

Notes on a Conjecture Related to the Distribution of Prime Numbers

Germán Andrés Paz

Instituto de Educación Superior N°28 Olga Cossettini
(2000) Rosario, Santa Fe, Argentina
E-mail: `germanpaz_ar@hotmail.com`

Abstract

In this paper we discuss Legendre's, Brocard's, Andrica's, and Oppermann's conjectures. We propose a conjecture regarding the distribution of prime numbers and we also prove that if it is true, then the previously mentioned conjectures follow. Moreover, we also show that if the conjecture in question holds, then there is at least one prime number in the interval $[n, n + 2 \lfloor \sqrt{n} \rfloor - 1]$ for every positive integer n .

Keywords: *Andrica's conjecture, Brocard's conjecture, Legendre's conjecture, Oppermann's conjecture, prime numbers*

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1 Introduction

The well-known Bertrand's postulate states that for every integer $n > 3$ there always exists a prime number p such that $n < p < 2n - 2$ (another formulation of this theorem is that for every $n > 1$ there always exists a prime number p such that $n < p < 2n$). This statement, which had been conjectured by Joseph Bertrand in 1845, was first proved by P. L. Chebyshev in 1850. In 1919, Ramanujan [14] gave a simpler proof, from which the concept of Ramanujan primes would later arise. Erdős published another simple proof of Bertrand's postulate in 1932 [6].

After Bertrand's postulate was proved in 1850, better results have been obtained by using both elementary and nonelementary methods. In 1952, Jitsuro Nagura [10] showed that there exists a prime in the interval $[n, 6n/5]$ for every $n \geq 25$. In 1976, Lowell Schoenfeld [15] proved that the interval $(n, n + n/16597)$ contains a prime for all $n \geq 2010760$. Iwaniec and Pintz [8] showed that there is always a prime in the interval $[n - n^{23/42}, n]$ for sufficiently large n . In 1998, Pierre Dusart [3] showed in his doctoral thesis that there exists a prime in the interval $[n, n + n/(2 \ln^2 n)]$ for every $n \geq 3275$. In 2006,

M. El Bachraoui [5] proved that there is a prime in the interval $[2n, 3n]$. Dusart improved his previous result in 2010 [4], when he proved that for $n \geq 396738$ there exists a prime in the interval $[n, n + n/(25 \ln^2 n)]$. In 2011, Andy Loo [9] showed that there is a prime in the interval $[3n, 4n]$. Moreover, the fact that $\pi[n, n + n/(2 \ln^2 n)] \geq 1$ for $n \geq 3275$ implies that each of the intervals $[4n, 5n]$, $[5n, 6n]$, $[6n, 7n]$, and $[7n, 8n]$ contains a prime for every integer $n \geq 5$ [13].

Although much progress has been made towards finding shorter and shorter intervals containing at least one prime, there are still many open problems in Number Theory regarding the existence of prime numbers in certain intervals. Some of these problems are: 1) Legendre's conjecture (see Sect. 2), 2) Brocard's conjecture (Sect. 3), 3) Andrica's conjecture (Sect. 4), and 4) Oppermann's conjecture (Sect. 5).

In this paper we make the following conjecture regarding the distribution of prime numbers:

Conjecture 1. *If n is any positive integer and we take n consecutive integers located between n^2 and $(n + 1)^2$, then among those n integers there is at least one prime number. In other words, if $a_1, a_2, a_3, a_4, \dots, a_n$ are n consecutive integers such that $n^2 < a_1 < a_2 < a_3 < a_4 < \dots < a_n < (n + 1)^2$, then at least one of those n integers is a prime number.*

Remark 1. Throughout this paper, whenever we say that a number b is *between* a number a and a number c , it means that $a < b < c$, which means that b is never equal to a or c . Moreover, the number n that we use in this document is always a positive *integer*.

Let us see some cases in which Conjecture 1 is true:

- If we consider $n = 1$, we have $1^2 < 2 < 3 < (1 + 1)^2$, and we can see that the numbers 2 and 3 are both prime numbers.
- If we consider $n = 2$, we have $2^2 < 5 < 6 < 7 < 8 < (2 + 1)^2$. If we take any sequence of 2 consecutive integers greater than 2^2 and smaller than $(2 + 1)^2$, then at least one of those 2 integers is a prime number. This is true because each of the sequences $\{5, 6\}$, $\{6, 7\}$, and $\{7, 8\}$ contains at least one prime number.
- If we consider the case where $n = 3$, we have $3^2 < 10 < 11 < 12 < 13 < 14 < 15 < (3 + 1)^2$. It is easy to verify that each of the sequences $\{10, 11, 12\}$, $\{11, 12, 13\}$, $\{12, 13, 14\}$, and $\{13, 14, 15\}$ contains at least one prime number.

We can easily prove that Conjecture 1 is also true for $n = 4$, $n = 5$, $n = 6$, and larger values of n .

Let us suppose that p and q are two consecutive prime numbers such that $p < q$. It is easy to verify that the amount of composite numbers between p and q is equal to $q - p - 1$. In other words, the gap between p and q is equal to $q - p - 1$.

A gap between two consecutive prime numbers is said to be *maximal* if it is larger than all gaps between smaller primes. Let P denote a prime number followed by a maximal gap G . This means that G is a maximal gap between two consecutive primes P and Q . Taking into account that $G = Q - P - 1$, we can use the following method to find out the values of n for which Conjecture 1 is true:

- We calculate $\lfloor \sqrt{P} \rfloor$.
- If $G < \lfloor \sqrt{P} \rfloor$, then Conjecture 1 holds for every n such that $G - 1 \leq n \leq \lfloor \sqrt{P} \rfloor - 1$ (in this paper, the gap between two consecutive prime numbers greater than 2 is an odd number).
- If $G = \lfloor \sqrt{P} \rfloor$ and $(\lfloor \sqrt{P} \rfloor + 1)^2 - P - 1 < \lfloor \sqrt{P} \rfloor$, then Conjecture 1 holds for $n = \lfloor \sqrt{P} \rfloor - 1$ and for $n = \lfloor \sqrt{P} \rfloor$.
- If $G > \lfloor \sqrt{P} \rfloor$, $(\lfloor \sqrt{P} \rfloor + 1)^2 - P - 1 < \lfloor \sqrt{P} \rfloor$, and the previous maximal gap is less than $\lfloor \sqrt{P} \rfloor$, then Conjecture 1 holds for $n = \lfloor \sqrt{P} \rfloor$.

By using this method in combination with tables of maximal gaps (see [1] and [11]) and lists of prime numbers, we verify that Conjecture 1 holds at least for every n such that $1 \leq n \leq 1193806023$ (we have $10^9 < 1193806023$). In other words, the conjecture holds over the first $1193806024^2 = 1425172822938688576$ positive integers. This number is greater than 10^{18} .

Remark 2. Andersen [1] and Nicely [11] define gaps between consecutive primes p and q as $g = q - p$. In this paper, we define a gap between consecutive primes as $g = q - p - 1$. This means that in order to work with the method described above in combination with Andersen and Nicely's tables of maximal gaps, we need to subtract 1 from the gaps listed on those tables. Therefore, a maximal gap of x in Andersen and Nicely's tables is considered as a maximal gap of $x - 1$ in our paper.

In Sects. 2, 3, 4, and 5 we will show that if Conjecture 1 is true, then Legendre's, Brocard's, Andrica's, and Oppermann's conjectures follow. Moreover, we will also show that if the mentioned conjecture holds, then there exists a prime in the interval $[n, n + 2 \lfloor \sqrt{n} \rfloor - 1]$ for every positive integer n (see Sect. 6).

2 Legendre's Conjecture

Legendre's conjecture [19] states that for every positive integer n there exists at least one prime number p such that $n^2 < p < (n + 1)^2$. This conjecture is one of Landau's problems [7, 18].

It is easy to verify that the amount of integers located between n^2 and $(n + 1)^2$ is equal to $2n$.

Proof. We have

$$\begin{aligned}(n+1)^2 - n^2 &= 2n + 1 \\ n^2 + 2n + 1 - n^2 &= 2n + 1 \\ 2n + 1 &= 2n + 1.\end{aligned}$$

We need to exclude the number $(n+1)^2$ because we are taking into consideration the integers that are greater than n^2 and smaller than $(n+1)^2$. Therefore, we get

$$2n + 1 - 1 = 2n. \quad \square$$

According to this result, between n^2 and $(n+1)^2$ there are two groups of n consecutive integers each that do not have any integer in common. Example for $n = 3$:

$$\begin{array}{c} 3^2 \quad \underbrace{10 \quad 11 \quad 12}_{\substack{\text{Group A} \\ (n \text{ consecutive integers})}} \quad \underbrace{13 \quad 14 \quad 15}_{\substack{\text{Group B} \\ (n \text{ consecutive integers})}} \quad (3+1)^2 \\ \underbrace{\hspace{15em}}_{2n \text{ consecutive integers}} \end{array}$$

Group A and Group B do not have any integer in common. Now, according to Conjecture 1, Group A contains at least one prime number and Group B also contains at least one prime number, which implies that between 3^2 and $(3+1)^2$ there are at least two prime numbers. This is true because the numbers 11 and 13 are both prime.

All this means that if Conjecture 1 is true, then there are at least two prime numbers between n^2 and $(n+1)^2$ for every positive integer n . As a result, if Conjecture 1 is true, then Legendre's conjecture is also true.

3 Brocard's Conjecture

Brocard's conjecture [17] states that if p_n and p_{n+1} are consecutive prime numbers greater than 2, then between $(p_n)^2$ and $(p_{n+1})^2$ there are at least four prime numbers.

Since $2 < p_n < p_{n+1}$, we have $p_{n+1} - p_n \geq 2$. This means that there is at least one positive integer a such that $p_n < a < p_{n+1}$. As a result, there exists at least one positive integer a such that $(p_n)^2 < a^2 < (p_{n+1})^2$.

Conjecture 1 states that between $(p_n)^2$ and a^2 there are at least two prime numbers and that between a^2 and $(p_{n+1})^2$ there are also at least two prime numbers. In other words, if Conjecture 1 is true, then there are at least four prime numbers between $(p_n)^2$ and $(p_{n+1})^2$. As a consequence, if Conjecture 1 is true, then Brocard's conjecture is also true.

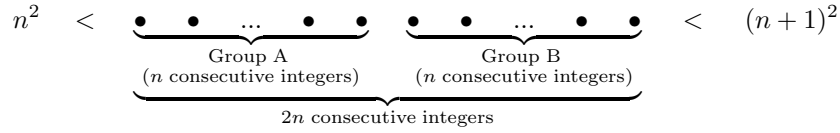
4 Andrica's Conjecture

Andrica's conjecture [2, 16] states that $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ for every pair of consecutive prime numbers p_n and p_{n+1} (of course, $p_n < p_{n+1}$).

Obviously, every prime number is located between two consecutive perfect squares. Now, let us suppose that p is any prime number and that q is the prime number immediately following p . If we take into account that p is obviously located between n^2 and $(n+1)^2$ for some n , two things may happen:

Case 1. *The number p is among the first n consecutive integers that are located between n^2 and $(n+1)^2$. These n integers form what we call 'Group A,' and the following n integers form what we call 'Group B.'*

Let us look at the following graphic.



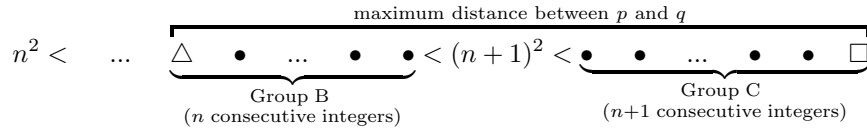
If p is located in Group A and Conjecture 1 is true, then q is either located in Group A or in Group B. In both cases we have $\sqrt{q} - \sqrt{p} < 1$, since $\sqrt{(n+1)^2} - \sqrt{n^2} = 1$ and the numbers \sqrt{q} and \sqrt{p} are closer to each other than $\sqrt{(n+1)^2}$ in relation to $\sqrt{n^2}$.

Case 2. *The prime number p is located in Group B.*

If p is located in Group B and Conjecture 1 is true, it may happen that q is also located in Group B. In this case, it is very easy to verify that $\sqrt{q} - \sqrt{p} < 1$, as explained before.

Otherwise, if q is not located in Group B, then q is located in 'Group C.' In this case, the largest value that q can have is $q = (n+1)^2 + n + 1 = n^2 + 2n + 1 + n + 1 = n^2 + 3n + 2$, while the smallest value p can have is $p = n^2 + n + 1$ (in order to make the process easier, we are not taking into account the fact that in this case the numbers p and q have different parity, which means that they cannot be both prime at the same time).

This means that the largest possible difference between \sqrt{q} and \sqrt{p} is $\sqrt{q} - \sqrt{p} = \sqrt{n^2 + 3n + 2} - \sqrt{n^2 + n + 1}$.



$$\triangle = n^2 + n + 1 = p$$

$$\square = n^2 + 3n + 2 = q$$

It is easy to prove that $\sqrt{n^2 + 3n + 2} - \sqrt{n^2 + n + 1} < 1$.

Proof. We have

$$\begin{aligned}
\sqrt{n^2 + 3n + 2} - \sqrt{n^2 + n + 1} &< 1 \\
\sqrt{n^2 + 3n + 2} &< 1 + \sqrt{n^2 + n + 1} \\
n^2 + 3n + 2 &< \left(1 + \sqrt{n^2 + n + 1}\right)^2 \\
n^2 + 3n + 2 &< 1 + 2\sqrt{n^2 + n + 1} + n^2 + n + 1 \\
n^2 + 3n + 2 - n^2 - n - 1 &< 1 + 2\sqrt{n^2 + n + 1} \\
2n + 1 &< 1 + 2\sqrt{n^2 + n + 1} \\
2n &< 2\sqrt{n^2 + n + 1} \\
n &< \frac{2\sqrt{n^2 + n + 1}}{2} \\
n &< \sqrt{n^2 + n + 1} \\
n^2 &< n^2 + n + 1,
\end{aligned}$$

which is true for every positive integer n . □

Remark 3. In general, to prove that an inequality is correct, we can solve that inequality step by step. If we get a result which is obviously correct, then we can start with that correct result, ‘work backwards from there’ and prove that the initial statement is true.

We can see that even when the difference between q and p is the largest possible difference, we have $\sqrt{q} - \sqrt{p} < 1$. If the difference between q and p were smaller, then of course it would also happen that $\sqrt{q} - \sqrt{p} < 1$.

According to Cases 1 and 2, if Conjecture 1 is true, then Andrica’s conjecture is also true.

5 Oppermann’s Conjecture

Oppermann’s conjecture [12] states that for any integer $n > 1$ there is a prime number in the interval $[n^2 - n, n^2]$ and another prime in the interval $[n^2, n^2 + n]$. Now, if Conjecture 1 holds, then there exists a prime in the interval $[n^2 - n + 1, n^2 - 1]$ and another prime in the interval $[n^2 + 1, n^2 + n]$, for $n > 1$. This means that if Conjecture 1 is true, then so is Oppermann’s conjecture.

Note that according to Oppermann’s conjecture there is a prime in the interval $[n^2 - n + 1, n^2 - 1]$ and another prime in the interval $[n^2 + 1, n^2 + n]$ for $n > 1$, whereas we ask the question whether *any* sequence of n consecutive integers greater than n^2 and smaller than $(n + 1)^2$ contains at least one prime number. In other words, if Conjecture 1 is true, then Oppermann’s conjecture follows, whereas the reciprocal is not necessarily true.

6 On the Interval $[n, n + 2 \lfloor \sqrt{n} \rfloor - 1]$

It is easy to verify that if Conjecture 1 is true, then in the interval $[n^2 + n + 1, n^2 + 3n + 2]$ there are at least two prime numbers for every positive integer n .

Now, the number $n^2 + n + 1$ is always an odd integer.

Proof.

- If n is even, then n^2 is also even. Then we have

$$\begin{aligned} (\text{even integer} + \text{even integer}) + 1 &= \text{even integer} + \text{odd integer} = \\ &= \text{odd integer}. \end{aligned}$$

- If n is odd, then n^2 is also odd. Then we have

$$\begin{aligned} (\text{odd integer} + \text{odd integer}) + 1 &= \text{even integer} + \text{odd integer} = \\ &= \text{odd integer}. \quad \square \end{aligned}$$

Since the number $n^2 + n + 1$ is always an odd integer, then it may be prime or not. Now, the number $n^2 + 3n + 2$ can never be prime, since this number is always an even integer greater than 2.

Proof.

- If $n = 1$ (smallest value n can have), then $n^2 + 3n + 2 = 1 + 3 + 2 = 6$.
- If n is even, then n^2 and $3n$ are both even integers. The number 2 is also an even integer, and we know that

$$\text{even integer} + \text{even integer} + \text{even integer} = \text{even integer}.$$

- If n is odd, then n^2 and $3n$ are both odd integers, and we know that

$$\begin{aligned} (\text{odd integer} + \text{odd integer}) + \text{even integer} &= \\ = \text{even integer} + \text{even integer} &= \text{even integer}. \quad \square \end{aligned}$$

From all this we deduce that if Conjecture 1 is true, then the maximum distance between two consecutive prime numbers is the one from the number $n^2 + n + 1$ to the number $n^2 + 3n + 2 - 1 = n^2 + 3n + 1$, which means that in the interval $[n^2 + n + 1, n^2 + 3n + 1]$ there are at least two prime numbers. In other words, in the interval $[n^2 + n + 1, n^2 + 3n]$ there is at least one prime number.

The difference between the numbers $n^2 + n + 1$ and $n^2 + 3n$ is $n^2 + 3n - (n^2 + n + 1) = n^2 + 3n - n^2 - n - 1 = 2n - 1$. In addition to this, $\lfloor \sqrt{n^2 + n + 1} \rfloor = n$. This means that in the interval $[n^2 + n + 1, n^2 + n + 1 + 2 \lfloor \sqrt{n^2 + n + 1} \rfloor - 1]$ there is at least one prime number. In other words, if $a = n^2 + n + 1$, then the interval $[a, a + 2 \lfloor \sqrt{a} \rfloor - 1]$ contains at least one prime number.

Remark 4. The symbol $\lfloor \cdot \rfloor$ represents the *floor function*. The floor function of a given number is the largest integer that is not greater than that number. For example, $\lfloor 3.5 \rfloor = 3$.

Now, if Conjecture 1 is true, then the following statements are all true:

Statement 1. *If a is a perfect square, then in the interval $[a, a + \lfloor \sqrt{a} \rfloor]$ there is at least one prime number.*

Statement 2. *If a is an integer such that $n^2 < a \leq n^2 + n + 1 < (n + 1)^2$, then in the interval $[a, a + \lfloor \sqrt{a} \rfloor - 1]$ there is at least one prime number.*

Statement 3. *If a is an integer such that $n^2 < n^2 + n + 2 \leq a < (n + 1)^2$, then in the interval $[a, a + 2 \lfloor \sqrt{a} \rfloor - 1]$ there is at least one prime number.*

We know that $a + 2 \lfloor \sqrt{a} \rfloor - 1 \geq a + \lfloor \sqrt{a} \rfloor$.

Proof. We have

$$\begin{aligned} a + 2 \lfloor \sqrt{a} \rfloor - 1 &\geq a + \lfloor \sqrt{a} \rfloor \\ 2 \lfloor \sqrt{a} \rfloor - 1 &\geq \lfloor \sqrt{a} \rfloor \\ 2 \lfloor \sqrt{a} \rfloor &\geq \lfloor \sqrt{a} \rfloor + 1 \\ \lfloor \sqrt{a} \rfloor + \lfloor \sqrt{a} \rfloor &\geq \lfloor \sqrt{a} \rfloor + 1 \\ \lfloor \sqrt{a} \rfloor &\geq 1, \end{aligned}$$

which is true for every positive integer a . □

And we also know that $a + 2 \lfloor \sqrt{a} \rfloor - 1 > a + \lfloor \sqrt{a} \rfloor - 1$.

Proof. We have

$$\begin{aligned} a + 2 \lfloor \sqrt{a} \rfloor - 1 &> a + \lfloor \sqrt{a} \rfloor - 1 \\ 2 \lfloor \sqrt{a} \rfloor &> \lfloor \sqrt{a} \rfloor, \end{aligned}$$

which is obviously true for every positive integer a . □

All this means that the interval $[a, a + 2 \lfloor \sqrt{a} \rfloor - 1]$ can be applied to the number a in Statement 1, to the number a in Statement 2, and to the number a in Statement 3. Therefore, if n is any positive integer and Conjecture 1 is true, then in the interval $[n, n + 2 \lfloor \sqrt{n} \rfloor - 1]$ there is at least one prime number (in order to provide more standardized notation, we are now replacing letter a with letter n). According to this, we can also say that if Conjecture 1 is true, then there is always a prime number in the interval $[n, n + 2 \lfloor \sqrt{n} \rfloor - 1]$ for every positive integer n .

7 Conclusion

We have proved that if Conjecture 1 is true, then Legendre's, Brocard's, Andrica's, and Oppermann's conjectures follow. In addition, we have shown that if the mentioned conjecture holds, then the interval $[n, n + 2\lfloor\sqrt{n}\rfloor - 1]$ contains a prime for every positive integer n .

Now, the number $2\lfloor\sqrt{n}\rfloor - 1$ is always an odd number. Suppose that Conjecture 1 is true and that b is an odd integer greater than 1. Then the interval $[b, b + 2\lfloor\sqrt{b}\rfloor - 2]$ contains at least one prime number. Under the same assumption we can state that if c is any positive even integer, then there exists a prime in the interval $[c + 1, c + 2\lfloor\sqrt{c}\rfloor - 1]$.

Conjecture 1 implies that if n is any positive integer greater than 1, then at least one of the intervals $[n - \lfloor\sqrt{n}\rfloor + 1, n]$ and $[n, n + \lfloor\sqrt{n}\rfloor - 1]$ contains a prime. Note that we can state the following:

- If d is a certain positive odd integer such that d is not a perfect square and $\lfloor\sqrt{d}\rfloor$ is even, then the fact that there is a prime in $[d - \sqrt{d}, d]$ does not imply that there is a prime in $[d - \lfloor\sqrt{d}\rfloor + 1, d]$. Similarly, the fact that there exists a prime in $[d, d + \sqrt{d}]$ does not imply that there exists a prime in $[d, d + \lfloor\sqrt{d}\rfloor - 1]$.
- If f is a certain positive even integer such that f is not a perfect square and $\lfloor\sqrt{f}\rfloor$ is odd, then the fact that there is a prime in $[f - \sqrt{f}, f]$ does not imply that there is a prime in $[f - \lfloor\sqrt{f}\rfloor + 1, f]$. Similarly, the fact that there exists a prime in $[f, f + \sqrt{f}]$ does not imply that there exists a prime in $[f, f + \lfloor\sqrt{f}\rfloor - 1]$.

This means that if n is not a perfect square and n and $\lfloor\sqrt{n}\rfloor$ have different parity, then the fact that $\pi[n - \sqrt{n}, n] \geq 1$ does not imply that $\pi[n - \lfloor\sqrt{n}\rfloor + 1, n] \geq 1$, and the fact that $\pi[n, n + \sqrt{n}] \geq 1$ does not imply that $\pi[n, n + \lfloor\sqrt{n}\rfloor - 1] \geq 1$.

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