The orbit motion in the gravity field

Sangwha Yi
Department of Math, Taejon University  300-716

ABSTRACT
In the general relativity theory, using Einstein’s gravity field equation, find the solution of the orbit motion in the general relativity theory. Therefore the solution has the angular velocity. According to the solution, the matter rotates in the gravity force.

PACS Number: 04.40.20, 04.90.+e
Key words: The general relativity theory
The gravity force
The orbit motion
The angular velocity

e-mail address: sangwha1@nate.com
Tel: 051-624-3953
I. Introduction

This paper is that in the general relativity theory, find the term that treat that the matter does the orbit motion in the gravity.

The general relativity theory’s field equation is written completely.

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - \frac{8\pi G}{c^4} T_{\mu\nu} \]  

Eq (1) multiply \( g^{\mu\nu} \) and does contraction,

\[ g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = - \frac{8\pi G}{c^4} T^\lambda_{\lambda} \]  

Therefore, Eq (1) is

\[ R_{\mu\nu} = - \frac{1}{2} g_{\mu\nu} \frac{8\pi G}{c^4} T^\lambda_{\lambda} = - \frac{8\pi G}{c^4} T_{\mu\nu} \]  

\[ R_{\mu\nu} = - \frac{8\pi G}{c^4} (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda_{\lambda}) \]  

(3)

In this time, the spherical coordinate system’s vacuum solution is by \( T_{\mu\nu} = 0 \)

\[ R_{\mu\nu} = 0 \]  

(4)

The spherical coordinate system’s invariant time is

\[ d\tau^2 = A(t,r)dt^2 - \frac{1}{c^2} [B(t,r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2] \]  

(5)

Using Eq(5)’s metric, save the Riemannian-curvature tensor, and does contraction, save Ricci-tensor.

\[ R_{tt} = - \frac{A''}{2B} + \frac{A'A''}{4B^2} - \frac{A'}{4AB} + \frac{A^2}{2B} - \frac{B}{4B^2} - \frac{\dot{A}\dot{B}}{4AB} = 0 \]  

(6)

\[ R_{tr} = \frac{A''}{2A} - \frac{A^2}{4A^2} - \frac{A'B'}{4AB} - \frac{B'}{4AB} - \frac{\dot{B}}{2A} + \frac{\dot{A}\dot{B}}{4A^2} + \frac{\dot{B}}{4AB} = 0 \]  

(7)

\[ R_{\theta\theta} = -1 + \frac{1}{B} \frac{rB'}{2B} + \frac{rA'}{2AB} = 0 \]  

(8)

\[ R_{\phi\phi} = R_{\theta\theta} \sin^2 \theta = 0 \]  

(9)

\[ R_r = \frac{\dot{B}}{Br} = 0 \]  

(10)  \[ R_{t\theta} = R_{t\phi} = R_{r\theta} = R_{r\phi} = R_{\phi\theta} = 0 \]  

(11)

In this time, \( \dot{t} = \frac{\partial}{\partial \tau}, \dot{r} = \frac{1}{c} \frac{\partial}{\partial \tau} \)
II. Additional chapter

By Eq(10),

\[ \dot{B} = 0 \quad (12) \]

By Eq(6) and Eq(7),

\[ \frac{R_{\parallel}}{A} + \frac{R_{\perp}}{B} = -\frac{1}{Br} \left( \frac{A'}{A} + \frac{B'}{B} \right) = -\frac{(AB)'}{rAB^2} = 0 \quad (13) \]

Therefore,

\[ A = \frac{1}{B} \quad (14) \]

If Eq(8) is inserted by Eq(14),

\[ R_{\infty} = -1 + \frac{1}{B} - \frac{rB'}{2B^2} + \frac{rA'}{2AB} = -1 + \left( \frac{r}{B} \right)' = 0 \quad (15) \]

If solve Eq(15)

\[ \frac{r}{B} = r + C_1 \rightarrow \frac{1}{B} = 1 + \frac{C_1}{r} \quad (16) \]

In this time, in the gravity, to find the term of the orbit motion in the general relativity theory, in Eq(16), if \( C_1 \) is

\[ C_1 = -\frac{2V_0T}{2\pi} \left( \frac{V_0}{c} \right)^2 = -\frac{2GM}{c^2} \quad (17) \]

\( V_0 \) is the orbit velocity. \( T \) is the orbit motion period. \( c \) is the light velocity in the gravity field.

In this time, it treat that the orbit velocity is constant.

Therefore, Eq(16) is

\[ A = \frac{1}{B} = 1 - \frac{2V_0T}{2\pi} \left( \frac{V_0}{c} \right)^2, \quad r\omega = V_0 \]

\[ = 1 - \frac{2\omega T}{2\pi} \left( \frac{V_0}{c} \right)^2, \quad \omega T = \theta \]

\[ = 1 - \frac{2\theta}{2\pi} \left( \frac{V_0}{c} \right)^2 \quad (18), \quad \text{Therefore} \quad \theta = \omega T = \frac{V_0T}{r} \]

\( \omega \) is the angular velocity.

\[ A = \frac{1}{B} = 1 - \frac{2V_0T}{2\pi r} \left( \frac{V_0}{c} \right)^2 = 1 - \frac{2GM}{rc^2} \quad (19) \]

In the Keppler 3—law,

\[ T^2 = \frac{4\pi^2 R^3}{GM} \quad (20), \quad R \text{ is the radius of the matter’s orbit.} \]

In this theory, in Eq(19)
\[ \frac{V_0^3 T}{2 \pi} = \frac{T}{2 \pi} \left( \frac{2 \pi R}{T} \right)^3 = \frac{4 \pi^2 R^3}{T^2} = GM \]  
(21), \( R \) is the radius of the matter’s orbit.

Therefore, Eq(19) satisfy Eq(20). And \( \frac{V_0^3 T}{2 \pi} = GM \) is constant.

To know Eq(18)’s second term, does Newton’s approximation

\[ F(r) \approx m_0 \frac{d^2 r}{dt^2} \approx \frac{1}{2} c^2 m_0 \frac{\partial(-A)}{\partial r} = -\frac{1}{2} c^2 m_0 \frac{2V_0 T}{2\pi r^2} \frac{V_0}{c}, r \omega = V_0 \]

\[ = -m_0 \frac{\omega T}{2\pi} V_0, \theta = \omega T \]

\[ = -m_0 \frac{V_0^2 \theta}{2\pi} \]  
(22)

In this time, if the matter rotate the orbit, \( \theta = 2\pi \)

\[ F(r) \approx m_0 \frac{d^2 r}{dt^2} \approx \frac{m_0 V_0^2}{r} \]

\[ F(r) \approx m_0 \frac{d^2 r}{dt^2} \approx \frac{1}{2} c^2 m_0 \frac{\partial(-A)}{\partial r} = -\frac{1}{2} c^2 m_0 \frac{2GM}{r^2 c^2} = -\frac{GM m_0}{r^2} \]

\[ F(r) \approx \frac{m_0 V_0^2}{r} \approx -\frac{GM m_0}{r^2} \]  
(23)

III. Conclusion

Therefore, the spherical coordinate system’s invariant time include the term that treat that the matter move in the orbit in the gravity is

\[ d\tau^2 = (1 - \frac{2V_0 T}{2\pi} (\frac{V_0}{c})^2) dt^2 - \frac{1}{c^2} \left[ \frac{1}{1 - \frac{2V_0 T}{2\pi} (\frac{V_0}{c})^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \]

\[ = (1 - \frac{2\omega T}{2\pi} (\frac{V_0}{c})^2) dt^2 - \frac{1}{c^2} \left[ \frac{1}{1 - \frac{2\omega T}{2\pi} (\frac{V_0}{c})^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \]

\[ = (1 - \frac{\theta}{2\pi} T (\frac{V_0}{c})^2) dt^2 - \frac{1}{c^2} \left[ \frac{1}{1 - \frac{\theta}{2\pi} T (\frac{V_0}{c})^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \]

\( V_0 \) is the orbit velocity, \( T \) is the orbit motion period. \( c \) is the light velocity in the no gravity field.

(24)

The spherical coordinate system’s invariant time(vacuum solution) is

\[ d\tau^2 = (1 - \frac{2GM}{rc^2}) dt^2 - \frac{1}{c^2} \left[ \frac{1}{1 - \frac{2GM}{rc^2}} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] (25) \]
Specially, if the matter move in the orbit,

\[
\begin{align*}
\frac{d\tau^2}{c^2} &= \left(1 - \frac{2GM}{rc^2}\right)dt^2 - \frac{1}{c^2} \left[ \frac{dr^2}{\left(1 - \frac{2GM}{rc^2}\right)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \\
&= \left(1 - \frac{2V_0}{2\pi T} \left(\frac{V_0}{c}\right)^2\right)dt^2 - \frac{1}{c^2} \left[ \frac{dr^2}{\left(1 - \frac{2V_0}{2\pi T} \left(\frac{V_0}{c}\right)^2\right)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] (26)
\end{align*}
\]

\(V_0\) is the orbit velocity, \(T\) is the orbit motion period, \(c\) is the light velocity in the no gravity field.

**Reference**


