A New Conjecture towards the solution of the Hodge conjecture

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Abstract

In this paper, we review important facts related to the Hodge conjecture. Also, we review Chow classes and their importance to the problem. At the end of this survey, we pose a new conjecture that would advance work on it if proven true, to further the development of the important Millennium prize problem.

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Introduction

The Hodge conjecture is one of the most important open problems in modern mathematics, and in fact, one of the hardest. In particular, it states the following [Del]:

Hodge conjecture. $\mathit{On\ a\ projective, nonsingular\ algebraic\ variety\ over\ } \mathbb{C}\mathit{, any\ Hodge\ class\ is\ a\ rational}$

$\mathit{linear\ combination\ of\ classes\ } \mathrm{cl}(Z) \mathit{\ of\ algebraic\ cycles}.$

In this paper, we review facts related to the Hodge conjecture. Also, we review Chow classes and their importance to the problem. At the end of this survey, we pose a new conjecture that would advance work on it if proven true, to further the algebraic development of the important Millennium prize problem. But before we do so, we review general facts about Hodge classes in the next section, along with the properties of the Chow classes.

Preliminaries

A $Hodge\ structure$ [At-Duv] is a decomposition of a vector space $V(\mathbb{C}) = \oplus_{p,q\in\mathbb{Z}} V^{p,q}$ such that $V^{p,q} = \overline{\overline{V}^{q,p}}$, where the accent denotes complex conjugation given by $\overline{v\otimes w} = \bar{v}\otimes w$. When V is a rational Hodge structure of weight $k = 2n$, it becomes the space of Hodge classes $B(V) = V \cap V^{n,n}$. A well-known example of a structure admitting this class is the k-th Betti number, which is given by

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

Under this structure, we have another decomposition, the $Hodge\ filtration$.[Cat, Del, Kap]

$$F^p = \bigoplus_{a\geq p} V^{a,b}$$

These decompositions of vector bundles are holomorphic; thus they induce a holomorphic connection on manifolds. The Hodge conjecture suggests an equivalence to rational linear combinations of algebraic cycles (see introduction for more details). Algebraic cycles are governed by the $Chow\ ring$, which satisfies the following properties[Tot]:

1. There is a contravariant functor $X \to \bigoplus_p CH^p(X)$ from smooth algebraic varieties to projective algebraic varieties.
2. $\deg \sum_i n_i P_i = \sum_i n_i$
3. Given a proper morphism of projective varieties, we have a projection map
$$\bigoplus_p CH^p(X) \to \bigoplus_p CH^p(Y)$$
4. A similar construction holds for vector bundles, more specifically we have the following map:
$$f^*: CH^*(X) \to CH^*(V)$$
5. $c_0(V) = 1, c_p(V) = 0$
6. $c(V_2) = c(V_1)c(V_3)$
, where c represents the character classes of the vector bundle
7. When $k = \mathbb{C}$, $CH^p(X) = H^{2p}(X, \mathbb{Z})$
8. Note that they are sometimes denoted by $A^k(X) = \bigoplus_{i=0}^{\dim S} A^*(X)$.

Our conjecture will concern the relation between these classes and their relation to the Chern classes of Hermitian vector bundles on smooth manifolds. However, before the presentation of this important problem, we review recent research results on the Hodge conjecture.

The Problem

We first introduce the following conjecture:

Conjecture 1.
The Hodge conjecture is equivalent to solving the following systems of Chern classes and algebraic cycles:

$$\begin{cases} \det\left(\frac{it\Omega}{2\pi} + I\right) = c_n(Z) \\ A^k(X) = \bigoplus_{i=0}^{\dim S} A^*(X) \end{cases}$$

, where Ω is the curvature form associated to a manifold and $c_n(Z)$ is a class number. Moreover, the computations of the $H^{2p}(X, \mathbb{C})$ are equivalent to the computations of the $H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$.

There is no known algorithm to compute such systems, and the classes in the Hodge conjecture themselves. Thus, we also pose the following problem:

Problem 2.
Given that Conjecture 1 holds, construct an algorithm to compute such classes to show that they satisfy the statement of the Hodge conjecture.

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problems that have been posed, and hopes that they will continue to inspire minds in future generations. He is also only 15 years old, and has a great passion for the subject.

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