The Mathematical Basis of the Fine Structure Constant

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It can be shown that the Fine Structure Constant can be defined as:

$$\alpha \equiv \frac{1}{\sqrt{-2e\pi - e^{2\pi} + 2\pi e + e^{\pi^2} + \pi^2}} = 0.007297352558.$$  

The Fine Structure Constant, \( \alpha \), gives the magnitude of spectral line splitting (Fine Structure) due to spin-orbit coupling and the consequent difference in energy between electron spin and orbital angular momentum vectors being parallel and anti-parallel. It is also the ratio of the Hartree potential energy, \( H_a \), or twice the Rydberg ionisation energy, \( R_y \), to the electron rest energy, \( m_e c^2 \):

$$\alpha = \frac{\sqrt{H_a}}{\sqrt{m_e c^2}} = \frac{\sqrt{2R_y}}{\sqrt{m_e c^2}}$$

The description of the state of the electron undergoing spin-orbit coupling should show the relationship:

\[
\text{(total electron energy)} = (\text{electron rest energy}) + ([\text{orbit}] \times [\text{spin}] \times [\text{hydrogen ground state}])
\]

A complete orbit is \( 2\pi \) radians, a change from spin parallel to spin anti-parallel is \( \pi \) radians, and the hydrogen ground state is \(-1\) Rydberg:

\[
E_{\text{total}} = (m_e c^2) + ([2\pi] \times [\pi] \times [-R_y]) = m_e c^2 - 2\pi^2 R_y
\]

The relationship between action \( S \) (energy \( \times \) time), the reduced Planck constant \( \hbar \), and the phase of the electron wave is given by:

$$e^{iS/\hbar}$$

and the Canonical Commutation Relation is:

$$[E, t] = Et - tE = -i\hbar$$

The difference in action associated with rotating the electron wave’s reference-frame by \( \theta \) will be:

$$\Delta S\theta = (Et - tE)\theta$$

giving:
The imaginary unit is therefore cancelled out. A further difference from classical rotations comes due to the Uncertainty Relations between position $x$ and momentum $p$, and between energy $E$ and time $t$:

\[ e^{i \Delta \theta \hbar / \hbar} = e^{i (Et - tE) \hbar / \hbar} = e^{i (-i) \theta} = e^{i \theta} \]

This condition can nevertheless be understood in classical terms by appealing to General Relativity Theory, in which the presence of mass-energy produces space-time curvature characterised by a metric tensor, $g_{\mu\nu}$, the mechanism behind gravity and gravitational waves. Since the Uncertainty Relations must also apply to uncertainty in fluctuations of the $g_{\mu\nu}$ field, an expression of the relative error in position:

\[ \eta = \frac{\Delta x}{x} \]

becomes effectively interchangeable with the equation for the strain amplitude of a gravitational wave:

\[ a = \frac{\Delta l}{l} \]

That is:

\[ a \Delta p \geq \frac{\hbar}{2} \]

\[ \Delta E a / c \geq \frac{\hbar}{2} \]

Crucially, the distortion in length identified with gravitational waves, $\Delta l$, oscillates alternately between orthogonal axes. The corollary is that a full treatment of the increment in position, $\Delta x$, should involve transposing orthogonal axes. Some considerations following from this are:

- In Feynman diagrams, the change of sign of electrical energy from electron to anti-electron,
"-" → "+", is associated with transposing the space and time axes relative to the arrows representing these particles.

- in polar coordinates and the polar form of complex numbers, a transposing of orthogonal coordinates exchanges radius and angle:

\[(r)e^{(in\theta)} \rightarrow (\theta)e^{(inr)}\]

- because the electron wave flows around the atom, and there are two switch-like states of spin alignment, the atom can be treated as a "point-circuit" and transposing orthogonal dimensions will produce a transposing of "series" and "parallel" which, in C. E. Shannon's mathematical model of switching circuits, is associated with an exchange of the operations "+" and "×".

Therefore, the $e^{i\phi/h}$ components to be added are:

- spin alignment ($\pi$):

\[-e^{(i(\pi-tE)\theta/h)} \rightarrow \theta e^{(i(\pi-tE)/h)}\]

\[= -e^{\pi} \rightarrow \pi e\]

and:

\[-e^{(i(\pi-tE)\phi/h)} \rightarrow \phi e^{(i(\pi-tE)/h)}\]

\[= -e^{\pi} \rightarrow \pi e\]

- orbit (2$\pi$):

\[-e^{(i(2\pi-tE)\pi/h)} \rightarrow e^{(i(2\pi-tE)\pi/h)}\]

\[= -e^{2\pi} \rightarrow e^{\pi^2}\]

giving:

\[-2e^{\pi}Ha + 2\pi eHa - e^{2\pi}Ha + e^{\pi^2}Ha = m_ec^2 - 2\pi^2Ry\]

The Inverse Fine Structure Constant is:

\[\frac{1}{\alpha} = \frac{\sqrt{m_ec^2}}{Ha} = \frac{m_ec^2}{2Ry}\]
so that dividing all terms by 1 Ha (or 2 Ry) gives a value for the Fine Structure Constant of:

\[
\alpha \equiv \frac{1}{\sqrt{\left(-2e^{\pi} - e^{2\pi} + 2\pi e + e^{\pi^2} + \pi^2\right)}} = 0.007297352558
\]