbol{F}_l + \boldsymbol{F}_t \tag{5}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{F}_t \rangle = 0 \tag{6}$$

$$\boldsymbol{\nabla} \times \boldsymbol{F}_l = \boldsymbol{0} \tag{7}$$

So, this split indeed conforms to the Helmholtz decomposition.

This interpretation relies on idealized circumstance in which the decomposition runs along straight lines. This idealized condition is not provided in a curved parameter space. In curved parameter space the decomposition and the interpretation via Fourier transformation only work locally and with reduced accuracy.

4.2.1 Quaternionic Fourier transform split

The longitudinal Fourier transform represents only part of the full quaternionic Fourier transform. It depends on the selection of a radial line $\boldsymbol{k}(q)$ in p space that under ideal conditions runs along a straight line.

$$\mathcal{F}_{\mathbf{k}}\big(g(q)\big) = \mathcal{F}(g(q), \boldsymbol{k}(q)) \tag{1}$$

Or

$$\mathcal{F}_{\parallel}\big(g(q)\big) \stackrel{\text{def}}{=} \mathcal{F}\left(g_{\parallel}(q)\right) \tag{2}$$

It relates to the full quaternionic Fourier transform Ƒ

$$\mathcal{F}\big(g(q)\big) = \tilde{g}(p) \tag{3}$$

The inverse Fourier transform runs:

$$\mathcal{F}^{-1}(\tilde{g}(p)) = g(q) \tag{4}$$

The split in longitudinal and transverse Fourier transforms corresponds to a corresponding split in the multi-dimensional Dirac delta function.

4.3 Fourier integral

For the bra-ket inner product holds:

$$< q\big|\breve{P}\, f > = \hbar \cdot \nabla_q < q\big|f > = \hbar \cdot \nabla_q f^*(q) = \mathrm{g}(q) \tag{1}$$

$$= \int_{\boldsymbol{p}} < q|p > \cdot < p|g >$$

The static imaginary part is

$$< q\big|\breve{\boldsymbol{P}}\, f > = \hbar \cdot \boldsymbol{\nabla}_q < q\big|f > = \hbar \cdot \boldsymbol{\nabla}_q \boldsymbol{f}^*(q) = \mathbf{g}(q) \tag{2}$$

$$= Im\left(\int\limits_{\boldsymbol{p}} < q|p >\cdot< p|\boldsymbol{g} >\right) = \int\limits_{\boldsymbol{p}} Im(< q|p >\cdot< p|\boldsymbol{g} >)$$

$$= \int\limits_{\boldsymbol{p}} Im(< q|p >\cdot< p|\boldsymbol{g_l} >) + \int\limits_{\boldsymbol{p}} Im(< q|p >\cdot< p|\boldsymbol{g_t} >)$$

$$= \int\limits_{\boldsymbol{p}} Im(< q|p >\cdot \boldsymbol{\tilde{g}_l}(p)) + \int\limits_{\boldsymbol{p}} Im(< q|p >\cdot \boldsymbol{\tilde{g}_t}(p))$$

The left part is the longitudinal inverse Fourier transform of field $\boldsymbol{\tilde{g}}(p)$**.**

The right part is the transverse inverse Fourier transform of field $\boldsymbol{\tilde{g}}(p)$**.**

For the Fourier transform of $\mathbf{g}(q)$ holds the split:

$$\boldsymbol{\tilde{g}}(p) = \int\limits_{\boldsymbol{q}} Im(< p|q >\cdot \boldsymbol{g_l}(q)) + \int\limits_{\boldsymbol{p}} Im(< p|q >\cdot \boldsymbol{g_t}(q)) \tag{3}$$

$$= \int\limits_{\boldsymbol{q}} Im(< p|q >\cdot \boldsymbol{g}(q))$$

The longitudinal direction is a one dimensional (radial) space. The corresponding transverse direction is tangent to a sphere in 3D. Its direction depends on the field $\mathbf{g}(q)$ or alternatively on the combination of field f and the selected (ideal) coordinate system $\breve{Q}$.

For a weakly curved coordinate system Ϙ̆ the formulas hold with a restricted accuracy and within a restricted region.

4.3.1 Alternative formulation

The reference S. Thangavelu[1] provides an alternative specification of the multidimensional Fourier transform .

4.4 Functions invariant under Fourier transform

In this section we confine to a complex part of the Hilbert space.

See http://en.wikipedia.org/wiki/Hermite_polynomials.

There exist two types of Hermite polynomials: (1, 2)

1. The probalist's Hermite polynomials:

$$H_n^{prob}(z) = (-1)^n \exp(½z^2)\, \frac{d^n}{dz^n} \exp(-½z^2).$$

2. The physicist's Hermite polynomials

$$H_n^{phys}(z) = (-1)^n \exp(z^2) \frac{d^n}{dx^n} \exp(-z^2) = \exp(½z^2) \left(z - \frac{d}{dz}\right) \exp(-½z^2)$$

These two definitions are *not* exactly equivalent; either is a rescaling of the other:

$$H_n^{phys}(z) = 2^{n/2}\, H_n^{prob}\left(z\sqrt{2}\right) \tag{3}$$

In the following we focus on the physicist's Hermite polynomials.

The Gaussian function ϕ(z) defined by

$$\varphi(x) \; = \; exp(-\pi\, z^2) \tag{4}$$

is an eigenfunction of $\mathcal{F}$. It means that its Fourier transform has the same form.

[1] http://www.math.iitb.ac.in/atm/faha1/veluma.pdf

As $\mathcal{F}^4 = \mathrm{I}$ any λ in its spectrum $\sigma\,(\mathcal{F})$ satisfies $\lambda^4 = 1$: Hence,

$$\sigma\,(\mathcal{F}) = \{1; -1;\ i; -i\}. \tag{5}$$

We take the Fourier transform of the expansion:

$$exp(-½\, z^2 + 2\, z\, c - c^2) = \sum_{n=0}^{\infty} exp(-½\, z^2)\, H_n(z)\, c^n/n! \tag{6}$$

First we take the Fourier transform of the left hand side:

$$\frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} exp(-\boldsymbol{k}\, z\, p_z)\, exp(-½\, z^2 + 2\, z\, c - c^2)\ dz \tag{7}$$

$$= exp(-½\, p_z^2 - 2\, \boldsymbol{k}\, p_z\, c + c^2)$$

$$= \sum_{n=0}^{\infty} exp(-½\, p_z^2)\, H_n(p_z)\, (-\boldsymbol{k}\, c)^n/n!$$

The Fourier transform of the right hand side is given by

$$\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \int_{z=-\infty}^{\infty} exp(-\boldsymbol{k}\, z\, p_z) \cdot exp(-½\, z^2)\, H_n(z)\, c^n/n!\ dz \tag{8}$$

Equating like powers of c in the transformed versions of the left- and right-hand sides gives

$$\frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} \exp(-\boldsymbol{k}\, z\, p_z) \cdot exp(-½\, z^2)\, H_n(z)\, c^n/n!\ dz \tag{9}$$

$$= (-\boldsymbol{k})^n \cdot \exp(-½\, p_z^2)\ H_n(p_z)\, \frac{c^n}{n!}$$

Let us define the Hermite functions $\psi_n(z)$

$$\psi_n(z) \stackrel{\text{def}}{=} \ < z|\psi_n > = \mathrm{c}_n \exp(-½\, z^2)\ H_n(z) \tag{10}$$

$$|\mathcal{F}\,\psi_n >= \ |\psi_n > \ (-\boldsymbol{k})^n \qquad (11)$$

with suitably chosen c_n so as to make

$$\|\psi_n\|^2 \ = \ 1 \qquad (12)$$

$$\mathrm{c}_n = \frac{1}{\sqrt{2^n n!\,\sqrt{\pi}}} \qquad (13)$$

The importance of the Hermite functions lie in the following theorem.

"The Hermite functions ψ_n; n $\in \mathbb{N}$ form an orthonormal basis for $L^2(\mathbb{R})$"

Consider the operator

$$H \ = \ -½\frac{d^2}{dz^2} + \ ½\, z^2 \qquad (14)$$

Apply this to $\psi_n(z)$:

$$H \cdot \psi_n(z) = \ (½ \ + \ n)\, \psi_n(z) \qquad (15)$$

Thus, ψ_n is an eigenfunction of H.

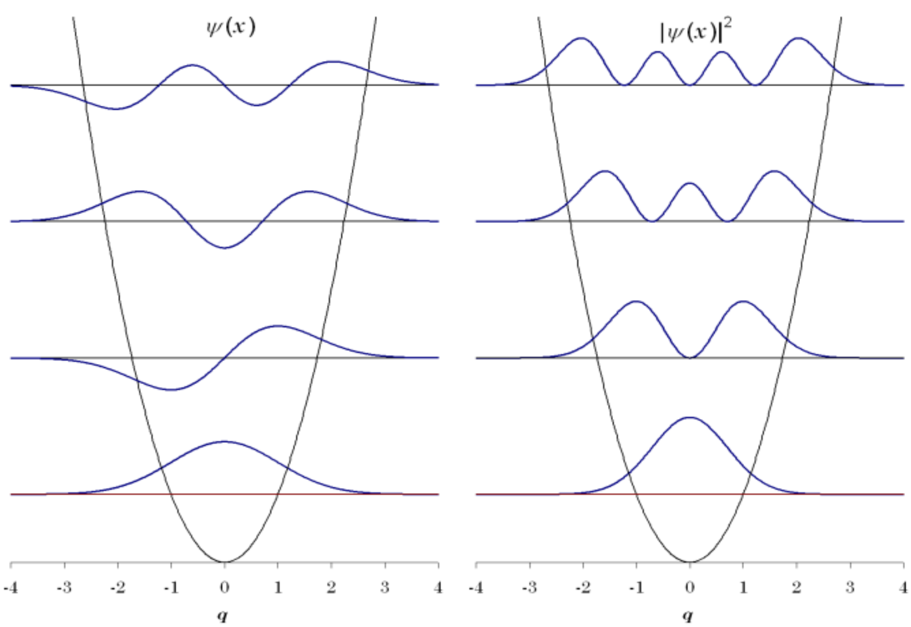
Let $\mathrm{f} \ = \ \psi_{4\mathrm{k}+\mathrm{j}}$ be any of the Hermite functions. Then we have

$$\sum_{n=-\infty}^{\infty} f(y + n) \cdot \exp\big(-2\,\pi\,\boldsymbol{k}\,x\,(y+n)\big) \tag{16}$$

$$= (-\boldsymbol{k})^j \sum_{n=-\infty}^{\infty} f(x + n)\, exp(2\,\pi\,\boldsymbol{k}\,n\,y)$$

The vectors $|\psi_n>$ are eigenvectors of the Fourier transform operator with eigenvalues $(-\boldsymbol{k})^n$. The eigenfunctions $\psi_n(x)$ represent eigenvectors $|\psi_n>$ that span the complex Hilbert space $\mathbf{H}_{\boldsymbol{k}}$.

For higher n the central parts of $\psi_n(x)$ and $|\psi_n(x)|^2$ become a sinusoidal form.



A **coherent state**[2] is a specific kind of state[3] of the quantum harmonic oscillator whose dynamics most closely resemble the oscillating behavior of a classical harmonic oscillator system. The ground state is a squeezed coherent state[4].

[2] http://en.wikipedia.org/wiki/Coherent_state
[3] States
[4] Canonical conjugate: Heisenberg's uncertainty

4.5 Special Fourier transform pairs

Functions that keep the same form through Fourier transformation are:

$$f(q) = \exp(-|q|^2) \tag{1}$$

$$f(q) = \frac{1}{|q|} \tag{2}$$

$$f(q) = comb(q) \tag{3}$$

The comb function consists of a set of equidistant Dirac delta functions.

Other examples of functions that are invariant under Fourier transformation are the linear and spherical harmonic oscillators and the solutions of the Laplace equation.

4.6 Complex Fourier transform invariance properties

Each even function $f(q) \Leftrightarrow \tilde{f}(p)$ induces a Fourier invariant:

$$h(q) = \sqrt{2\pi}\, f(q) + \tilde{f}(q). \tag{1}$$

$$\tilde{h}(q) = \sqrt{2\pi}\, h(q) \tag{2}$$

Each odd function $f(q) \Leftrightarrow \tilde{f}(p)$ induces a Fourier invariant:

$$h(q) = \sqrt{2\pi}\, f(q) - \tilde{f}(q). \tag{3}$$

A function $f(q)$ is invariant under Fourier transformation *if and only if* the function f satisfies the differential equation

$$\frac{\partial^2 f(q)}{\partial q^2} - t^2 f(q) = \alpha\, f(q), \text{ for some scalar } \alpha \in C. \tag{4}$$

The Fourier transform invariant functions are fixed apart from a scale factor. That scale factor can be 1, ***k***, -1 or –***k***. ***k*** is an imaginary base number in the longitudinal direction.

Fourier-invariant functions show iso-resolution, that is, $\Delta_\text{p} = \Delta_\text{q}$ in the Heisenberg's uncertainty relation.

For proves see: http://www2.ee.ufpe.br/codec/isoresolution_vf.pdf.

5 Quaternionic probability amplitude distributions

Continuous quaternionic distributions contain a scalar field in their real part and an associated vector field in their imaginary part. In a quaternionic probability amplitude distribution (QPAD), the scalar field can be interpreted as a distribution of the density of property carriers. The associated vector field can be interpreted as a distribution of the current density of these carriers.

5.1 Differential equation

For QPAD's the equation for the differential can be interpreted as a differential continuity equation. Another name for continuity equation is balance equation. The differential continuity equation is paired by an integral continuity equation. The differential equation runs:

$$g(q) = g_0(q) + \boldsymbol{g}(q) = \nabla f(q)$$

$$= \nabla_0 f_0(q) \mp \langle \boldsymbol{\nabla}, \boldsymbol{f}(q) \rangle$$

$$\pm \nabla_0 \boldsymbol{f}(q) + \boldsymbol{\nabla} f_0(q) \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{f}(q) \right)$$

5.2 Continuity equation

Let us approach the balance equation from the integral variety of the balance equation.

When $\rho_0(q)$ is interpreted as a charge density distribution, then the conservation of the corresponding charge[5] is given by the continuity equation:

[5] Also see Noether's laws: http://en.wikipedia.org/wiki/Noether%27s_theorem

Total change within V = flow into V + production inside V (1)

In formula this means:

$$\frac{d}{d\tau}\int\limits_V \rho_0\, dV = \oint_S \hat{\boldsymbol{n}}\rho_0 \frac{\boldsymbol{v}}{c}\, dS + \int\limits_V s_0\, dV \tag{2}$$

$$\int\limits_V \nabla_0\rho_0\, dV = \int\limits_V \langle \boldsymbol{\nabla}, \boldsymbol{\rho}\rangle\, dV + \int\limits_V s_0\, dV \tag{3}$$

The conversion from formula (2) to formula (3) uses the Gauss theorem[6]. Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface S, $\boldsymbol{v}(\tau, \boldsymbol{q})$ is the velocity at which the charge density $\rho_0(\tau, \boldsymbol{q})$ enters volume V and s_0 is the source density inside V. In the above formula $\boldsymbol{\rho}$ stands for

$$\boldsymbol{\rho} = \rho_0 \boldsymbol{v}/c \tag{4}$$

It is the flux (flow per unit area and unit time) of ρ_0 .

The combination of $\rho_0(\tau, \boldsymbol{q})$ and $\boldsymbol{\rho}(\tau, \boldsymbol{q})$ is a quaternionic skew field $\rho(\tau, \boldsymbol{q})$ and can be seen as a probability amplitude distribution (QPAD).

$$\rho \stackrel{\text{def}}{=} \rho_0 + \boldsymbol{\rho} \tag{5}$$

$\rho(\tau, \boldsymbol{q})\rho^*(\tau, \boldsymbol{q})$ can be seen as an overall probability density distribution of the presence of the carrier of the charge. $\rho_0(\tau, \boldsymbol{q})$ is a charge density distribution. $\boldsymbol{\rho}(\tau, \boldsymbol{q})$ is the current density distribution.

This results in the law of charge conservation:

$$s_0(\tau, \boldsymbol{q}) = \nabla_0\rho_0(\tau, \boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \big(\rho_0(\tau, \boldsymbol{q})\boldsymbol{v}(\tau, \boldsymbol{q}) + \boldsymbol{\nabla} \times \boldsymbol{a}(\tau, \boldsymbol{q})\big)\rangle \tag{6}$$

[6] http://en.wikipedia.org/wiki/Divergence_theorem

$$= \nabla_0 \rho_0(\tau, \boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(\tau, \boldsymbol{q}) + \boldsymbol{A}(\tau, \boldsymbol{q}) \rangle$$

$$= \nabla_0 \rho_0(\tau, \boldsymbol{q}) \mp \langle \boldsymbol{v}(\tau, \boldsymbol{q}), \boldsymbol{\nabla} \rho_0(\tau, \boldsymbol{q}) \rangle \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(\tau, \boldsymbol{q}) \rangle \, \rho_0(\tau, \boldsymbol{q})$$

$$\mp \langle \boldsymbol{\nabla}, \boldsymbol{A}(\tau, \boldsymbol{q}) \rangle$$

The blue colored $\pm$ indicates quaternionic sign selection through conjugation of the field $\rho(\tau, \boldsymbol{q})$. The field $\boldsymbol{a}(\tau, \boldsymbol{q})$ is an arbitrary differentiable vector function.

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{a}(\tau, \boldsymbol{q}) \rangle = 0 \tag{7}$$

$\boldsymbol{A}(\tau, \boldsymbol{q}) \stackrel{\text{def}}{=} \boldsymbol{\nabla} \times \boldsymbol{a}(\tau, \boldsymbol{q})$ is always divergence free. In the following we will neglect $\boldsymbol{A}(\tau, \boldsymbol{q})$.

Equation (6) represents a balance equation for charge density. What this charge actually is, will be left in the middle. It can be one of the properties of the carrier or it can represent the full ensemble of the properties of the carrier.

Up to this point the investigation only treats the real part of the full equation. The full continuity equation runs:

$$s(\tau, \boldsymbol{q}) = \nabla \rho(\tau, \boldsymbol{q}) = s_0(\tau, \boldsymbol{q}) + \boldsymbol{s}(\tau, \boldsymbol{q}) \tag{8}$$

$$= \nabla_0 \rho_0(\tau, \boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(\tau, \boldsymbol{q}) \rangle \pm \nabla_0 \boldsymbol{\rho}(\tau, \boldsymbol{q}) + \boldsymbol{\nabla} \rho_0(\tau, \boldsymbol{q}) \pm \left(\pm \boldsymbol{\nabla} \times \boldsymbol{\rho}(\tau, \boldsymbol{q}) \right)$$

$$= \nabla_0 \rho_0(\tau, \boldsymbol{q}) \mp \langle \boldsymbol{v}(\tau, \boldsymbol{q}), \boldsymbol{\nabla} \rho_0(\tau, \boldsymbol{q}) \rangle \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(\tau, \boldsymbol{q}) \rangle \, \rho_0(\tau, \boldsymbol{q})$$

$$\pm \nabla_0 \boldsymbol{v}(\tau, \boldsymbol{q}) + \nabla_0 \rho_0(\tau, \boldsymbol{q}) + \boldsymbol{\nabla} \rho_0(\tau, \boldsymbol{q})$$

$$\pm\left(\pm\left(\rho_0(\tau,\boldsymbol{q})\,\boldsymbol{\nabla}\times\boldsymbol{v}(\tau,\boldsymbol{q})-\boldsymbol{v}(\tau,\boldsymbol{q})\times\boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q})\right)\right)$$

$$s_0(\tau,\boldsymbol{q})=2\nabla_0\rho_0(\tau,\boldsymbol{q})\mp\langle\boldsymbol{v}(q),\boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q})\rangle\mp\langle\boldsymbol{\nabla},\boldsymbol{v}(\tau,\boldsymbol{q})\rangle\,\rho_0(\tau,\boldsymbol{q}) \tag{9}$$

$$\boldsymbol{s}(\tau,\boldsymbol{q})=\pm\nabla_0\boldsymbol{v}(\tau,\boldsymbol{q})\pm\ \boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q}) \tag{10}$$

$$\pm\left(\pm\left(\rho_0(\tau,\boldsymbol{q})\,\boldsymbol{\nabla}\times\boldsymbol{v}(\tau,\boldsymbol{q})-\boldsymbol{v}(\tau,\boldsymbol{q})\times\boldsymbol{\nabla}\rho_0(\tau,\boldsymbol{q})\right)\right)$$

The red sign selection indicates a change of handedness by changing the sign of one of the imaginary base vectors. Conjugation also causes a switch of handedness. It changes the sign of all three imaginary base vectors.

In its simplest form the full continuity equation runs:

$$s(\boldsymbol{q},\tau)=\nabla\rho(\boldsymbol{q},\tau)$$

Thus the full continuity equation specifies a quaternionic distribution s as a flat differential $\nabla\rho$.

When we go back to the integral balance equation, then holds for the imaginary parts:

$$\frac{d}{d\tau}\int_V \boldsymbol{\rho}\,dV=-\oint_S\widehat{\boldsymbol{n}}\rho_0\,dS-\oint_S\widehat{\boldsymbol{n}}\times\boldsymbol{\rho}\,dS+\int_V\boldsymbol{s}\,dV \tag{4}$$

$$\int_V\nabla_0\,\boldsymbol{\rho}\,dV=-\int_V\boldsymbol{\nabla}\rho_0\,dV-\int_V\boldsymbol{\nabla}\times\boldsymbol{\rho}\,dV+\int_V\boldsymbol{s}\,dV \tag{5}$$

For the full integral equation holds:

$$\frac{d}{d\tau}\int\limits_V \rho \, dV + \oint_S \widehat{\boldsymbol{n}}\rho \, dS = \int\limits_V s \, dV \tag{6}$$

$$\int\limits_V \nabla \rho \, dV = \int\limits_V s \, dV \tag{7}$$

Here $\widehat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface *S*, $\boldsymbol{v}(\tau, \boldsymbol{q})$ is the velocity at which the charge density $\rho_0(\tau, \boldsymbol{q})$ enters volume *V* and s_0 is the source density inside *V.* In the above formula ρ stands for

$$\rho = \rho_0 + \boldsymbol{\rho} = \rho_0 + \frac{\rho_0 \boldsymbol{v}}{c} \tag{8}$$

It is the flux (flow per unit of area and per unit of progression) of ρ_0 . t stands for progression (not coordinate time).

5.3 Fluid dynamics

The quaternionic continuity equation is the foundation of quaternionic fluid dynamics. Depending on the nature of the streaming medium, this branch of physics exists in two forms.

- In conventional fluid dynamics the streaming charge carriers are elements of a gas or a liquid.
- In quantum fluid dynamics the streaming charge carriers are tiny patches of the parameter space of the QPAD.

It means that in quantum fluid dynamics the coupling of QPAD's can affect the local curvature.

5.3.1 Coupling equation

In its simplest form the continuity equation runs:

$$\nabla\psi = \varphi$$

The continuity equation couples the local distribution ψ to a source φ.

The coupling strength can be made explicit. This results in the coupling equation.

$$\nabla\psi = m\,\phi$$

Here m is the coupling factor and ϕ is the adapted source.

6 Conservation laws

The following holds for all QPAD's!!!

Only the interpretation tells whether the QPAD concerns a quantum state function, a photon, a gluon or the field of a single charge, a field of a set of charges or a field corresponding to the density distribution of eventually moving charge carriers.

6.1 Differential potential equations

Let $\phi(q)$ define a quaternionic potential. The potential corresponds to a charge density distribution $\phi_0(q)$ and a current density distribution $\boldsymbol{\phi}(q)$.

Note: This means that the following holds for any QPAD!

$$\phi(q) = \rho_0(q) + \boldsymbol{\rho}(q) = \rho_0(q) + \rho_0(q)\boldsymbol{v}(q) \tag{1}$$

The gradient and curl of $\phi(q)$ are related. In configuration space holds:

$$\mathfrak{F}(q) \stackrel{\text{def}}{=} \nabla\phi(q) = \nabla_0\phi_0(q) \mp \langle\boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle \pm \nabla_0\boldsymbol{\phi}(q) \pm \boldsymbol{\nabla}\phi_0(q) \pm \left(\pm\boldsymbol{\nabla} \times \boldsymbol{\phi}(q)\right) \quad (2)$$

$$\mathfrak{E}(q) \stackrel{\text{def}}{=} -\boldsymbol{\nabla}\phi_0(q) \quad (3)$$

$$\mathfrak{B}(q) \stackrel{\text{def}}{=} \boldsymbol{\nabla} \times \phi(q) \quad (4)$$

$$\mathfrak{F}(q) \stackrel{\text{def}}{=} \nabla\phi(q) = \mathfrak{F}_0(q) + \boldsymbol{\mathfrak{F}}(q) \quad (5)$$

$$\mathfrak{F}_0(q) = \nabla_0\phi_0(q) \mp \langle\boldsymbol{\nabla}, \boldsymbol{\phi}(q)\rangle \quad (6)$$

$$\boldsymbol{\mathfrak{F}}(q) = \mp\mathfrak{E}(q) \pm \mathfrak{B}(q) \pm \nabla_0\boldsymbol{\phi}(q) \quad (7)$$

6.2 Flux vector

The longitudinal direction **k** of field $\mathfrak{E}(q)$ and the direction **i** of field $\mathfrak{B}(q)$ fix two mutual perpendicular directions. This generates curiosity to the significance of the direction $\mathbf{k} \times \mathbf{i}$. With other words what happens with $\mathfrak{E}(q) \times \mathfrak{B}(q)$.

The **flux vector** $\mathfrak{S}(q)$ is defined as:

$$\mathfrak{S}(q) \stackrel{\text{def}}{=} \mathfrak{E}(q) \times \mathfrak{B}(q) \quad (1)$$

6.3 Conservation of energy

$$\langle\boldsymbol{\nabla}, \mathfrak{S}(q)\rangle = \langle\mathfrak{B}(q), \boldsymbol{\nabla} \times \mathfrak{E}(q)\rangle - \langle\mathfrak{E}(q), \boldsymbol{\nabla} \times \mathfrak{B}(q)\rangle \quad (1)$$

$$= -\langle \mathfrak{B}(q), \nabla_0 \mathfrak{B}(q) \rangle - \langle \mathfrak{E}(q), \boldsymbol{\phi}(q) \rangle - \langle \mathfrak{E}(q), \nabla_0 \mathfrak{B}(q) \rangle$$

$$= -½\nabla_0(\langle \mathfrak{B}(q), \mathfrak{B}(q) \rangle + \langle \mathfrak{E}(q), \mathfrak{E}(q) \rangle) - \langle \mathfrak{E}(q), \boldsymbol{\phi}(q) \rangle$$

The **field energy density** is defined as:

$$u_{field}(q) = ½(\langle \mathfrak{B}(q), \mathfrak{B}(q) \rangle + \langle \mathfrak{E}(q), \mathfrak{E}(q) \rangle) = u_{\mathfrak{B}}(q) + u_{\mathfrak{E}}(q) \quad (2)$$

$\mathfrak{S}(q)$ can be interpreted as the **field energy current density**.

The continuity equation for field energy density is given by:

$$\nabla_0 u_{field}(q) + \langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle = -\langle \mathfrak{E}(q), \boldsymbol{\phi}(q) \rangle = -\phi_0(q)\langle \mathfrak{E}(q), \boldsymbol{v}(q) \rangle \quad (3)$$

This means that $\langle \mathfrak{E}(q), \boldsymbol{\phi}(q) \rangle$ can be interpreted as a source term.

6.3.1 Interpretation in physics

Despite the fact that the above equations hold for any QPAD, we give here the physical interpretations when $\mathfrak{E}$ is the electric field and $\mathfrak{B}$ is the magnetic field.

$\phi_0(q)\mathfrak{E}(q)$ represents **force** per unit volume.

$\phi_0(q)\langle \mathfrak{E}(q), \boldsymbol{v}(q) \rangle$ represents **work** per unit volume, or, in other words, the power density. It is known as the Lorentz power density and is equivalent to the time rate of change of the mechanical energy density of the charged particles that form the current $\boldsymbol{\phi}(q)$**.**

$$\nabla_0 u_{field}(q) + \langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle = -\nabla_0 u_{mechanical}(q) \quad (4)$$

$$\nabla_0 u_{mechanical} = \langle \mathfrak{E}(q), \boldsymbol{\phi}(q) \rangle = \phi_0(q)\langle \mathfrak{E}(q), \boldsymbol{v}(q) \rangle \quad (5)$$

$$\nabla_0 \left(u_{field}(q) + u_{mechanical}(q) \right) = -\langle \boldsymbol{\nabla}, \mathfrak{S}(q) \rangle \quad (6)$$

Total change within V = flow into V + production inside V (7)

$$u(q) = u_{field}(q) + u_{mechanical}(q) = u_B(q) + u_E(q) + u_{mechanical}(q) \quad (8)$$

$$U = U_{field} + U_{mechanical} = U_B + U_E + U_{mechanical} = \int_V u \, dV \quad (9)$$

$$\frac{d}{dt} \int_V u \, dV = \oint_S \langle \widehat{\boldsymbol{n}}, \mathfrak{S} \rangle dS + \int_V s_0 \, dV \quad (10)$$

Here the source s_0 is zero.

6.3.2 How to interprete U$_{mechanical}$

$U_{mechanical}$ is the energy of the private field (state function) of the involved particle(s).

6.4 Conservation of linear momentum

$\mathfrak{S}(q)$ can also be interpreted as the **field linear momentum density**. The time rate change of the field linear momentum density is:

$$\nabla_0 \mathfrak{S}(q) = \boldsymbol{g}_{field}(q) = \nabla_0 \, \mathfrak{E}(q) \times \mathfrak{B}(q) + \mathfrak{E}(q) \times \nabla_0 \mathfrak{B}(q) \quad (1)$$

$$= \left(\boldsymbol{\nabla} \times \mathfrak{B}(q) - \boldsymbol{\rho}(q) \right) \times \mathfrak{B}(q) - \mathfrak{E}(q) \times \boldsymbol{\nabla} \times \mathfrak{E}(q) \quad (2)$$

$$\boldsymbol{G}(\mathfrak{E}) = \mathfrak{E} \times (\boldsymbol{\nabla} \times \mathfrak{E}) = \langle \boldsymbol{\nabla} \mathfrak{E}, \mathfrak{E} \rangle - \langle \mathfrak{E}, \mathfrak{E} \rangle = ½ \boldsymbol{\nabla} \langle \mathfrak{E}, \mathfrak{E} \rangle - \langle \mathfrak{E}, \mathfrak{E} \rangle \quad (3)$$

$$= -\boldsymbol{\nabla}(\mathfrak{E}\mathfrak{E}) + ½ \boldsymbol{\nabla} \langle \mathfrak{E}, \mathfrak{E} \rangle + \langle \boldsymbol{\nabla}, \mathfrak{E} \rangle \mathfrak{E}$$

$$= -\boldsymbol{\nabla}(\mathfrak{E}\mathfrak{E} + ½\mathbf{1_3}\langle\mathfrak{E}\,,\mathfrak{E}\rangle) + \langle\boldsymbol{\nabla}\,,\mathfrak{E}\rangle\mathfrak{E}$$

$$\boldsymbol{G}(\mathfrak{B}) = \mathfrak{B} \times (\boldsymbol{\nabla} \times \mathfrak{B}) = -\boldsymbol{\nabla}(\mathfrak{B}\mathfrak{B} + ½\mathbf{1_3}\langle\mathfrak{B}\,,\mathfrak{B}\rangle) + \langle\boldsymbol{\nabla}\,,\mathfrak{B}\rangle\mathfrak{B} \tag{4}$$

$$\boldsymbol{H}(\mathfrak{B}) = -\boldsymbol{\nabla}(\mathfrak{B}\mathfrak{B} + ½\mathbf{1_3}\langle\mathfrak{B}\,,\mathfrak{B}\rangle) \tag{5}$$

$$\nabla_0\mathfrak{S}(q) = \boldsymbol{G}(\mathfrak{B}) + \boldsymbol{G}(\mathfrak{E}) - \boldsymbol{\rho}(q) \times \mathfrak{B}(q) \tag{6}$$

$$= \boldsymbol{H}(\mathfrak{E}) + \boldsymbol{H}(\mathfrak{B}) - \boldsymbol{\rho}(q) \times \mathfrak{B}(q) + \langle\boldsymbol{\nabla}\,,\mathfrak{B}\rangle\mathfrak{B} + \langle\boldsymbol{\nabla}\,,\mathfrak{E}\rangle\mathfrak{E}$$

$$= \boldsymbol{H}(\mathfrak{E}) + \boldsymbol{H}(\mathfrak{B}) - \boldsymbol{\rho}(q) \times \mathfrak{B}(q) - \rho_0(q)\,\mathfrak{E}(q)$$

$$= \boldsymbol{H}(\mathfrak{E}) + \boldsymbol{H}(\mathfrak{E}) - \boldsymbol{f}(q) = \boldsymbol{\mathcal{T}}(q) - \boldsymbol{f}(q)$$

$\mathcal{T}$(q) is the linear momentum flux tensor.

The linear momentum of the field contained in volume *V* surrounded by surface *S* is:

$$\boldsymbol{P}_{field} = \int_V \boldsymbol{g}_{field}\,dV = \int_V \rho_0\boldsymbol{\phi}\,dV + \int_V \langle\nabla\boldsymbol{\phi},\mathfrak{E}\rangle\,dV + \oint_S \langle\widehat{\boldsymbol{n}},\mathfrak{E}\boldsymbol{A}\rangle dS \tag{7}$$

$$\boldsymbol{f}(q) = \boldsymbol{\rho}(q) \times \mathfrak{B}(q) + \rho_0(q)\,\mathfrak{E}(q) \tag{8}$$

Physically, $\boldsymbol{f}(q)$ is the Lorentz force density. It equals the time rate change of the mechanical linear momentum density $\boldsymbol{g}_{mechanical}$.

$$\boldsymbol{g}_{mechanical}(q) = \rho_{0m}(q)\boldsymbol{v}(q) \tag{9}$$

The force acted upon a single particle that is contained in a volume *V* is:

$$\boldsymbol{F} = \int_V \boldsymbol{f}\, dV = \int_V (\boldsymbol{\rho} \times \mathfrak{B} + \rho_0\, \mathfrak{E})\, dV \tag{10}$$

Brought together this gives:

$$\nabla_0 \left(\boldsymbol{g}_{field}(q) + \boldsymbol{g}_{mechanical}(q)\right) = -\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{T}}(q)\rangle \tag{11}$$

This is the continuity equation for linear momentum.

The component $\mathcal{T}_{\mathrm{ij}}$ is the linear momentum in the i-th direction that passes a surface element in the j-th direction per unit time, per unit area.

Total change within *V* = flow into *V* + production inside *V* (12)

$$\boldsymbol{g}(q) = \boldsymbol{g}_{field}(q) + \boldsymbol{g}_{mechanical}(q) \tag{13}$$

$$\boldsymbol{P} = \boldsymbol{P}_{field} + \boldsymbol{P}_{mechanical} = \int_V \boldsymbol{g}\, dV \tag{14}$$

$$\frac{d}{dt}\int_V \boldsymbol{g}\, dV = \oint_S \langle \widehat{\boldsymbol{n}}, \boldsymbol{\mathcal{T}}\rangle dS + \int_V \boldsymbol{s}_{\boldsymbol{g}}\, dV \tag{15}$$

Here the source $\boldsymbol{s}_{\boldsymbol{g}} = 0$.

6.5 Conservation of angular momentum

6.5.1 *Field angular momentum*

The angular momentum relates to the linear momentum.

$$\boldsymbol{h}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}(q) \tag{1}$$

$$\boldsymbol{h}_{field}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}_{field}(q) \tag{2}$$

$$\boldsymbol{h}_{mechanical}(q) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{g}_{mechanical}(q) \tag{3}$$

$$\boldsymbol{\mathcal{K}}(\boldsymbol{q}_c) = (\boldsymbol{q} - \boldsymbol{q}_c) \times \boldsymbol{\mathcal{T}}(q) \tag{4}$$

This enables the balance equation for angular momentum:

$$\nabla_0 \left(\boldsymbol{h}_{field}(\boldsymbol{q}_c) + \boldsymbol{h}_{mechanical}(\boldsymbol{q}_c) \right) = -\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{K}}(\boldsymbol{q}_c) \rangle \tag{5}$$

Total change within V = flow into V + production inside V

$$\boldsymbol{J} = \boldsymbol{J}_{field} + \boldsymbol{J}_{mechanical} = \int_V \boldsymbol{h}\, dV \tag{6}$$

$$\frac{d}{dt} \int_V \boldsymbol{h}\, dV = \oint_S \langle \widehat{\boldsymbol{n}}, \boldsymbol{\mathcal{K}} \rangle dS + \int_V \boldsymbol{s_h}\, dV \tag{7}$$

Here the source $\boldsymbol{s_h} = 0$.

For a localized charge density contained within a volume *V* holds for the mechanical torsion:

$$\tau(\boldsymbol{q}_c) = \int\limits_V (\boldsymbol{q}' - \boldsymbol{q}_c) \times \boldsymbol{f}(q')dV \tag{8}$$

$$= \int\limits_V (\boldsymbol{q}' - \boldsymbol{q}_c) \times \left(\rho_0(q')\boldsymbol{\mathfrak{E}}(q') + \boldsymbol{j}(q') \times \boldsymbol{\mathfrak{B}}(q')\right)dV$$

$$= Q(\boldsymbol{q} - \boldsymbol{q}_c) \times (\boldsymbol{\mathfrak{E}}(q) + \boldsymbol{v}(q) \times \boldsymbol{\mathfrak{B}}(q))$$

$$\boldsymbol{J}_{field}(\boldsymbol{q}_c) = \boldsymbol{J}_{field}(\boldsymbol{0}) + \boldsymbol{q}_c \times \boldsymbol{P}(q) \tag{9}$$

Using

$$\langle \boldsymbol{\nabla a}, \boldsymbol{b} \rangle = \boldsymbol{n}_\nu \frac{\partial a_\mu}{\partial q_\nu} b_\mu \tag{10}$$

$$\langle \boldsymbol{b}, \boldsymbol{\nabla a} \rangle = \boldsymbol{n}_\mu \frac{\partial a_\mu}{\partial q_\nu} b_\mu \tag{11}$$

holds

$$\boldsymbol{J}_{field}(\boldsymbol{0}) = \int\limits_V \boldsymbol{q}' \times \boldsymbol{\mathfrak{S}}(q')dV = \int\limits_V \boldsymbol{q}' \times \boldsymbol{\mathfrak{E}}(q') \times \boldsymbol{\nabla} \times \boldsymbol{\phi}(q')\, dV \tag{12}$$

$$= \int\limits_V (\boldsymbol{q}' \times \langle (\boldsymbol{\nabla\phi}), \boldsymbol{\mathfrak{E}} \rangle - \langle \boldsymbol{q}' \times \boldsymbol{\mathfrak{E}}, (\boldsymbol{\nabla\phi}) \rangle)\, dV$$

$$= \int_V \boldsymbol{q}' \times \langle (\nabla \boldsymbol{\phi}), \mathfrak{E} \rangle dV$$

$$+ \int_V \mathfrak{E} \times \boldsymbol{\phi} \, dV - \int_V \langle \nabla, \mathfrak{E} \boldsymbol{q}' \times \boldsymbol{\phi} \rangle dV + \int_V (\boldsymbol{q}' \times \boldsymbol{\phi}) \langle \nabla, \mathfrak{E} \rangle dV$$

6.5.2 Spin

Define the non-local spin term, *which does not depend on* $\boldsymbol{q}'$ as:

$$\boldsymbol{\Sigma}_{field} = \int_V \mathfrak{E}(q) \times \boldsymbol{\phi}(q) dV \tag{13}$$

Notice

$$\boldsymbol{\phi}(q) \times \nabla \phi_0(q) = \phi_0 \nabla \times \boldsymbol{\phi}(q) + \nabla \times \left(\phi_0(q) \boldsymbol{\phi}(q) \right)$$

And

$$\boldsymbol{L}_{field}(\boldsymbol{0}) = \int_V \boldsymbol{q}' \times \langle (\nabla \boldsymbol{\phi}), \mathfrak{E} \rangle dV + \int_V \boldsymbol{q}' \times \rho_0 \boldsymbol{\phi} dV \tag{14}$$

Using Gauss:

$$\int_V \langle \nabla, \boldsymbol{a} \rangle dV = \oint_S \langle \widehat{\boldsymbol{n}}, \boldsymbol{a} \rangle dS \tag{15}$$

And

$$\rho_0 = \langle \nabla, \mathfrak{E} \rangle \tag{16}$$

Leads to:

$$\boldsymbol{J}_{field}(\mathbf{0}) = \boldsymbol{\Sigma}_{field} + \boldsymbol{L}_{field}(\mathbf{0}) + \oint_{S} \langle \widehat{\boldsymbol{n}}, \mathfrak{E}\boldsymbol{q}' \times \boldsymbol{\phi} \rangle dS \tag{17}$$

6.5.3 Spin discussion

The spin term is defined by:

$$\boldsymbol{\Sigma}_{field} = \int_{V} \mathfrak{E}(q) \times \boldsymbol{\phi}(q) dV \tag{1}$$

In free space the charge density ρ_0 vanishes and the scalar potential ϕ_0 shows no variance. Only the vector potential $\boldsymbol{\phi}$ may vary with q_0. Thus:

$$\mathfrak{E} = \boldsymbol{\nabla}\phi_0 - \nabla_0\boldsymbol{\phi} \approx -\nabla_0\boldsymbol{\phi} \tag{2}$$

$$\boldsymbol{\Sigma}_{field} \approx \int_{V} (\nabla_0\boldsymbol{\phi}(q)) \times \boldsymbol{\phi}(q) dV \tag{3}$$

Depending on the selected field $\boldsymbol{\Sigma}_{field}$ has two versions that differ in their sign. These versions can be combined in a single operator:

$$\boldsymbol{\Sigma}_{field} = \begin{bmatrix} \boldsymbol{\Sigma}^{+}{}_{field} \\ \boldsymbol{\Sigma}^{-}{}_{field} \end{bmatrix} \tag{4}$$

If $\frac{\boldsymbol{\phi}(q)}{|\boldsymbol{\phi}(q)|}$ can be interpreted as tantrix (q_0)) and $\frac{\nabla_0\boldsymbol{\phi}(q)}{|\nabla_0\boldsymbol{\phi}(q)|}$ can be interpreted as the principle normal $\boldsymbol{N}(q_0)$, then $\frac{(\nabla_0\boldsymbol{\phi}(q))\times\boldsymbol{\phi}(q)}{|(\nabla_0\boldsymbol{\phi}(q))\times\boldsymbol{\phi}(q)|}$ can be interpreted as the binormal $\mathfrak{B}(q_0)$.

From these quantities the curvature and the torsion[7] can be derived.

[7]Path characteristics

$$\begin{bmatrix} \dot{\boldsymbol{T}}(t) \\ \dot{\boldsymbol{N}}(t) \\ \dot{\boldsymbol{B}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(\mathrm{t}) & 0 \\ -\kappa(\mathrm{t}) & 0 & \tau(\mathrm{t}) \\ 0 & -\tau(\mathrm{t}) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}(t) \\ \boldsymbol{N}(t) \\ \boldsymbol{B}(t) \end{bmatrix} \tag{5}$$