Introduction. In global differential geometry, people want to get some global topological property of a manifold through the analysis of local differentiable structures. The global topological property can either be homotopy types or (co-)homology classes. In this dissertation, we mainly treat with homotopy property. We discuss some already-known results about the relation between Riemannian structure and the 1-homotopy group. As to the relation to homology, Gauss-Bonnet theorem is a good example and characteristic classes is a theory focusing on the homology classes of a certain manifold with real or complex metrics. The holonomy of Riemannian manifold is essential for my study. The De Rham decomposition has been discussed in the following. Besides, the Berger classification gives all the possible holonomy groups of a Riemannian manifold. We can also try to build up a specific manifold such that its holonomy is a given group. This is a question in the holonomy area. We may also generalize the Berger classification theorem to differentiable manifolds without giving Riemannian metrics, since parallel transports and connections can exist prior to metrics thus the holonomy is already well-defined on general differentiable manifolds. This dissertation is only a beginning of my further study in the area of global topology of Riemannian manifolds. After I acquire enough analyze tools and deeper knowledge in algebraic topology, I may merge local analysis and global topology together and discover many potential results in this attractive area.

Let $M$ be a riemannian manifold. A closed piecewise smooth curve $\gamma$ in $M$ gives an isometry on the tangent space $T_xM$ by parallel transports along $\gamma$, for any $x \in \gamma$. We denote this isometry as $\tilde{\gamma}(x)$. For a fixed point $x$ in $M$, all closed piecewise smooth curves with base point $x$ can derive a subgroup of $SO(T_xM)$. This subgroup is called the global holonomy group with base point $x$, denoted by $H^+_x(M)$. If $\gamma$ is homotopy to $x$ (here we regard the point $x$ as a constant curve), then $\tilde{\gamma}(x)$ is a map that could be smoothly deformed to $Id : T_xM \rightarrow T_xM$. i.e., there exists a smooth curve in $H^+_x(M)$ connecting $\tilde{\gamma}(x)$ and $Id$. 
Therefore, the collection of $\gamma_0$'s where $\gamma_0$'s are piecewise smooth curves containing $x$ and homotopy to $x$, is exactly the pathwise connected component of $H^*_x(M)$ that contains $Id$.

This connected component is also a Lie group. We denote it as $H^*_x(M)$ and call it the holonomy group with base point $x$. The Lie groups $H^*_x(M)$ and $H^*_y(M)$ have the same Lie algebra $\eta$.

The Lie algebra $\eta$ is called the holonomy algebra of $M$ at $x$.

Theorem 1 (Ambrose-Singer) Let $R$ be the curvature tensor of $M$. Then

$$\eta = \text{span}\left\{ \zeta \circ R(x)(\nu) \circ \zeta \big| X, Y \in T_xM, \zeta(0) = x^1 \right\}.$$ (This means $\zeta$ is a curve starting at $x$.)

Note. To understand this theorem, let us inspect the elements in $\eta$ first. An element $v$ in $\eta$ is a tangent vector at $Id$ of the Lie group $H^*_x(M)$: $v = \frac{d}{dt}\big|_{t=0}(\tilde{\gamma}_t)$. In the case $R = 0$ everywhere in a neighborhood of $x$, $M$ is locally isometric to the Euclidean space $R^n$. In $R^n$, parallel transports do not cause anything and the holonomy group is trivial. If we assume that the holonomy group of $M$ is not constantly trivial, then $M$ must has curvatures. Theorem 1 claims that $\eta$ is only related to the curvature tensors at the points path-wise connected with $x$.

The whole structure of $M$ has influence on $\eta$. This is because from $v = \frac{d}{dt}\big|_{t=0}(\tilde{\gamma}_t)$, $\tilde{\gamma}_0 = Id$, we can only know that $\tilde{\gamma}_0$ is the identity map of $T_xM$, but the closed curve $\gamma_0$ does not have to be the constant point. So the vectors $v$ in $\eta$ can be different when $\gamma_0$ is the constant point $x$ and $\gamma_0$ is not the constant point respectively. This is in consistent with that the structure of $\eta$ depends on curves $\zeta$ starting from $x$, in theorem 1.

Definition 1. If the operation of the global holonomy group $H^*_x(M)$ (or the holonomy group $H^*_x(M)$) on $T_xM$ is irreducible, (irreducible means that there does not exist non-trivial invariant subspace in $T_xM$. i.e., it is an irreducible representation of the Lie group $H^*_x(M)$), then $M$ is called an irreducible manifold.

Note. To make the definition sensible, we must show that the irreducibility of $M$ does not
depend on the choice of point \(x\). Actually, we can know this from the following theorem.

**Theorem 2.** Let \(TM\) be the tangent bundle of \(M\). A sub-bundle \(T\) of \(TM\) is called an invariant distribution, if for arbitrary points \(x, y\) in \(M\) and an arbitrary curve \(\zeta\) connecting \(x\) and \(y\), 
\[
\tilde{\zeta}(T_x) = T_y.
\]
Here \(\tilde{\zeta}\) is the isometry map from \(T_xM\) to \(T_yM\) determined by the parallel transport along \(\zeta\), and \(T_x\) and \(T_y\) denotes the fibre of \(T\) at \(x\) and \(y\) separately, i.e. \(T_x = \bigcap T_xM\) and \(T_y = \bigcap T_yM\). For an arbitrary point \(x \in M\), the collection of all \(H^\ast(M)\)-invariant subspaces of \(T_xM\) is in one-one correspondence with the collection of invariant distributions on \(M\). The bijection is given by the parallel transports of the subspaces in \(T_xM\) along all smooth curves on \(M\).

Therefore, whether \((H^\ast(M), T_xM)\) is an irreducible representation does not depend on the choice of \(x\). And the decompositions of \(T_xM\), into a direct sum of irreducible subspaces, are the same for all \(x\) in \(M\).

For a connected Riemannian manifold \(M\), its universal covering space \(\tilde{M}\) is simply connected.

Pulling back the metric from \(M\) to \(\tilde{M}\), \(\tilde{M}\) will also be a Riemannian manifold, and \(M\) and \(\tilde{M}\) are locally isometric. We will assume \(M\) to be simply connected and if \(M\) is not, we can consider its universal covering space. In this simply-connected case, \(H^\ast(M) = H(M)\).

**Theorem 3.** (De Rham). Let \(M\) be a completed simply-connected Riemannian manifold, with its holonomy group \(H\). For \(x \in M\), if \(T_xM\) can be decomposed as a direct sum of two orthogonal \(H\)-invariant subspaces: 
\[
T_xM = T_1 \oplus T_2,
\]
then \(T_1\) and \(T_2\) can generate two invariant distributions over \(M\). \(T_1\) and \(T_2\) are closed under the Lie bracket, so \(T_1\) and \(T_2\) are integrable. The maximal integral manifolds of distributions \(T_1\) and \(T_2\) are denoted by \(M_1\) and \(M_2\). Then, the bijection \(\phi : M_1 \times M_2 \to M\) is an isometry of Riemannian manifolds,
where the metric of \( M_1 \times M_2 \) is given by \( \left\langle \cdot, \cdot \right\rangle_{M_1} + \left\langle \cdot, \cdot \right\rangle_{M_2} \).

Note. If \( (H, T_x M) \) is reducible, since the orthogonal complement of an \( H \)-invariant subspace is also \( H \)-invariant, we can require the direct sum \( T_1 \oplus T_2 \) to be orthogonal.

The projections \( \pi_1: M_1 \times M_2 \to M_1 \), \( \pi_2: M_1 \times M_2 \to M_2 \) are continuous. Since \( M_1 \times M_2 = M \) is simply-connected, we can see that every closed curve \( \gamma \) in \( M_1 \) (or \( M_2 \)) is contractible. (the contraction of \( \gamma \) in \( M_1 \times M_2 \) can give the contraction in \( M_1 \) or \( M_2 \), through the projections) . So \( M_1 \) and \( M_2 \) in theorem 3 must be simply-connected manifolds. And their global; holonomy groups are equal to their holonomy groups. Conversely, if \( M_1 \) and \( M_2 \) are both simply-connected, then \( M_1 \times M_2 \) is simply-connected. This is because we can give a continuous deformation in \( M_1 \times M_2 \) by giving continuous deformations in each coordinates.

Theorem 4. Let \( H_1 \) and \( H_2 \) be the holonomy groups of \( M_1 \) and \( M_2 \) respectively. Then \( H_1 \times H_2 \) is isomorphic to \( H \), the holonomy group of \( M_1 \times M_2 \). The decomposition of a completed simply-connected Riemannian manifold preserves the decomposition of its holonomy group.

Proofs of theorem 1, theorem 3, and theorem 4 are all not short. I read these proofs in the book of Riemannian geometry by H.Wu. There exist alternative proofs.

From theorem 1 (Ambrose-singer), we know that the holonomy group is trivial if and only if the curvature tensor \( R = 0 \), if and only if \( M \) is isometric to \( R^n \), provided that \( M \) is simply-connected.

We decompose \( T_x M \) as an orthogonal sum:

\[
T_x M = V_0 \oplus V_1 \oplus \ldots \oplus V_m ,
\]

Where \( V_0 \) is the largest subspace satisfying that the operation of \( H_x (M) \) on \( V_0 \) is trivial, and \( V_1, \ldots, V_m \) are irreducible \( H_x (M) \)-invariant subspaces. The integral manifolds of the
distribution generated by $V_0$ must be isometric to $R^n$. Using theorem 3, theorem 4 and induction, we can get the following corollary.

Corollary 1. Let $M$ be a completed, simply-connected Riemannian manifold. Then $M$ is isometric to the orthogonal product:

$$M = R^k \times M_1 \times \cdots \times M_m$$

Where $k \geq 0$ and $M_1, \ldots, M_m$ are irreducible simply-connected completed Riemannian manifolds. And $H$, the holonomy group of $M$, satisfies

$$H = H_1 \times \cdots \times H_m,$$

where $H_1, \ldots, H_m$ are the holonomy groups of $M_1, \ldots, M_m$ respectively.

Theorem 5 (Uniqueness). The decomposition of $M$ in corollary 1 is unique, up to the order of $M_1, \ldots, M_m$.

Remark 1. in simply-connected case, $M$ has a unique maximal integer $k$, such that $M = R^k \times N$, where $N$ is another simply connected Riemannian manifold with its decomposition in corollary 1 not containing the component $R^k$. Here $\cong$ means an isometry. We will ask:

Is there a unique maximal integer $n$, such that $M$ is a vector bundle $\left(\pi, N, R^n\right)$? Here $\pi$ is the projection, $N$ is the base manifold, and $R^n$ is the fibre type. Obviously, $n \geq k$. Fibre bundle is a natural generalization of Cartesian product. If the answer to the first question is true, then to what extent is the vector bundle parallelizable? If $n$ is constantly equal to $k$, then the vector bundle is parallelizable. But what will happen if $n > k$?

Remark 2. Are there similar conclusions for not simply connected cases?

Remark 3. We know that the homology group $H$ of the product manifolds $M_1 \times M_2$ satisfies

$$H = H_1 \times H_2,$$

for $M_1$ and $M_2$ simply-connected. Since fibre bundle $\left(\pi, M_1, M_2\right)$ is still a Riemannian manifold with the metric $\left\langle \cdot, \cdot \right\rangle_{M_1} + \left\langle \cdot, \cdot \right\rangle_{M_2}$, could we have a way to compute the homology group of the fibre bundle $\left(\pi, M_1, M_2\right)$? We know that to compute the homology group of the bundle manifold is much more complex than to compute the homology group of the Cartesian product manifold. We still want similarities here.
Notice that the fibre bundle \((\pi, M_1, M_2)\) is simply-connected, provided that \(M_1\) and \(M_2\) are both simply-connected. Our reasons are:

For an arbitrary closed curve \(\gamma\) in the bundle manifold, by the continuation of projection \(\pi\), we can firstly contract \(\gamma\) into the fibre-space at one point of the base manifold. Since the fibre is \(M_2\), we can secondly contract the curve in the fibre-space to a point. So \(\gamma\) in \((\pi, M_1, M_2)\) is contractible.

Therefore, for the holonomy of \((\pi, M_1, M_2)\), we still have \(H^* = H\).

Now we turn to another famous theorem.

Theorem 6 (Cartan-hadamard). Let \(M\) be a completed, simply-connected Riemannian manifold, with non-positive sectional curvature. Then the exponential map \(Exp_x : T_x M \to M\) is an diffeomorphism, for any \(x \in M\).

Note. If a non-positive-sectional-curvature manifold \(M\) is not simply-connected, then the exponential map is a covering map, since \(\tilde{M}\), the universal covering manifold of \(M\), is simply-connected, and locally has the same metric and curvature with \(M\). The projection of the exponential map of \(\tilde{M}\) gives the exponential map of \(M\). If the sectional curvature of \(M\) is not constantly non-positive, then \(Exp_x : T_x M \to M\) may not be a covering map. For example, consider sphere \(M = S^n\). The fibre \(Exp^{-1}(x)\) is not discrete in \(T_x M\).

To ensure that the exponential map is a covering map, a sufficient condition is that there do not exist conjugate points on the completed manifold \(M\). This is much weaker than the condition of non-positive sectional curvature.

Theorem 7. For a completed, non-positive sectional-curved manifold \(M\), the homotopy group higher than \(\pi_1(M)\) are all trivial.

Note. This is because \(R^n\) is the covering space of \(M\). And the homotopy groups \(\pi_2, \ldots, \pi_n\) of a topological space and its covering space are the same.

For a non-positive sectional-curved manifold, since the homotopy groups other than \(\pi_1(M)\) are all trivial, we can expect \(\pi_1(M)\) to control some geometric structures of \(M\).
Theorem 8 (Gromoll-Lawson-Wolf-Yau). Let $M$ be a compact Riemannian manifold whose sectional curvature is non-positive. If $\pi_1(M) = \Gamma_1 \times \Gamma_2$, where $\Gamma_1$ and $\Gamma_2$ are normal subgroups of $\pi_1(M)$, and there do not exist non-trivial elements $x \neq e$ commutative with all elements in $\pi_1(M)$, then $M$ is isometric to $M_1 \times M_2$, and $\pi_1(M_1) = \Gamma_1$, $\pi_1(M_2) = \Gamma_2$.

Note. Sectional curvature of $M_1$ (or $M_2$) does not have to be non-positive, however, they can be positive at some points. So we cannot decompose $M$ completely as $M = M_1 \times \ldots \times M_m$, simply by induction, as in the case of De Rham’s theorem.

Let $\tilde{M}$, $\tilde{M}_1$, $\tilde{M}_2$ be the universal covering manifolds of $M$, $M_1$, $M_2$ separately, and $P_1$, $P_2$ the corresponding projections. Then $\tilde{M}$ is isometric to $\tilde{M}_1 \times \tilde{M}_2$. We can inspect the operations of $\pi_1(M) = \pi_1(M_1) \times \pi_1(M_2)$, $\pi_1(M_1)$, $\pi_1(M_2)$ on the fibres $P^{-1}((m_1, m_2)) = P_1^{-1}(m_1) \times P_2^{-1}(m_2)$, $P_1^{-1}(m_1)$, $P_2^{-1}(m_2)$ respectively. Could the operation of $\pi_1(M)$ be decomposed as $\pi_1(M_1)$ on $P_1^{-1}(m_1)$ and $\pi_1(M_2)$ on $P_2^{-1}(m_2)$?

The following theorem shows the relation between the De Rham decomposition of holonomy theory and fundamental groups of non-positive sectional-curved manifolds.

Theorem 9 (Eberlein). Let $M$ be a compact Riemannian manifold with non-positive sectional curvature. The universal covering manifold of $M$ is denoted by $\tilde{M}$, with the pull-back metric from $M$. Then from corollary 1, $\tilde{M}$ is isometric to $R^k \times N$, where $k$ is the largest integer and the manifold $N$ does not contain the Euclidean component $R^k$. For the fundamental group $\pi_1(M)$, there is also a largest integer $l$ such that there exists a normal subgroup of $\pi_1(M)$, isomorphic to $Z^l$. Of course, this subgroup is commutative, since $Z^l$ commutes. Eberlein claims that $k = l$.

Note. In the proof of this theorem, the maximal normal subgroup of $\pi_1(M)$ that isomorphic to
$Z'$ can be regarded as the fundamental group of a maximal subtori. We confront with maximal subtori here, as in the Lie group theory: a Cartan subalgebra of a semisimple Lie algebra is the Lie algebra of a maximal subtori, and all maximal subtoris in a Lie group are conjugate. What a strange thing that toris emerge everywhere! I may research this Eberlein’s theorem in deep in my Ph.D study.