General Spin Dirac Equation (II)

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Abstract

In the reading Nyambuya (2009), it is shown that one can write down a general spin Dirac equation by modifying the usual Einstein energy-momentum equation *via* the insertion of the quantity "s" which is identified with the spin of the particle. That is to say, a Dirac equation that describes a particle of spin $\frac{1}{2}s\hbar\boldsymbol{\sigma}$ where $\hbar$ is the normalised Planck constant, $\boldsymbol{\sigma}$ are the Pauli 2×2 matrices and $s = (\pm 1, \pm 2, \pm 3, \ldots etc)$. What is not clear in this reading (*i.e.* Nyambuya 2009) is how such a modified energy-momentum relation would arise in *Nature*. At the end of the day, the insertion by *lathe of hand* of the quantity "s" into the usual Einstein energy-momentum equation, would then appear to be nothing more than speculation. In the present reading – by making use of the curved spacetime Dirac equations proposed in the work Nyambuya (2008), we move the exercise of Nyambuya (2009) from the realm of speculation to that of plausibility.

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1 Introduction

In the reading Nyambuya (2009), it is shown that the dispersion relation or the Einstein energy-momentum equation $E^2 = s^2\boldsymbol{p}^2c^2 + m_0^2c^4$ leads to a General Spin Dirac Equation. That is to say, the resulting Dirac equation describes a particle of spin $\frac{1}{2}s\hbar\boldsymbol{\sigma}$ where $\hbar$ is the normalised Planck constant, $\boldsymbol{\sigma} = \sigma^1\boldsymbol{i} + \sigma^2\boldsymbol{j} + \sigma^3\boldsymbol{k}$ where σ^k are the usual 2×2 Pauli matrices and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the three orthonormal basis on the x, y, z grid. In the dispersion relation $E^2 = s^2\boldsymbol{p}^2c^2 + m_0^2c^4$, E is the total energy of the particle, $\boldsymbol{p}$ is its momentum, m_0 its rest mass and $c = 2.99792458 \times 10^8\,\mathrm{ms}^{-1}$ is the speed of light in vacuum. What was not clear in this reading (*i.e.* Nyambuya 2009) is how such an energy-momentum would arise. At the end of the day, the insertion by *lathe of hand* of the quantity "s" into the usual Einstein energy-momentum equation $E^2 = \boldsymbol{p}^2c^2 + m_0^2c^4$, would then appear to be nothing more than speculation. Herein, by making use of the curved spacetime Dirac equations proposed in Nyambuya (2008), we move the exercise of Nyambuya (2009) from the realm of speculation to that of plausibility.

In the equation $E^2 = s^2\boldsymbol{p}^2c^2 + m_0^2c^4$, it is not clear why the quantity "s" has to take integral values $s = (\pm 1, \pm 2, \pm 3, \ldots etc)$. Because spin has to take integral and half integral values, it was assumed without proof that this quantity "s" has to take integral values. This off cause is a hole in the theory that needs to be filled. This reading will furnish this dearth in the general spin Dirac

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equation theory set in Nyambuya (2009). We not only demonstrate how "s" comes to be part of the dispersion relation $E^2 = \boldsymbol{p}^2c^2 + m_0^2c^4$, but how and why this quantity comes to take only integer values. In summary, the aim or achievement of this work if twofold *i.e.*:

(1). We unambiguously demonstrate how the quantity "s" becomes a part of the Einstein energy-momentum dispersion relation.

(2). We prove that "s" can only take integral values $s = (\pm 1, \pm 2, \pm 3, \ldots etc)$.

(3). We generalise the notion of a 'general spin Dirac equation' to include all the three curved spacetime Dirac equations.

Now, the synopsis of this reading is as follows: in the next section, we are going to give a brief exposition of the curved spacetime Dirac equation first presented in Nyambuya (2008). In the successive section, we are going to dwell on the main thrust of the present reading by demonstrating how "s" comes to be part of the dispersion relation $E^2 = s^2\boldsymbol{p}^2c^2 + m_0^2c^4$ and as-well how and why "s" comes to take only integer values. Thereafter, we give a general discussion and the conclusions drawn thereof.

2 Curved Spacetime Dirac Equations

As is well known, the Dirac equation is derived from the equation $\eta_{\mu\nu}p^\mu p^\nu = m_0^2c^2$ where $\eta_{\mu\nu}$ is the usual flat Minkowski metric with spacetime signature $[-1, +1, +1, +1]$. We know that its equivalent in curved spacetime is given by:

$$g_{\mu\nu}p^\mu p^\nu = m_0^2c^2, \tag{1}$$

where the four momentum p^μ is given by $p^\mu = (E, \boldsymbol{p})$ and $g_{\mu\nu}$ is the metric of spacetime. In order to aid the reader in visualizing (1) in a way that conforms to the end that we seek, we have to this equation in its equivalent matrix form, *i.e.*:

$$m_0^2c^2 = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}^T \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{21} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}. \tag{2}$$

Above, the 'T' in the superscript of the column vector denotes the transpose operation on that column vector.

Now, in writing down the curved spacetime version of the Dirac equation (in the reading Nyambuya 2008), we made a novel suggestion of writing down the spacetime metric tensor $g_{\mu\nu}$ as:

$$g_{\mu\nu}^{(a)} = \frac{1}{2}\left\{\gamma_\mu^{(a)}, \gamma_\nu^{(a)}\right\} A_\mu A_\nu, \tag{3}$$

where A_μ is some four vector and $a = (1, 2, 3)$. In general, the metric $g_{\mu\nu}^{(a)}$ is such that:

$$[g_{\mu\nu}^{(a)}] = \begin{pmatrix} A_0A_0 & \lambda A_0A_1 & \lambda A_0A_2 & \lambda A_0A_3 \\ \lambda A_1A_0 & -A_1A_1 & \lambda A_1A_2 & \lambda A_1A_3 \\ \lambda A_2A_0 & A_2A_1 & -A_2A_2 & \lambda A_2A_3 \\ \lambda A_3A_0 & \lambda A_3A_1 & \lambda A_3A_2 & -A_3A_3 \end{pmatrix}, \tag{4}$$

where for $(a = 1; \lambda = 0)$, $(a = 2; \lambda = +1)$ and $(a = 3; \lambda = -1)$. In the case $(a = 1)$, there are no off-diagonal terms in the metric, while for the cases $a = (2, 3)$, we have off diagonal terms (see Nyambuya 2008). As shown there-in Nyambuya (2008), the resulting three curved spacetime Dirac equations are given by:

$$\left[i\hbar A^\mu \gamma_{(a)}^\mu \partial_\mu - m_0c\right]\psi = 0, \tag{5}$$

where:

$$\gamma_0^{(a)} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma_k^{(a)} = \frac{1}{2}\begin{pmatrix} 2\lambda I_2 & i^\lambda\sqrt{1+\lambda^2}\sigma^k \\ -i^\lambda\sqrt{1+\lambda^2}\sigma^k & -2\lambda I_2 \end{pmatrix}. \tag{6}$$

In the above (and hereafter), I_2 is the 2×2 identity matrix, σ^k is the usual 2×2 Pauli matrices and the 0's are 2×2 null matrices. It is not a difficult exercise to show that multiplication of (5) from the left handside by the operator $\left[i\hbar A^\mu \gamma_{(a)}^{\mu\dagger}\partial_\mu + m_0 c\right]$ leads us to the Klein-Gordon equation $g_{\mu\nu}\partial^\mu\partial^\nu\psi = (m_0c^2/\hbar)^2\psi$ provided $\partial_\mu A^\mu = \partial^\mu A_\mu = 0$. The condition $\partial_\mu A^\mu = \partial^\mu A_\mu = 0$, should be taken as a gauge condition restricting this four vector. In the next section, we are going to demonstrate the Lorentz invariance of the curved spacetime Dirac equation (5).

2.1 Lorentz Invariance

To prove Lorentz invariance[1] two conditions must be satisfied:

(1). Given any two inertial observers O and O′ anywhere in spacetime, if in the frame O we have $[i\hbar A^\mu\gamma^\mu_{(a)}\partial_\mu - m_0c]\psi(x) = 0$, then $[i\hbar A'^\mu\gamma'^\mu_{(a)}\partial'_\mu - m_0c]\psi'(x') = 0$ is the equation describing the same state but in the frame O′.

(2). Given that $\psi(x)$ is the wavefunction as measured by observer O, there must be a prescription for observer O′ to compute $\psi'(x')$ from $\psi(x)$ and this describes to O′ the same physical state as that measured by O.

Now, since A^μ and ∂_μ are both vectors, the quantity $A^\mu\partial_\mu$ is obviously a scalar. From this, it follows that a Lorentz transformation is not going to affect ψ and $\gamma^\mu_{(a)}$ *i.e.*:

$$\psi'(x') = \begin{cases} \psi(x), & \text{Case (I)} \\ S\psi(x), & \text{Case (II)} \end{cases}, \quad \text{and} \quad \gamma'^\mu_{(a)} = \gamma^\mu_{(a)}. \tag{7}$$

The meaning of the above is that the matrices $\gamma^\mu_{(a)}$ are constant matrices and the Dirac four component ψ is in Case (I) scalar. The Dirac four component ψ is constrained to only be a scalar. In Case (II), we can have this transform under a multiplication of ψ by some constant matrix S. If $S = S(\boldsymbol{r}, t)$, then this matrix will have to be such that $A^\mu\gamma^\mu_{(a)}\partial_\mu S = 0$ in-order for Lorentz invariance to hold.

The present exercise to re-demonstrate the Lorentz invariance of (5) has been conducted to demonstrate the *all*-important difference that we must always take note of, that is, in the bare Dirac theory, the γ-matrices and as-well the four component function ψ, do transform under a Lorentz transformation. This is not the case here; $\gamma^\mu_{(a)}$ is a constant matrix and the Dirac four component function ψ is scalar. In the reading Nyambuya (2008), this very important fact that $\gamma^\mu_{(a)}$ is a constant matrix and that the Dirac four component function ψ can be scalar, was missed altogether, hence the need to make this clear at the present moment in the further development of the curved spacetime Dirac equation. Additionally, we have shown here that equation (5) is not *Lorentz covariant* but *Lorentz invariant*. The orginal Dirac equation is not *Lorentz invariant* but *Lorentz convariant* – this is something to be noted as it distinguishes the present effort from that of Dirac (1928a,b).

[1] There is a difference between *Lorentz invariance* and *Lorentz covariance*. In most cases as in the present, Lorentz invariance is used to mean Lorentz covariance. We are not going to go onto explaining what is the difference between the two. We sincerely believe that our target readership knows this and if they do not, they have access to consult any good textbook that deals with the theory of relativity (special/general). The usual Dirac equation is Lorentz covariant and not Lorentz invariant. We have chosen to use the term Lorentz invariance instead of Lorentz covariance because the term Lorentz invariance is what is usually used. In-order that we are on the same level of understanding with the general reader, we do not have to deviate from the standard terminology.

2.2 General Magnitude of a Four Vector

In this section, we are going to look into the issue of the magnitude of a four vector. For example, the square of the magnitude of the four momentum p^{μ} is such that $g_{\mu\nu}p^{\mu}p^{\nu} = m_0^2c^4$. If we take a general four vector V^{μ}, then $g_{\mu\nu}V^{\mu}V^{\nu} = \kappa^2$. Notice that in $g_{\mu\nu}p^{\mu}p^{\nu} = m_0^2c^4$, $m_0^2c^4$ is a constant, it takes the value everywhere all the times; so that in general we can assume that the κ in $g_{\mu\nu}V^{\mu}V^{\nu} = \kappa^2$, is a constant aswell. We ask, "*In general, does κ have to be a constant?*" The answer to this question is no, it only has to be a scalar since the quantity $g_{\mu\nu}V^{\mu}V^{\nu}$ is a scalar. A constant is a special kind of a scalar, it is a scalar that takes the same value everywhere all the times. If κ is a general scalar, then $\kappa = \kappa(\boldsymbol{r}, t)$.

Given the above *i.e.* $\kappa = \kappa(\boldsymbol{r}, t)$, what we seek here is a function that gives the value of κ at the different $(\boldsymbol{r}, t)$-points. Since ${g_0}^0 = {g_0}^0(\boldsymbol{r}, t)$ is itself a scalar, we propose that, in general, the magnitude of all four vectors in spacetime be such that $\kappa \propto {g_0}^0$, so that:

$$g_{\mu\nu}V^{\mu}V^{\nu} = \kappa_*^2 {g_0}^0, \tag{8}$$

where κ_* is a constant which takes the same value everywhere all the times for-all observers. The quantity κ_* has the dimensions of V^{μ}. One will ask the good question "*What is the motivation for* (8)?" Well, the motivation for the proposal (8) is that if we do not have such a setting, then contrary to experience, the rest mass of a particle in spacetime will have to depend on where the particle is, and when it is at that place where it is – simple, $m_0 = m_0(\boldsymbol{r}, t)$. To avoid this, we have no choice but to impose (8).

2.3 Energy Solutions

The energy-momentum equation for the particles described by equation (5) is:

$$(A^0)^2E^2 - \left(2\lambda A^0 A^k p_k c\right)E - \left(A^k\right)^2 p_k^2 c^2 + \lambda c^2 \left(A^j A^k p_j p_k\right)_{i\neq j} = m_*^2 c^4, \tag{9}$$

where in line with (8), we will have $m_*^2c^4 = m_0^2c^4A_0A^0 = m_0^2c^4(A^0)^2$, where m_0 is a constant; and is the rest mass of the particle in question. Now, dividing (9) throughout by $(A^0)^2$, we will have:

$$E^2 - \left(2\lambda \frac{A^k}{A_0} p_k c\right)E - \left(\frac{A^k}{A_0}\right)^2 p_k^2 c^2 - \lambda c^2 \left[\left(\frac{A^j}{A_0}\right)\left(\frac{A^k}{A_0}\right) p_j p_k\right]_{j\neq k} = m_0^2 c^4. \tag{10}$$

Notice that if m_* were a constant, then $m_0 = m_*/A^0(\boldsymbol{r}, t) = m_0(\boldsymbol{r}, t)$ which goes against experience. It is for this reason that we afore-proposed the condition (8).

Now, setting $s^k = A^k/A_0$; and inserting these settings into the above, we will have:

$$E^2 - \left(2\lambda s^k p_k c\right)E - \left(s^k\right)^2 p_k^2 c^2 - \lambda c^2 \left[s^j s^k p_j p_k\right]_{j\neq k} = m_0^2 c^4. \tag{11}$$

Making E the subject of the formula, we will have:

$$E = \lambda s^k p_k c \pm \sqrt{\left(s^k\right)^2 p_k^2 c^2 + (\lambda s^k p_k c)^2 + \lambda c^2 \left[s^j s^k p_j p_k\right]_{j\neq k} + m_0^2 c^4}. \tag{12}$$

From this, it is clear that we will have three negative energy particles and three positive energy particles. We are going to justify in the next section the insertion of "s" into the Einstein energy-momentum equation $E^2 = s^2\boldsymbol{p}^2c^2 + m_0^2c^4$.

3 Justification of $E^2 = s^2\boldsymbol{p}^2c^2 + m_0^2c^4 : s \in \mathbb{N}$

Let us consider the case $\lambda = 0$. Space is usually assumed to be isotropic. This assumption finds solid justification form experience since observations reveal no directional properties of space, the deeper meaning of which is that space must have no preferential direction or directional properties. In the case of the metric (3), isotropy would mean that the space parts of the four vector A_μ must all be equal or identical to each other, that is $A_k = A_{space}$ for-all $j = (1, 2, 3)$. If this were the case that $A_k = A_{space}$, then $s^k = s$ for-all $k = (1, 2, 3)$. From this, it follows that for the case $\lambda = 0$, we will have the energy-momentum equation:

$$E^2 = s^2 \boldsymbol{p}^2 c^2 + m_0^2 c^4. \tag{13}$$

Thus, the equation $E^2 = s^2\boldsymbol{p}^2c^2 + m_0^2c^4$ finds its sort for justification. What is left is to justify why and how "s" comes to take integral values $s = (\pm1, \pm2, 3, \ldots\ etc)$ *i.e.* why and how $s \in \mathbb{N}$ where $\mathbb{N}$ in the set of all positive and negative integers.

Before we go on to supply the above mentioned proof, let us write down the general spin dispersion relationship for a particle whose spacetime is isotropic. This we are going to do so that we supply not only the proof of why and how $s \in \mathbb{N}$ for the case $\lambda = 0$, but for the other two cases as-well *i.e.* $\lambda = \pm 1$. This general dispersion relationship a particle whose spacetime is isotropic is given by:

$$E = \lambda s \left(\sum_{k=1}^{3} p_k \right) c \pm \sqrt{s^2 \boldsymbol{p}^2 c^2 + \lambda s^2 c^2 \sum_{j=1}^{3} \sum_{k=1}^{3} \left[p_j p_k\right]_{j \neq k} + m_0^2 c^4}. \tag{14}$$

Now, (5) can be written in the general Schrödinger formulation as $\hat{\mathcal{H}}\Psi = \hat{\mathcal{E}}\Psi$ where $\hat{\mathcal{H}}$ and $\hat{\mathcal{E}}$ are the Hamiltonian and energy operators respectively. So doing, *i.e.* writing (5) said form, we will have:

$$\left[i\hbar\gamma^0 s^k \gamma^k_{(a)} \partial_k - \gamma^0 m_0 c \right] \psi = -i\hbar \frac{\partial \psi}{\partial t}. \tag{15}$$

From this, it follows that the new General Spin Dirac Hamiltonian $\mathcal{H}_D^{(a)}(s)$ is given by:

$$\mathcal{H}_D^{(a)}(s) = i\hbar\gamma^0 s^k \gamma^k_{(a)} \partial_k - \gamma^0 m_0 c. \tag{16}$$

This General Spin Dirac Hamiltonian commutes with the total angular momentum operator $\boldsymbol{\mathcal{J}}(s)$ *i.e.* $\left[\boldsymbol{\mathcal{J}}(s), \mathcal{H}_D^{(a)}(s)\right] = 0$ for-all $a = (1, 2, 3)$ and for-all $s = (\pm1, \pm2, \pm3, \ldots\ etc)$. The prove of this assertion is supplied in the Appendix. This fact that $\left[\boldsymbol{\mathcal{J}}(s), \mathcal{H}_D^{(a)}(s)\right] = 0$ is important as it tells us that $\boldsymbol{\mathcal{J}}(s)$ is the total angular momentum of the particle since it commutes with the Hamiltonian. The operator $\boldsymbol{\mathcal{J}}(s)$ is such that:

$$\boldsymbol{\mathcal{J}}(s) = \boldsymbol{\mathcal{L}}(s) + \boldsymbol{\mathcal{S}}(s), \quad \text{where,} \quad \boldsymbol{\mathcal{S}}(s) = \frac{1}{2}\hbar\boldsymbol{\Sigma}_s \quad \text{and} \quad \boldsymbol{\mathcal{L}}(s) = -i\hbar\boldsymbol{r} \times \boldsymbol{\nabla}_s, \tag{17}$$

where:

$$\boldsymbol{\Sigma}_s = s^1\mathcal{S}^1\boldsymbol{i} + s^2\mathcal{S}^2\boldsymbol{j} + s^3\mathcal{S}^3\boldsymbol{k} \quad \text{and} \quad \boldsymbol{\nabla}_s = \boldsymbol{i}s^1\frac{\partial}{\partial x} + \boldsymbol{j}s^2\frac{\partial}{\partial y} + \boldsymbol{k}s^3\frac{\partial}{\partial z}. \tag{18}$$

The $\mathcal{S}^k$'s are 4×4 matrices such that:

$$\mathcal{S}^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \implies \mathcal{S}^i\mathcal{S}^j = \delta^{ij} I_4, \tag{19}$$

where δ^{ij} is the Kronecker-delta function which is such that $\delta^{ij} = 1$ for $i = j$, and $\delta^{ij} = 0$ for $i \neq j$ and I_4 is (and hereafter) the 4×4 identity matrix. Clearly, $\boldsymbol{\mathcal{L}}(s)$ is the orbital angular momentum of the particle and likewise, $\boldsymbol{\mathcal{S}}(s)$ is the associated spin matrix.

Now, to prove that $s \in \mathbb{N}$, as a first step, let us define the 4×4 spin-operators $\hat{\mathcal{S}}_x = \hbar\mathcal{S}^1$, $\hat{\mathcal{S}}_y = \hbar\mathcal{S}^2$ and $\hat{\mathcal{S}}_z = \hbar\mathcal{S}^3$. Further, let us define the 4×4 spin-ladder operators $\hat{\mathcal{S}}_\pm$ which are such that:

$$\begin{array}{rcl}\hat{\mathcal{S}}^x_\pm & = & \hat{\mathcal{S}}_y \pm i\hat{\mathcal{S}}_z\\ \hat{\mathcal{S}}^y_\pm & = & \hat{\mathcal{S}}_z \pm i\hat{\mathcal{S}}_x\\ \hat{\mathcal{S}}^z_\pm & = & \hat{\mathcal{S}}_x \pm i\hat{\mathcal{S}}_y\end{array}. \tag{20}$$

In the above (and hereafter), (x, y, z) represent $(k = 1, 2, 3)$ respectively. **NB:** hereafter, we shall without notice interchange the labels or indices *i.e.*, sometimes we shall use $k = (1, 2, 3)$ and sometimes $k = (x, y, z)$.

Now, these 4×4 spin-ladder operators are related to the operators $\hat{S}^k$ by the commutator relationship:

$$\left[\hat{\mathcal{S}}^i, \hat{\mathcal{S}}^j_\pm\right] = \pm\hbar\delta^{ij}\hat{S}^j_\pm. \tag{21}$$

Now, we propose the following eigenvalue equation:

$$\hat{\mathcal{S}}^k\psi = s^k\hbar\mathcal{S}^k\psi. \tag{22}$$

How does such an eigenvalue equation come about? Well, in-order to have this eigenvalue equation, the operator $\hat{\mathcal{S}}^k$ should be defined such that:

$$\hat{\mathcal{S}}^k = i\hbar^2\mathcal{S}^k\frac{\partial}{\partial S^k}, \tag{23}$$

where S_k is the k^{th}-phase of the particle. That is, if p_μ is the four momentum of a particle and x^μ is its four position in spacetime, then, the phase of this particle S is such that $S = p_\mu x^\mu$. This phase can be split into four components as $S = p_0x^0 + p_1x^1 + p_2x^2 + p_3x^3$. The components S^k then are such that $S^0 = p_0x^0$ and $S^1 = p_1x^1$, $S^2 = p_2x^2$, $S^3 = p_3x^3$, so, we can write $S^k = p_kx^k$ and the k's are not summing up as is the case in the usual Einstein summation convention. Now, the wavefunction of any particle is a function of the phase, that is, $\psi \propto e^{iS/\hbar}$. Further, the phase of a curved spacetime Dirac particle is given by $S = S(s, \boldsymbol{p}, p_0) = s_0p_0x^0 + s_1p_1x^1 + s_2p_2x^2 + s_3p_3x^3$ so that $\partial S/\partial S^k = s_k$. With all this, it is now clear, how the eigenvalue equation (22) arises or comes about.

Now, multiplying (22) by $\mathcal{S}^k$ from the left, we will have $\mathcal{S}^k\hat{\mathcal{S}}^k\psi = s^k\hbar\psi$. From this, it follows that we can rewrite (15) as:

$$\left[i\gamma^0\mathcal{S}^k\hat{\mathcal{S}}^k\gamma^k_{(a)}\partial_k - \gamma^0m_0c\right]\psi = -i\hbar\frac{\partial(\mathcal{S}^k\hat{\mathcal{S}}^k\psi)}{\partial t}. \tag{24}$$

Acting on this equation from the left by $\hat{S}^z_\pm$, one can easily show by using the fact (21) (namely $\left[\hat{\mathcal{S}}^z, \hat{\mathcal{S}}^z_\pm\right] = \pm\hbar\hat{\mathcal{S}}^z_\pm$) that the resulting equation is:

$$\left[i\hbar\gamma^0\left\{s^x\gamma^x_{(a)}\partial_x + s^y\gamma^y_{(a)}\partial_y + (s^z \pm 1)\gamma^z_{(a)}\partial_z\right\} - \gamma^0m_0c\right]\psi_{s_z\pm1} = -i\hbar\frac{\partial\psi_{s_z\pm1}}{\partial t}, \tag{25}$$

where $\psi_{s_z\pm1} = \hat{S}^z_\pm\mathcal{S}^k\hat{\mathcal{S}}^k\psi$: in this equation *i.e.* (25) s^x and s^y remain unchanged by the application of the operation $\hat{\mathcal{S}}^z_\pm$, while s_z changes by one unit. The above equation describes a particle of spin $\frac{1}{2}\hbar\boldsymbol{\sigma}_{s_z\pm1}$ where $\boldsymbol{\sigma}_{s_z\pm1} = s^x\sigma^x\boldsymbol{i} + s^y\sigma^y\boldsymbol{j} + (s^z \pm 1)\sigma^z\boldsymbol{k}$. The operator $\hat{S}_+$ increases the s^z by one unit, while the operator $\hat{S}_-$ decreases this quantity by one unity. If we want to simultaneously raise or lower the spin for-all the $s^k : k = (x, y, z)$, then we have to act on (25) using all the three operators *i.e.* $\hat{\mathcal{S}}^x_\pm$, $\hat{\mathcal{S}}^y_\pm$ and $\hat{\mathcal{S}}^z_\pm$. This means we can define the operator:

$$\hat{\Pi}\left(\hat{\mathcal{S}}^k_\pm\right) = \hat{\mathcal{S}}^x_\pm\hat{\mathcal{S}}^y_\pm\hat{\mathcal{S}}^z_\pm, \tag{26}$$

which then acts on (25). That is, acting from the left on (25) using this new operator $\hat{\Pi}\left(\hat{\mathcal{S}}^k_\pm\right)$, and thereafter performing the necessary algebraic operations, the resulting equation is:

$$\left[i\hbar\gamma^0(s^k \pm 1)\gamma^k_{(a)}\partial_k - \gamma^0 m_0 c\right]\psi_{s^k\pm 1} = -i\hbar\frac{\partial\psi_{s^k\pm 1}}{\partial t}, \tag{27}$$

where $\psi_{s^k\pm 1} = \hat{\Pi}\left(\hat{\mathcal{S}}^k_{\pm}\right)\mathcal{S}^k\hat{\mathcal{S}}^k\psi$, that is, $\psi_{s^k\pm 1}$ is the wavefunction of the particle ψ where the spin quantum s^k of ψ has either been increased $(+)$ or decreased $(-)$ by one unit for-all the three directions xyz.

Now, to prove that "s^k" only takes integral values, we simple have to prove that one of the values of "s^k" is an integer. Since "s^k" only changes by integral values, if just one of the values of "s^k" is an integer, then, all the other values of this quantity must be integers too – surely, this is not difficult to understand. To prove that just one of the values of "s^k" is an integer is not a difficult task to perform either. We know that in Minkowski spacetime where $|A_\mu| \equiv 1$: $\forall\,\mu = 0, 1, 2, 3$, the energy-momentum dispersion relation is given by the Einstein energy-momentum equation $E^2 = \boldsymbol{p}^2c^2 + m_0^2c^4$; in this equation $s^k \equiv 1$ for-all $k = (1, 2, 3)$. If the Minkowski spacetime is envisaged as the lowest energy state for any quantum configuration, then $s^k = 1$ for-all $k = (1, 2, 3)$ is one of the quantum mechanical states for any particle. Clearly, this is sufficient proof that one of the values of "s^k" for-all $k = (1, 2, 3)$, is an integer. From the foregoing, it thus follows that "s^k" will take only integral values *i.e.* $s^k = (\pm 1, \pm 2, \pm 3, \dots\ etc)$. This completes the proof that $s^k \in \mathbb{N}$ for-all $k = (1, 2, 3)$. We have not only proved that "s^k" is an integer, but in so doing, we have also proved why spin is a quantised physical quantity.

4 Metric of a General Spin Dirac Particle

From the above findings, we can compute the general spacetime metric of a general spin Dirac particle. We have argued that the four vector A_μ is such that $s_k = A_k/A_0$. From this, we can write down a four spin quantum number s_μ. To do this, we note that the four vector A_μ can be written with its components as $A_\mu = (A_0, A_k)$. Further, this can be written as $A_\mu = A_0(1, A_k/A_0) = A_0(1, s_k)$. The quantity $(1, s_k)$ is the four spin quantum number that we seek *i.e.*, $s_\mu = (1, s_k)$ where $s_0 = 1$. For our convenience, let us set $A_0 = \Phi$. From this, the four vector A_μ can now be written as $A_\mu = \Phi(1, s_k) = \Phi s_\mu$. Now, substituting $A_\mu = \Phi s_\mu$ into (3), we will have:

$$g^{(a)}_{\mu\nu} = \frac{1}{2}\Phi^2\left\{\gamma^{(a)}_\mu, \gamma^{(a)}_\nu\right\}s_\mu s_\nu. \tag{28}$$

Written in full, $g^{(a)}_{\mu\nu}$ is such that:

$$[g^{(a)}_{\mu\nu}] = \Phi^2\begin{pmatrix} 1 & \lambda s_1 & \lambda s_2 & \lambda s_3 \\ \lambda s_1 & -s_1^2 & \lambda s_1 s_2 & \lambda s_1 s_3 \\ \lambda s_2 & \lambda s_2 s_1 & -s_2^2 & \lambda s_2 s_3 \\ \lambda s_3 & \lambda s_3 s_1 & \lambda s_3 s_2 & -s_3^2 \end{pmatrix}. \tag{29}$$

From this, we see that the metric is controlled by one variable function $\Phi = \Phi(\boldsymbol{r}, t)$ since λ and s_k are all constants. Thus, (29) is the metric of a general spin curved spacetime Dirac particle.

The usual metric of spacetime $g_{\mu\nu}$ has ten potentials. This was reduced to four potential by the introduction of the four vector A_μ. Now, these for potential has been reduced to just one potential. This is a tremendous simplification – from ten potentials to just one potential! At this point, the reader may legitimately want to ask if $g_{\mu\nu}$ has the same meaning as in Einstein's General Theory *of* Relativity (GTR)? To answer this question, one has to visit the reading Nyambuya (2010). It is shown therein (Nyambuya 2010) the vector A_μ gives raise to the nuclear force non-abelian gauge field. The details of the Unified Field Theory presented in Nyambuya (2010) are still being worked. What the reader can do for now is simple take A_μ as a four vector and nothing else. As to whether this vector represents a gravitational, electric or any force field for that matter is of no consequence here since we are not concerned with the force field which this four vector represents.

5 Discussion and Conclusion

We strongly believe that this reading justifies the assertion made in Nyambuya (2009), namely that the modified Einstein dispersion relation $E^2 = s^2\boldsymbol{p}^2c^2 + m_0^2c^4$ leads to a general spin Dirac equation. When this assertion was made in Nyambuya (2009), it was not clear then, as to how such a dispersion relation would arise in *Nature*. We have shown that the curved spacetime Dirac equation proposed in Nyambuya (2008) can be used to justify the modified Einstein dispersion relation $E^2 = s^2\boldsymbol{p}^2c^2 + m_0^2c^4$. Not only have we justified this, we have also argued that "s" must take integral values. This means that, the work presented in Nyambuya (2009) has been put on a much more acceptable pedestal. The reason we say this is because we believe that despited the fact that the true meaning and significance the curved spacetime Dirac equation derived in Nyambuya (2008) has not been found yet, these curved spacetime Dirac equations are credible, mathematically and physically legitimate equations. Actually, it has been demonstrated that these curved spacetime Dirac equation are key to the attainment of a general spin Dirac equation.

Insofar as the unification programme of physics is concerned, we believe that the writing down of an acceptable general spin Dirac equation is a step in the right direction. If discovered, the final unified theory is expected to be such that a "single" equation/principle will explain about every observable phenomenon. Amongst others, it is expected that a single equation must be able to explain all particles from a simple unifying principle. In the light of the aforestated, it is somewhat sad to say that the current state of physics *vis* the equations purporting to explain particles – is very "ugly". For example, the Schrödinger equation describes spin-0 atoms and molecules (Schrödinger 1926), the Klein-Gordon equation describes spin-0 particles (that is carriers of forces), while the Dirac equation describes spin-1/2 particles, and the Rarita-Schwinger equation describes spin-3/2 particles (Ratita *&* Schwinger 1941). From this rather "ugly" trend, does it mean we have to look for another equation to describe spin-2 particles, and then another for spin-5/2 particles *etc*? This does not look beautiful, simple, or at the very least suggest at the far and deeper end, a unification of the Natural Laws. It is on this note that we feel the present endeavours are worthwhile.

Another interesting outcome is that (5) is no longer restricted to the description of Fermions, but Bosons aswell. If this equation proves successful as happened with Dirac's original equation, then, it will perhaps be the first equation in physics to describe both Fermions and Bosons. Further, this equation shares some common ground with super-symmetry theories – that is, theories that try and unify quantum mechanics and gravitation; in that it allows for the transmutation of a Fermion to a Boson and *vice-versa*. We believe this equation might be of interest to physicist working in this field. To transform a Fermion to a Boson and *vice-versa*, one simple acts on the wavefunction ψ with the operator $\hat{\Pi}\left(\hat{\mathcal{S}}^k_{\pm}\right)$. In physical terms, we have no idea what an operation on ψ with $\hat{\Pi}\left(\hat{\mathcal{S}}^k_{\pm}\right)$ is. For all we know is that from an abstract mathematical standpoint, this is what one must do.

5.1 Conclusion

Assuming the acceptability (correctness) of the ideas propagated herein, we hereby make the following conclusions:

(1). *We have demonstrated that the curved spacetime Dirac equations naturally lead to a general spin Dirac equation.*

(2). *The spin of these curved spacetime Dirac particles is found to be naturally quantised i.e. it comes in integral multiples of a fundamental basic unit of spin. This spin quantization strongly appears to be wholly a part and parcel of the fabric of spacetime itself.*

(3). *The fact that the spin of a particle is measured to be the same independent of the orientation; this fact suggests very strongly that spacetime must be isotropic on a quantum scale. If this were not the case that space is isotropic on the quantum scale, then, according to the ideas propagated herein, a particles' spin will be different when measured in different directions.*

(4). *It has been shown that the curved spacetime Dirac equation leads to a Dirac wavefunction that is a scalar, i.e., the resulting four component wavefunction ψ – together with the $\gamma^{\mu}_{(a)}$-matrices – is not affected by a Lorentz transformation. Effectively, the resulting curved spacetime Dirac equation is not Lorentz covariant, but truly Lorentz invariant in the true and stickiest sense of Lorentz invariant.*

Appendix

We are going to prove the crucial assertion that we stated in on page (5) without any proof, that is: $\left[\boldsymbol{\mathcal{J}}(s), \mathcal{H}^{(a)}_D(s)\right] = 0$, for-all $a = 1, 2, 3$. To begin, we know that $\boldsymbol{\mathcal{J}}(s) = \boldsymbol{\mathcal{L}}(s) + \boldsymbol{\mathcal{S}}(s)$, from this, it follows that $[\boldsymbol{\mathcal{L}}(s), \mathcal{H}_D(s)] + [\boldsymbol{\mathcal{S}}(s), \mathcal{H}_D(s)] = 0$, and further, since $\boldsymbol{\mathcal{S}}(s) = \mathcal{S}_x(s)\boldsymbol{i} + \mathcal{S}_y(s)\boldsymbol{j} + \mathcal{S}_z(s)\boldsymbol{k}$ and $\boldsymbol{\mathcal{L}}(s) = \mathcal{L}_x(s)\boldsymbol{i} + \mathcal{L}_y(s)\boldsymbol{j} + \mathcal{L}_z(s)\boldsymbol{k}$. Combining these facts, one obtains that:

$$\left[\mathcal{J}_j(s), \mathcal{H}^{(a)}_D(s)\right] = 0, \tag{A.1}$$

where $j = x, y, z$ and $\mathcal{J}_j = \mathcal{L}_j + \mathcal{S}_j$. So, if we can prove (A.1) for-all $j = x, y, z$ and for-all $a = 1, 2, 3$, we will have proved that $\left[\boldsymbol{\mathcal{J}}(s), \mathcal{H}^{(a)}_D(s)\right] = 0$ for-all $a = 1, 2, 3$. We only have to prove this for just one of the three cases $j = x, y, z$, this prove is sufficient as prove for the remaining two cases. We shall prove this for the case $j = x$.

$$\mathcal{L}_x(s) = -i\hbar I_4 \begin{vmatrix} y & z \\ s_y\frac{\partial}{\partial y} & s_z\frac{\partial}{\partial z} \end{vmatrix} = -i\hbar I_4\left(y s_z\frac{\partial}{\partial z} - z s_y\frac{\partial}{\partial y}\right) = -i\hbar I_4 L_x(s), \tag{A.2}$$

so that:

$$\mathcal{J}_x(s) = -\frac{1}{2}i\hbar\begin{pmatrix} 2I_2L_x(s) - s_x\sigma_x & 0 \\ 0 & 2I_2L_x(s) + s_x\sigma_x \end{pmatrix}, \tag{A.3}$$

where $L_x(s) = y s_z\partial_z - z s_y\partial_y$. Now, since $\mathcal{H}^{(a)}_D(s) = i\hbar\gamma^0 s^k\gamma^k_{(a)}\partial_k - \gamma^0 m_0 c$, (A.1) implies that for the case $(a = 2)$, we will have:

$$\left[\mathcal{J}_x(s), i\hbar\gamma^0 s^k\gamma^k_{(2)}\partial_k\right] - \left[\mathcal{J}_x(s), \gamma^0 m_0 c\right] = 0. \tag{A.4}$$

In this way, our task is now much easier, if we can show that $\left[\mathcal{J}_x(s), \gamma^0 s^k\gamma^k_{(2)}\partial_k\right] = 0$ and $\left[\mathcal{J}_x(s), \gamma^0\right] = 0$, we accomplish our mission. Let us start with the easier of the two, that is, show that $(-\frac{1}{2}i\hbar)^{-1}\left[\mathcal{J}_x(s), \gamma^0\right] = 0$.

$$\left(-\frac{1}{2}i\hbar\right)^{-1}\mathcal{J}_x(s)\gamma^0 = \begin{pmatrix} 2I_2L_x(s) + s_x\sigma_x & 0 \\ 0 & 2I_2L_x(s) + s_x\sigma_x \end{pmatrix}\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{A.5}$$

so that:

$$\left(-\frac{1}{2}i\hbar\right)^{-1}\mathcal{J}_x(s)\gamma^0 = \begin{pmatrix} 2I_2L_x(s) + s_x\sigma_x & 0 \\ 0 & -2I_2L_x(s) - s_x\sigma_x \end{pmatrix}, \tag{A.6}$$

and:

$$\left(-\frac{1}{2}i\hbar\right)^{-1}\gamma^0\mathcal{J}_x(s) = \begin{pmatrix} 2I_2L_x(s) + s_x\sigma_x & 0 \\ 0 & -2I_2L_x(s) - s_x\sigma_x \end{pmatrix}. \tag{A.7}$$

Now, subtracting (A.7) from (A.6), one obtains the desired result, namely $(-\frac{1}{2}i\hbar)^{-1}\left[\mathcal{J}_x(s), \gamma^0\right] \equiv 0$, hence $\left[\mathcal{J}_x(s), \gamma^0 m_0 c\right] = 0$. We are now left with demonstrating that $\left[\mathcal{J}_x(s), \gamma^0 s^k\gamma^k_{(2)}\partial_k\right] = 0$.

$$\gamma^0 s^k \gamma^k_{(2)} \partial_k = \frac{1}{2} \begin{pmatrix} I_2 s^k \partial_k & i\sqrt{2} s^k \sigma_k \partial_k \\ i\sqrt{2} s^k \sigma_k \partial_k & I_2 s^k \partial_k \end{pmatrix} \tag{A.8}$$

so that $\left(-\frac{1}{2}i\hbar\right)^{-1} [\mathcal{J}_x(s)] \left[\gamma^0 s^k \gamma^k_{(2)} \partial_k\right]$ is such that:

$$\frac{1}{2} \begin{pmatrix} 2I_2 L_x(s) + s_x\sigma_x & 0 \\ 0 & -2I_2 L_x(s) - s_x\sigma_x \end{pmatrix} \begin{pmatrix} I_2 s^k \partial_k & i\sqrt{2} s^k \sigma_k \partial_k \\ i\sqrt{2} s^k \sigma_k \partial_k & I_2 s^k \partial_k \end{pmatrix} \tag{A.9}$$

which is equal to:

$$\frac{1}{2} \begin{pmatrix} [2I_2 L_x(s) + s_x\sigma_x] (s^k \partial_k) & i\sqrt{2} [2I_2 L_x(s) + s_x\sigma_x] (s^k \sigma_k \partial_k) \\ i\sqrt{2} [2I_2 L_x(s) + s_x\sigma_x] (s^k \sigma_k \partial_k) & - [2I_2 L_x(s) + s_x\sigma_x] (s^k \partial_k) \end{pmatrix} \tag{A.10}$$

so that $\left[\gamma^0 s^k \gamma^k_{(2)} \partial_k\right] \left[\left(-\frac{1}{2}i\hbar\right)^{-1} \mathcal{J}_x(s)\right]$ is such that:

$$\frac{1}{2} \begin{pmatrix} I_2 s^k \partial_k & i\sqrt{2} s^k \sigma_k \partial_k \\ i\sqrt{2} s^k \sigma_k \partial_k & I_2 s^k \partial_k \end{pmatrix} \begin{pmatrix} 2I_2 L_x(s) + s_x\sigma_x & 0 \\ 0 & -2I_2 L_x(s) - s_x\sigma_x \end{pmatrix}, \tag{A.11}$$

which implies $\left[\left(-\frac{1}{2}i\hbar\right)^{-1} \mathcal{J}_x(s)\right] \left[\gamma^0 s^k \gamma^k_{(2)} \partial_k\right] = \left[\gamma^0 s^k \gamma^k_{(2)} \partial_k\right] \left[\left(-\frac{1}{2}i\hbar\right)^{-1} \mathcal{J}_x(s)\right]$, hence we arrive at our desired result namely $\left[\mathcal{J}_x(s), \gamma^0 s^k \gamma^k_{(2)} \partial_k\right] = 0$. According to our earlier arguments, if follows that the main result $\left[\boldsymbol{\mathcal{J}}(s), \mathcal{H}^{(a)}_D(s)\right] = 0$ for-all $a = (1, 2, 3)$ and for-all $s = (\pm 1, \pm 2, \pm 3 ... etc)$ is attained.

Q.E.D.

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