Proof of the SYZ Conjecture

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Abstract

In this short paper, we prove that the Strominger-Yau-Zaslow (SYZ) conjecture holds by showing that mirror symmetry is equivalent to T- duality under fibrations from Lagrangian tori. In order to do this, we use some recent developments on Ooguri- Vafa spaces to construct such fibers. Moreover, we show that this is only possible under the trivial vector bundle {0}, thus giving an equivalence between the triangulated categories $D^b Fuk_0(Y,\omega)$ and $D_0^b(\check{Y})$.

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Introduction

In this paper, we prove that the Strominger-Yau-Zaslow (SYZ) conjecture holds by showing that mirror symmetry is equivalent to T- duality under fibrations from Lagrangian tori. In order to do this, we use some recent developments on Ooguri- Vafa spaces to construct such fibers. Such spaces were recently studied in [Lu] , and are important to this solution. For example, we have the following theorem:

Theorem 1.1 [Lu]
$The\ following\ data\ for\ an\ Ooguri-Vafa\ metric\ satsifies\ the\ 4d\ wall\ crossing\ formalism$:

$$\Omega(q\varsigma_e + p\varsigma_m) = \delta_{p,0}\Omega_q$$

$Moreover, the\ nontrivial\ transformations\ from\ BPS\ \varkappa - rays\ are\ \varkappa - factors, satsifying\ the\ trivial\ 4d\ wall\ crossings.$

The reader can refer to [Cer] for the detailed description of the invariant Ω. This theorem will serve as the starting point of our proof. However, we will need some more algebraic details, such as this theorem:

Theorem 1.2[Bru] $The\ following\ statments\ are\ equivalent$:

1. $M \cong T^*B\backslash\Lambda\ as\ symplectic\ manifolds\ fibered\ over\ B\ in\ Lagrangian\ submanifolds$

2. $The\ bundle\ \pi: M \to B\ admits\ a\ global\ Lagrangian\ section\ \sigma: B \to M$

3. $The\ Chern\ class\ v = \delta([\mu]) vanishes\ , and\ for\ any\ section\ s: B \to M, the\ 2-form\ s^*\omega\ on\ B\ vanishes.$

Before we move more in depth into these details and any other important definitions, we recall the statement of the SYZ conjecture:

Conjecture 1.3 (Strominger- Yau- Zaslow)
$X\ and\ its\ mirror\ \check{X}\ carry\ dual\ fibrations\ by\ special\ Lagrangian\ tori; i.e$:

$$\check{X} = \{(L, \nabla), L\ fiber\ of \pi, \nabla \in \hom(\pi_1 T, U(1))$$

We will show the implications of our proof after we prove the main result. But, before we do so, we recall some important facts about these objects.

Preliminaries

Recall that the *Hitchin moduli space* consists of the following [Cha]:

-A Riemannian surface C

- The solutions to the equations

$$F_A - \phi \wedge \phi = 0$$

$$d_A \star \phi = 0$$

The solutions to these equations are well known to exist with a flat *hyperkahler* metric, given by

$$ds^2 = -\frac{1}{4\pi} \int |d^2 z|\, Tr(\delta A_z \otimes \delta A_{\bar{z}} + \delta A_{\bar{z}} \otimes \delta A_z + \delta \phi_z \otimes \delta \phi_{\bar{z}} + \delta \phi_{\bar{z}} \otimes \delta \phi_z)$$

A more detailed description can be found in [Swa]. As a general fact, recall that these metrics are basically a combination of the symplectic and complex forms. Thus, they will serve to be a major part of our proof. Some recent works have introduced a notion from theoretical physics and algebraic geometry known as *wall crossing*. In particular, we have the following two formulas[Cer]:

- $4d$ *wall crossing formula*: Given the birational automorphism $\varkappa_\gamma : X_{\gamma'} \to (1 - X_{\gamma'})^{\langle \gamma, \gamma' \rangle} X_{\gamma'}$, we take the product of all the geodesics γ that are near the wall or *participating*, raised to the powers of the invariant $\Omega(\gamma)$ (see [Neit] for a more detailed description). More specifically, we have
$$: \prod_\gamma \varkappa_\gamma^{\Omega(\gamma)} :$$
,where the :'s are multiplied by the order of a phase Z_γ.
- $2d - 4d$ *wall crossing formula*: We can extend the first formula by introducing two automorphisms of a vector bundle V over torus T (see [Neit] for the exact formulations of these automorphisms) , thus extending the first formula to the following:
$$: \prod_{\gamma, \gamma_{ij}} \varkappa_\gamma^\omega S_{\gamma_{ij}}^\mu :$$

Before we move on to the presentation of the proof, we rewrite the invariant $\Omega(\gamma)$ in a discrete version for the purposes so it will satisfy the metric mentioned above:

Definition 2.1. The *discrete wall crossing invariant* is given by the following data:

$$\zeta_q = \begin{cases} 1\ if\ saddle\ connection\ with\ lift\ \gamma \\ 0\ otherwise \end{cases}$$

Now we prove the Strominger-Yau-Zaslow (SYZ) conjecture in the next section.

Main Result

Our main result is the following theorem, which we have divided into 2 parts for clarity:

Theorem 3.1. $The\ Strominger - Yau - Zaslow\ conjecture\ holds\ for\ the\ following$:

$a)$ $The\ invariant\ \Omega\ defined\ in$ [Neit]

$b)$ $In\ the\ general\ case, implied\ by\ the\ structure\ of\ the\ Hitchin\ moduli\ space.$

$Proof$. We break down the proof into several pieces, as follows:

Theorem 3.2.
$The\ discrete\ invariants\ (of\ Definition\ 2.1) induce\ a\ holomorphic\ connection\ on\ the\ torus.$

$Moreover, the\ bundle\ inducing\ this\ connection\ is\ the\ trivial\ bundle.$

$Proof$. We can induce a foliation $\Delta \subset TS$, where Δ is equivalent to the modified invariant introduced in the last section. Since we are working under a Riemannian structure, there must exist a holomorphic subbundle to $T_*(P)$, where P is the principal bundle of the manifold[John]. As a side note, it was shown in [Bis, Iyer] that holomorphic bundles over a manifold admit a holomorphic connection on the manifold. This is an important fact as we continue to construct this implication. This bundle is the trivial bundle as it follows from the definition of holomorphicity. Finally, we use these trivialization maps to construct fibers $\pi^{-1}(U)$, which are equivalent to the set $\{0, -i^2\}$.

Theorem 3.3. $This\ construction\ in\ Theorem\ 3.2\ satisfies\ the\ hyperkahler\ metric\ structure.$

$Proof$. First we check that this is a Riemmanian structure. This follows since we have the holomorphic connection induced by the trivial bundle in Theorem 3.2, and that we have a foliation. Moreover, since it is isotropic, property (1) is implied. From this, the rest of the proof of the Riemmanian part is trivial. It is obvious that we have a complex structure. Finally, we check that it is Lagrangian. So, we take the Cartesian product of the metric of the Lagrangian tori and the trivial bundle itself to obtain

$$(\{0\} \times \{0\}, \omega \times -\omega) = (\{0\}, \{0, 0\}, \{\omega\}, \{\omega, -\omega\})$$

(Here, we omitted the metric for the purposes of the reader's clarity). Since the metric $\omega \in \Delta$, and since there exists a projection $\pi: \mathbb{Z} \times \{0\} \to \mathbb{Z}$ it satisfies the properties of a Lagrangian manifold, thus completing the proof.

Theorem 3.4.$The\ dual\ fibration\ \pi^*\ is\ equivalent\ to\ a\ subset\ \pi_0\ of\ the\ fibration\ \pi^{-1}(U)$

$Proof$. First let $\pi_0: \{0\}, \{0, 0\} \to \{0\}$ be a subset of the projection $\pi: \{\mathbb{Z}\}, \{\mathbb{Z}, 0\} \to \{\mathbb{Z}\}$. Then since the dual fibrations are given by the map $\pi^*: \hom(\{\mathbb{Z}\}, \{\mathbb{Z}, 0\}, V) \to \hom(\{0\}, V)$, where V denotes a vector space, we realize that there exists an equivalent projection π_0 because 0 is an ideal of a vector space and of the integers, thus completing the proof.

Theorem 3.5.
$The\ structure\ described\ in\ Theorems\ 3.2-$
$3.4\ is\ equivalent\ to\ that\ described\ in\ the\ \ SYZ\ conjecture.$

$Proof$. First we check that the discrete invariant described earlier is in $\hom\left(\pi_1 T, U(1)\right)$. This follows from the fact that it is equal to $\hom(\mathbb{Z}\times\mathbb{Z},*) = (\pi_0,*)$, where the $*$ denotes multiplication as usual. It is well known that a line bundle is a part of a fiber bundle. Next we check that $rank\ L = 1$. This follows from the existence of the two elements in the set described in Theorem 3.2. Moreover, by Theorem 3.4 of [Bru], it follows that the dual of the torus T^* is equivalent to $\pi^*(\mathbb{Z})$, and over the fibration on the Lagrangian submanifold, we obtain $M \cong \pi^*(\mathbb{Z})\mathbb{Z}/\Lambda$, thus completing the proof.

$Proof\ of\ Theorem$ 3.1. Part a) of the proof is trivial because it is already proven in [Cha]. For part b), the result simply follows from Theorems 3.1-3.5.

Theorem 3.6. $There\ exists\ an\ equivalence\ of\ triangulated\ categories\ D^b Fuk_0(Y,\omega) and\ D_0^b(\check{Y})$.

$Proof$. This is a consequence of the Main Theorem in [Cha].

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