International Journal of Mechanical Engineering
ISSN : 2277-7059 Volume 2 Issue 2
http://www.ijmejournal.com/ https://sites.google.com/site/journalijme/

Free Vibration Analysis of Rectangular Plates Using Galerkin-Based Finite Element Method

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Abstract. In the present work a study of free vibration of thin isotropic rectangular plates with various edge conditions is conducted. This study involves the obtaining of natural frequencies by solving the mathematical model that governs the vibration behavior of the plate using a Galerkin-based finite element method. Cubic quadrilateral serendipity subparametric elements with twelve degrees of freedom are used in this analysis. Even though the order of polynomial used is the lowest possible, the effectiveness of the method for calculating the natural frequencies accurately is demonstrated by comparing the solution obtained against the existing analytical results. The effect of the aspect ratio, the number of elements, and the number of sampling points on the accuracy of the solution is also presented.

Keywords: Galerkin finite element; Thin plate; Free vibrations, Serendipity element.

1. Introduction

Plates, as structural elements, are extensively used in many fields of engineering including aerospace, civil structures, hydraulic structures, containers, ships, instruments, and machine parts. When in service, they are subjected to dynamic loadings the effect of which is very critical. Much research has been conducted into plate behavior, using a wide range of methods. An excellent monograph of the early literature relating to vibration analysis of plates was published by Leissa [1]. Most researchers, e.g. [1], [2], [3], have used classical thin plate theory in their formulations to study the plate response; where the flexural vibration of the thin plate is characterized by a fourth-order partial differential equation. A direct solution of such equation might be difficult and most of the reported solutions are based on numerical methods such as finite difference method [4], and finite element method [5], [6].
Despite the fact that the Galerkin finite element approach is very powerful, easy to understand, and effectively applicable to the spectrum of engineering problems, no much attention was given to it in the literature. This paper is devoted for the presentation of the approach as well as the demonstration of its adequacy for the solution of bending of isotropic plates. In this chapter, a four-node twelve degrees of

freedom, (4N-12DOF), subparametric quadrilateral element is developed and used for the frequency response analysis of a thin rectangular isotropic plate.

2. Thin Plate Model

The governing equation that describes the flexural vibration of thin plates subjected to transverse loading, based on classical plate theory, is expressed as[2]:

$$\rho(x,y)h\frac{\partial^2 w(x,y,t)}{\partial t^2}+D\left(\frac{\partial^4 w(x,y,t)}{\partial x^4}+2\frac{\partial^4 w(x,y,t)}{\partial x^2\partial y^2}+\frac{\partial^4 w(x,y,t)}{\partial y^4}\right)=P_z(x,y,t). \quad \textbf{(1)}$$

Where, $w(x,y,t)$, is the out of plane motion in positive z -direction, P_z is the exciting load per unit area, E, h, ν and $\rho(x,y)$, are the modulus of elasticity, plate thickness, the Poisson's ratio, and density respectively. In order to obtain the natural frequencies of the plate, the exciting load P_z is set equal to zero. The flexural rigidity is expressed as

$$D=\frac{Eh^3}{12(1-\nu)}. \quad \textbf{(2)}$$

3. Finite Element Formulation

The governing equation of a vibrating thin plate, Eq.(1), is a fourth-order partial differential equation which requires the continuity of both deflection and slope with respect to both x- and y-directions, namely w, θ and ϕ. In other words, at least three degrees of freedom are required at each node of the selected element to get a unique solution. Thus, the plate model requires C^1 -continuous parameter function.

3.1 Parametric Shape Functions

A linear quadrilateral element was selected to represent the geometrical, or shape, functions. The coordinates (x and y) of the selected linear element can be written in terms of parent element coordinates (ξ, and η)as,

$$x=\sum_1^4 \Phi_i(\xi,\eta)x_i$$
$$y=\sum_1^4 \Phi_i(\xi,\eta)y_i. \quad \textbf{(3)}$$

The shape function can be evaluated anywhere inside the parent element using the following expression

$$\Phi_i(\xi,\eta,\alpha)=\alpha_{1i}+\alpha_{2i}\xi+\alpha_{3i}\eta+\alpha_{4i}\xi\eta. \quad \textbf{(4)}$$

3.2 Parametric Trial Functions

A cubic or third order quadrilateral serendipity subparametric element with twelve degrees of freedom was chosen as a C^1-continuous parameter function. The parametric trial functions used in this analysis may be the same as those used in the commercial codes; however this subsection is devoted for the presentation of such functions.

For a linear quadrilateral element, the generalized displacement can be expressed as

$$u^e = \{u_1 \quad u_2 \quad u_3 \quad u_4\}^T, \tag{5}$$

where;

$$u_i = \{w_i \quad \theta_i \quad \phi_i\}^T = \left\{w_i \quad \frac{\partial w_i}{\partial x} \quad \frac{\partial w_i}{\partial y}\right\}^T. \tag{6}$$

The parameter function, w, can be expressed in terms of nodal displacement, slopes, and trial functions as follows

$$w(x,y,t) = [\Theta^e]^T u^e, \tag{7}$$

where;

$$\Theta^e = \{\Theta_1 \quad \Theta_2 \quad \Theta_3 \quad \Theta_4\}^T. \tag{8}$$

The parametric trial functions $\Theta_j(x,y)$ have values of unity at their respective nodes and zeros at the rest of nodes. However, when expressed in local coordinates, ξ and η, the nodal values of these trial functions are expressed as:

$$\begin{aligned}\Theta_j(\xi_j,\eta_j) &= \{\Theta_j^w(\xi_j,\eta_j) \quad \Theta_j^\theta(\xi_j,\eta_j) \quad \Theta_j^\phi(\xi_j,\eta_j)\}^T \\ &= \left\{\Theta_j^w(\xi_j,\eta_j) \quad \frac{\partial\Theta_j^w(\xi_j,\eta_j)}{\partial\xi} \quad \frac{\partial\Theta_j^w(\xi_j,\eta_j)}{\partial\eta}\right\}^T.\end{aligned} \tag{9}$$

It should be noted that for the cases where

$$\Theta_j^\theta(x,y) = 1 \Rightarrow \Theta_j^\theta(\xi,\eta) = \frac{\partial x}{\partial\xi}, \qquad \Theta_j^\phi(\xi,\eta) = \frac{\partial x}{\partial\eta}, \tag{10}$$

$$\Theta_j^\phi(x,y) = 1 \Rightarrow \Theta_j^\theta(\xi,\eta) = \frac{\partial y}{\partial\xi}, \qquad \Theta_j^\phi(\xi,\eta) = \frac{\partial y}{\partial\eta}. \tag{11}$$

For convenience, the parametric trial functions are denoted as $N_i(\xi,\eta)$, so that,

$$N_{(n*j-2)} = \Theta_j^w, N_{(n*j-1)} = \Theta_j^\theta, \; N_{(n*j)} = \Theta_j^\phi; \; n=3; \quad i=1.2, \ldots, 12; \; j=1,2,3,4. \tag{12}$$

The parametric trial function are evaluated inside the parent element using the following expression

$$N_j(\xi,\eta,\beta) = \beta_{1j} + \beta_{2j}\xi + \beta_{3j}\eta + \beta_{4j}\xi^2 + \beta_{5j}\xi\eta + \beta_{6j}\eta^2 + \beta_{7j}\xi^3 + \beta_{8j}\xi^2\eta + \beta_{9j}\xi\eta^2 + \beta_{10j}\eta^3 + \beta_{11j}\xi^3\eta + \beta_{12j}\xi\eta^3, \quad j = 1,2,.....12. \tag{13}$$

4. Exact and Classical Solutions

For simply supported (SSSS) thin isotropic plates; Leissa[1], presented the exact natural frequencies mathematically from the following closed form

$$\omega_{mn} = \frac{\pi^2}{a^2}\sqrt{\frac{D}{\rho h}}\left(m^2 + \left(\frac{a}{b}\right)^2 n^2\right). \tag{14}$$

where ω_{mn} is the natural frequency (rad/sec), a is the plate dimension measured in x-direction, b is the dimension of the plate measured in the y-direction, h is the plate thickness, ρ is the material density, D is the flexural rigidity, and m and n are the number of half waves in x and y directions respectively. The normalized natural frequency can be expressed as

$$\varpi = \left(m^2 + \left(\frac{a}{b}\right)^2 n^2\right) = \omega_{mn}\frac{a^2}{\pi^2}\sqrt{\frac{\rho h}{D}}. \tag{15}$$

For the cases of clamped plates (CCCC), Leissa[3] presented the nondimensional frequency parameter, λ, based on the classical Voigt[7] solution considering sinusoidal time response considering 0.3 as the value of Poisson's ratio. The respective classical natural frequencies are defined by

$$\omega_{mn} = \frac{\lambda}{a^2}\sqrt{\frac{D}{\rho h}}, \tag{16}$$

or

$$\varpi = \frac{\lambda}{\pi^2}. \tag{17}$$

5. Finite Element Equations

The residual of Eq.(1) can be expressed as

$$\Re(x,y,t) = \rho(x,y)h\frac{\partial^2 w_a}{\partial t^2} + D\left(\frac{\partial^4 w_a}{\partial x^4} + 2\frac{\partial^4 w_a}{\partial x^2 \partial y^2} + \frac{\partial^4 w_a}{\partial y^4}\right) - P_z(x,y,t) \neq 0. \tag{18}$$

The Galerkin weighted residual equation is given by

$$\int_{\Omega} N_i(x,y)\Re(x,y)dA = 0. \quad \textbf{(19)}$$

The approximate trial solutions, w_a, is expressed as

$$w_a(x,y,t) = \sum_{j=1}^{4} N_j(x,y)w_j(t). \quad \textbf{(20)}$$

In a matrix form, the weak formulation over the element can be expressed as

$$[M]\{\ddot{w}\} + [K]\{w\} = \{F\}. \quad \textbf{(21)}$$

The local mass matrix, $[M]$, the local stiffness matrix, $[K]$, and the local load matrix, $\{F\}$, can be respectively presented as

$$M_{ij} = -\rho h \int_A N_i N_j dA, \quad \textbf{(22)}$$

$$K_{ij} = \int_A -D\left[\frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial x^2} + v\frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial y^2} + \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial y^2} + v\frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x^2} + 2(1-v)\frac{\partial^2 N_i}{\partial x\partial y}\frac{\partial^2 N_j}{\partial x\partial y}\right]dA, \quad \textbf{(23)}$$

$$F_i = -\oint_S N_i Q_n ds + \oint_S M_n \frac{\partial N_i}{\partial n} ds - \oint_S N_i \frac{\partial M_{ns}}{\partial s} ds + [N_i M_{ns}]_s - \int_A N_i p dA. \quad \textbf{(24)}$$

5.1 Numerical Integration

In order to obtain exact numerical integration over the element, it is recommended [8, 9] that at least nine Gauss sampling points be used for the polynomial depicted in Eq. 13. Thus, at least three sampling points are needed in $\xi-$ direction, and three points in $\eta-$direction along with the respective weights, $\mathbf{w_k}$ and $\mathbf{w_l}$.

The stiffness and load matrix coefficients, for the element, are written as[9].

$$K_{ij} = \sum_{k=1}^{n}\sum_{l=1}^{n}\left(-|J|w_k w_l D\left(\frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial x^2} + \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial y^2} + 4\frac{\partial^2 N_i}{\partial x\partial y}\frac{\partial^2 N_j}{\partial x\partial y} + \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial y^2} + \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x^2}\right)\right)_{\xi_l,\eta_k}, \quad \textbf{(25)}$$

$$F_i = \sum_{k=1}^{n}\sum_{l=1}^{n}\left(-|J|w_k w_l (pN_i)\right)_{\xi_l,\eta_k} - \oint_S N_i Q_n ds + \oint_S M_n \frac{\partial N_i}{\partial n} ds - \oint_S N_i \frac{\partial M_{ns}}{\partial s} ds + [N_i M_{ns}]_s. \quad \textbf{(26)}$$

In a similar fashion, the mass matrix is

$$M_{ij} = \sum_{k=1}^{n}\sum_{l=1}^{n}\left(-|J|w_k w_l \rho h N_i N_j\right)_{\xi_k,\eta_l}, \quad \textbf{(27)}$$

Where;

$$|J| = \left(\frac{\partial x}{\partial \xi}\right)\left(\frac{\partial y}{\partial \eta}\right) - \left(\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\right). \quad \textbf{(28)}$$

6. Results and Discussion

Rectangular plates were modeled considering aspect ratios of 1. and 1.5 for simply supported plates and with an aspect ratio of 1. for the clamped plate. The plate thickness was considered to be constant and the material had a Poisson's ratio of 0.3 The plate was discretized with a progressively refined mesh, i.e. 2×2, 3×3, 4×4, 5×5, 6×6, 7×7, 8×8 (4N-12DOF) quadrilateral elements.
The normalized natural frequencies,ϖ, were considered instead of their dimensional values.

6.1 Numerical Results

For the simply supported plate, comparison of the normalized natural frequencies with the exact values is presented in Tables 1-3. The convergence of the six lowest modes of vibrations for the simply supported rectangular plate with aspect ration a/b = 1 is presented in Table 1. It is clear that the results converge to the exact solution using 64 elements which mounts to 243 degrees of freedom. Increasing the number of sampling points enhances the rate of convergence as can be inferred from Table 2. For a rectangular plate with aspect ratio of 1.5, the results are depicted in Table 3 and they indicate convergence towards the exact solution as well.

Table 1. Normalized natural frequency ϖ compared to exact results, Eq.(14), for (SSSS) Rectangular plate with aspect ration a/b = 1, and nsp = 3.

#	m	n	Exact	Elements						
				2×2	3×3	4×4	5×5	6×6	7×7	8×8
1	1	1	2	1.81	1.90	1.94	1.96	1.97	1.98	1.98
2	2	1	5	4.32	4.69	4.80	4.86	4.90	4.92	4.94
3	1	2	5	4.32	4.69	4.80	4.86	4.90	4.92	4.94
4	2	2	8	4.37	6.88	7.27	7.47	7.61	7.70	7.76
5	3	1	10	6.87	9.59	9.71	9.76	9.81	9.85	9.88
6	1	3	10	6.87	10.11	9.71	9.77	9.81	9.85	9.88

Table 2. Normalized natural frequency ϖ compared to exact results, Eq.(14), for (SSSS) Rectangular plate with aspect ration a/b = 1, and nsp = 5

#	m	n	Exact	Elements						
				2×2	3×3	4×4	5×5	6×6	7×7	8×8
1	1	1	2	1.92	1.95	1.97	1.98	1.98	1.99	1.99
2	2	1	5	4.84	5.08	5.03	5.01	5.00	4.93	5.00
3	1	2	5	4.84	5.08	5.03	5.01	5.00	4.93	5.00
4	2	2	8	6.28	7.66	7.75	7.81	7.86	7.89	7.91
5	3	1	10	6.28	10.87	10.68	10.42	10.27	10.19	10.14
6	1	3	10	7.35	10.93	10.70	10.42	10.28	10.20	10.14

Table 3. Normalized natural frequency ϖ compared to exact results, Eq.(14), for (SSSS) rectangular plate with a/b = 1.5 and nsp = 3

#	m	n	Exact	Elements						
				2×2	3×3	4×4	5×5	6×6	7×7	8×8
1	1	1	3.25	2.88	3.08	3.14	3.18	3.20	3.21	3.22
2	2	1	6.25	4.44	5.72	5.92	6.03	6.09	6.13	6.15
3	1	2	10	5.80	9.34	9.60	9.73	9.80	9.85	9.88
4	3	1	11.25	7.98	9.99	10.72	10.87	10.96	11.02	11.07
5	2	2	13	9.04	10.53	11.61	12.04	12.30	12.46	12.58
6	3	2	18	9.99	11.40	15.38	16.23	16.23	16.96	17.17

For the clamped plate, the normalized natural frequencies are compared with classical values quoted in reference [3] and the results are depicted in Table 4, and 5. The convergence towards the classical solution is similar to that in the case of simply supported plate. However, no enhancement in the solution was achieved by increasing the number of sampling points.

Table 4. Normalized natural frequency ϖ compared to classical results, Eq.(17), for (CCCC) rectangular plate with a/b = 1. and nsp = 3

#	classical	Elements						
		2×2	3×3	4×4	5×5	6×6	7×7	8×8
1	3.56	3.57	3.42	3.48	3.52	3.56	3.58	3.59
2	7.39	9.04	7.11	7.10	7.16	7.22	7.27	7.30
3	7.39	9.04	7.11	7.10	7.16	7.22	7.27	7.30
4	10.89	*	10.30	9.94	10.09	10.26	10.40	10.51
5	13.34	*	14.86	12.97	13.01	13.03	13.07	13.11
6	13.34	*	16.20	13.17	13.15	13.14	13.17	13.20

Table 5. Normalized natural frequency ϖ compared to classical results, Eq.(17), for (CCCC) rectangular plate with a/b = 1. and nsp = 5

#	classical	Elements						
		2×2	3×3	4×4	5×5	6×6	7×7	8×8
1	3.56	4.69	3.89	3.76	3.72	3.70	3.58	3.68
2	7.39	11.71	8.85	8.13	7.84	7.70	7.27	7.63
3	7.39	11.71	8.85	8.13	7.84	7.70	7.27	7.63
4	10.89	*	13.23	11.60	11.27	11.15	10.40	11.09
5	13.34	*	18.63	16.08	15.03	14.44	13.07	14.12
6	13.34	*	20.58	16.08	15.14	14.53	13.17	14.19

To emphasize the accuracy of the results obtained by the Galerkin-based finite element, the error percentages were compared with those obtained using (4N-16DOF) and (16N-64DOF) elements that were presented in reference [10] . The results are listed in Table 6, and 7.
In the present work, the maximum degrees of freedom used were 243 to get 0.8% error for the first mode and 1.24% error for the sixth mode. For the (4N-16DOF) element, 1024 degrees of freedom were necessary to get 0.9% error for the first mode and 0.4% error for the sixth mode. However, using (16N-64DOF) elements, 1024 degrees of freedom needed to get 3.6% error for the first mode and 1.4% error for the sixth mode.
It should be emphasized at this point that in the present work (4N-12DOF), Gauss-Legender formulation was used to get exact integration, while for the (4N-16DOF) and (16N-64DOF) works, the elements' matrices were obtained in closed forms to avoid errors introduced by numerical integration.

Table 6. Comparison of % Error for first mode of (SSSS) for aspect ratio of a/b=1.

No. of Elements	1	4	9	16	25	36	49	64	81	144	225
(4N-12DOF)	–	9.59	4.97	3.01	2	1.41	1.05	0.8	–	–	–
(4N-16DOF)	–	–	3.7	–	–	2	–	–	1.4	1.1	0.9
(16N-64DOF)	4.0	8.0	5.5	4.3	3.6	–	–	–	–	–	–

Table 7. Comparison of % Error for sixth mode of (SSSS) for aspect ratio of a/b=1.

No. of Elements	1	4	9	16	25	36	49	64	81	144	225
(4N-12DOF)	–	31.3	-1.12	2.87	2.34	1.89	1.52	1.24	–	–	–
(4N-16DOF)	–	–	15.8	–	–	1.8	–	–	0.7	0.5	0.4
(16N-64DOF)	10.2	6.8	1.7	1.8	1.4	–	–	–	–	–	–

7. Conclusion

The suitability of the Galerkin-based finite element for studying the convergence of natural frequencies of isotropic thin rectangular plates with various edge conditions is confirmed. This is accomplished even though a partial third order polynomial was used which represents the lowest possible order using quadrilateral serendipity elements.
Amongst the advantages of the method is that only a relatively small mesh size is needed to get accurate results. Progressive mesh refinement resulted in pronounced decrease in the error involved in the analysis. The results recorded in the case of refined (8x8) meshes, is within acceptable and practical range.

The increase in the number of sampling points was advantageous in the case of (SSSS) while for the case of (CCCC) such an increase worsen such accuracy.

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