Two Uncertain Things

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Sample Space (Frame of Discernment)

- A set of possible worlds/states/elementary outcomes
  \( \mathcal{W} = \{w_1, \ldots, w_n\} \)

- An agent considers some subset of \( \mathcal{W} \) possible and this subset qualitatively measures her uncertainty

- The more worlds an agent considers possible, the more uncertain she is, and the less she knows

- When throwing a dice, the sample space would be
  \( \mathcal{W} = \{w_1, w_2, w_3, w_4, w_5, w_6\} \), \( w_i \) means the dice lands \( i \)

- An agent can consider the dice landing on an even number possible, that is the subset \( \mathcal{W} = \{w_2, w_4, w_6\} \)
Sample Space

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A set of possible worlds/states/elementary outcomes
\[ W = \{ w_1, \ldots, w_n \} \]

An agent considers some subset of \( W \) possible and this subset qualitatively measures her uncertainty.

The more worlds an agent considers possible, the more uncertain she is, and the less she knows.

When throwing a dice, the sample space would be
\[ W = \{ w_1, w_2, w_3, w_4, w_5, w_6 \}, \quad w_i \text{ means the dice lands } i \]

An agent can consider the dice landing on an even number possible, that is the subset \( W = \{ w_2, w_4, w_6 \} \)
A method for representing uncertainty

Given a sample space $W = \{w_1, \ldots, w_n\}$, a probability measure assigns to each world $w_i$ a number (a probability)

This probability describes the likelihood that the world $w_i$ is the actual world
Algebra

- An algebra over $W$ is a set $F$ of subsets of $W$ that contains $W$ and is closed under union and complementation.
- If $A$ and $B$ are in $F$, then so are $A \cup B$ and $\overline{A}$.

Probability Measure

- Given a sample space $W$, a probability measure is a function $\mu : F \rightarrow [0, 1]$ that satisfies the following two properties:
  - $\mu(W) = 1$
  - Finite additivity: $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A$ and $B$ are disjoint sets in $F$. 
Algebra

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- If $A$ and $B$ are in $F$, then so are $A \cup B$ and $\overline{A}$.

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Probability Measures

- When flipping a coin, the sample space would be \( \mathcal{W} = \{ w_H, w_T \} \)
- An algebra: \( F = \{ \{ w_H \}, \{ w_T \}, \{ w_H, w_T \} \} \)
- A probability measure: \( \mu : F \to [0, 1] \)
  - \( \mu(\{ w_H \}) = 0.7, \mu(\{ w_T \}) = 0.3 \)
  - \( \mu(\{ w_H, w_T \}) = \mu(\{ w_H \}) + \mu(\{ w_T \}) = 1 \)
• Probability measures are not good at representing uncertainty because of the finite additivity property

• Ignorance is difficult to express - an agent has to assign a probability to \( \{w_H\} \)

• An agent may not have the computational power to compute all the probabilities
**Belief Measure**

Given a sample space \( \mathcal{W} \), a belief measure is a function \( Bel : 2^\mathcal{W} \to [0, 1] \) that satisfies the following three properties:

- \( Bel(\emptyset) = 0 \)
- \( Bel(\mathcal{W}) = 1 \)
- Inclusion-exclusion rule:
  \[
  Bel(A_1 \cup A_2 \cup \ldots \cup A_n) \geq \sum_{j} Bel(A_j) - \sum_{j<k} Bel(A_j \cap A_k) + \ldots + (-1)^{n+1} Bel(A_1 \cap A_2 \cap \ldots \cap A_n) \Omega_1
  \]

In \( \Omega_1 \), let \( A_1 = A \) and \( A_2 = \overline{A} \) for \( n = 2 \). Then

\[
Bel(A \cup \overline{A}) \geq Bel(A) + Bel(\overline{A}) - Bel(A \cap \overline{A}) \]

which gives

\[
Bel(A) + Bel(\overline{A}) \leq 1
\]
Belief and Plausibility Measures

Belief Measure

Given a sample space $W$, a belief measure is a function $Bel : 2^W \rightarrow [0, 1]$ that satisfies the following three properties:

- $Bel(\emptyset) = 0$
- $Bel(W) = 1$
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  $$Bel(A_1 \cup A_2 \cup \ldots \cup A_n) \geq \sum_{j} Bel(A_j) - \sum_{j<k} Bel(A_j \cap A_k) + \ldots + (-1)^{n+1} Bel(A_1 \cap A_2 \cap \ldots \cap A_n) \quad (\Omega_1)$$

In $(\Omega_1)$, let $A_1 = A$ and $A_2 = \overline{A}$ for $n = 2$. Then
$$Bel(A \cup \overline{A}) \geq Bel(A) + Bel(\overline{A}) - Bel(A \cap \overline{A})$$
which gives
$$Bel(A) + Bel(\overline{A}) \leq 1$$
Plausibility Measure

Given a sample space $W$, a plausibility measure is a function $Pl : 2^W \rightarrow [0, 1]$ that satisfies the following three properties:

- $Pl(\emptyset) = 0$
- $Pl(W) = 1$
- Inclusion-exclusion rule: $Pl(A_1 \cap A_2 \cap \ldots \cap A_n) \leq \sum_j Pl(A_j) - \sum_{j<k} Pl(A_j \cup A_k) + \ldots + (-1)^{n+1} Bel(A_1 \cup A_2 \cup \ldots \cup A_n) (\Omega_2)$

In $(\Omega_2)$, let $A_1 = A$ and $A_2 = \overline{A}$ for $n = 2$. Then $Pl(A \cap \overline{A}) \leq Pl(A) + Pl(\overline{A}) - Pl(A \cup \overline{A})$ which gives $Pl(A) + Pl(\overline{A}) \geq 1$
Belief and Plausibility Measures

**Plausibility Measure**

Given a sample space $W$, a plausibility measure is a function $PI : 2^W \rightarrow [0, 1]$ that satisfies the following three properties:

- $PI(\emptyset) = 0$
- $PI(W) = 1$
- Inclusion-exclusion rule: $PI(A_1 \cap A_2 \cap ... \cap A_n) \leq \sum_{j} PI(A_j) - \sum_{j<k} PI(A_j \cup A_k) + ... + (-1)^{n+1} Bel(A_1 \cup A_2 \cup ... \cup A_n) \ (\Omega_2)$

In $(\Omega_2)$, let $A_1 = A$ and $A_2 = \overline{A}$ for $n = 2$. Then $PI(A \cap \overline{A}) \leq PI(A) + PI(\overline{A}) - PI(A \cup \overline{A})$ which gives $PI(A) + PI(\overline{A}) \geq 1$
Given a sample space $W$, a basic belief assignment is a function $m : 2^W \rightarrow [0, 1]$ that satisfies the following two properties:

- $m(\emptyset) = 0$
- $\sum_{A \in 2^W} m(A) = 1$

$m(A) \geq 0$

If $m(A) > 0$ then $A$ is a focal set

Let $\mathcal{F}$ be the set of all focal sets induced by $m$, then $\langle \mathcal{F}, m \rangle$ is a body of evidence

It is clear that a bba resembles a probability distribution function
Basic Belief Assignment (Mass Function) (Möbius Representation)

- Given a sample space $W$, a basic belief assignment is a function $m : 2^W \rightarrow [0, 1]$ that satisfies the following two properties:
  - $m(\emptyset) = 0$
  - $\sum_{A \in 2^W} m(A) = 1$

- $m(A) \geq 0$
- If $m(A) > 0$ then $A$ is a focal set
- Let $\mathcal{F}$ be the set of all focal sets induced by $m$, then $\langle \mathcal{F}, m \rangle$ is a body of evidence
- It is clear that a bba resembles a probability distribution function
Belief and Plausibility Measures

Some Formulas

- $Pl(A) = 1 - Bel(\bar{A})$
- $Bel(A) \leq Pl(A)$
- $Bel(A) = \sum_{B | B \subseteq A} m(B)$
- $Pl(A) = \sum_{B | A \cap B \neq \emptyset} m(B)$
- $Q(A) = \sum_{B | A \subseteq B} m(B)$
- $m(A) = \sum_{B | B \subseteq A} (-1)^{|A - B|} Bel(B)$
- $m(A) = \sum_{B | B \subseteq A} (-1)^{|A - B|} [1 - Pl(\bar{B})]$
Belief and Plausibility Measures

- \( Bel(A) \) is the **total belief** that the actual world is in the set \( A \) which is obtained by adding degrees of evidence for the set itself, as well as for any of its **subsets**

- \( Pl(A) \) is the **total belief** that the actual world is in the set \( A \), and also the **partial evidence** for the set that is associated with any set that overlaps with \( A \)

- \( m(A) \) is the **degree of belief** that the actual world is in the set \( A \), but it **does not** take into account any additional evidence for the various subsets of \( A \)
Belief and Plausibility Measures

- $Q(A)$ is the **total belief** that can move freely to every point of $A$.
- The interval $[Bel(A), Pl(A)]$ describes the range of possible values of the likelihood of $A$.
- If $W$ is finite, then there is **one-to-one** correspondence between belief measures and bbas.
Belief and Plausibility Measures

Total Ignorance

- When no evidence is available about the actual world
- Capture it using a vacuous bba $m_{vac} : 2^W \rightarrow [0, 1]$ where $m_{vac}(W) = 1$ and $m_{vac}(A) = 0$ for all $A \in 2^W \setminus W$
- Capture it using a vacuous belief measure $Bel_{vac} : 2^W \rightarrow [0, 1]$ where $Bel_{vac}(W) = 1$ and $Bel_{vac}(A) = 0$ for all $A \in 2^W \setminus W$
- Capture it using a vacuous plausibility measure $Pl_{vac} : 2^W \rightarrow [0, 1]$ where $Pl_{vac}(\emptyset) = 0$ and $Pl_{vac}(A) = 1$ for all $A \neq \emptyset$
A bag contains 100 balls; 25 are known to be red, 25 are known to be either red or blue, and 50 are known to be either blue or yellow.

The sample space \( W = \{ \text{red}, \text{blue}, \text{yellow} \} \)

The bba \( m : 2^W \rightarrow [0, 1] \) where \( m(\{\text{red}\}) = 0.25 \), \( m(\{\text{red}, \text{blue}\}) = 0.25 \), \( m(\{\text{blue}, \text{yellow}\}) = 0.5 \), and \( m(\{\text{blue}\}) = m(\{\text{yellow}\}) = m(\{\text{red}, \text{yellow}\}) = m(W) = 0 \).
The belief measure $Bel : 2^W \rightarrow [0, 1]$ where
$Bel(\{red\}) = 0.25, \quad Bel(\{red, blue\}) = 0.5, \quad Bel(\{blue, yellow\}) = 0.5, \quad Bel(\{blue\}) = Bel(\{yellow\}) = 0, \quad Bel(\{red, yellow\}) = 0.25, \text{ and } Bel(W) = 1$

The plausibility measure $Pl : 2^W \rightarrow [0, 1]$ where
$Pl(\{red\}) = 0.5, \quad Pl(\{blue\}) = 0.75, \quad Pl(\{yellow\}) = 0.5, \quad Pl(\{red, blue\}) = 1, \quad Pl(\{blue, yellow\}) = 0.75, \quad Pl(\{red, yellow\}) = 1, \text{ and } Pl(W) = 1$
Belief and Plausibility Measures

**Rule of Combination**

- Used to *combine* evidence obtained from two independent sources
- Assume that the degrees of evidence 1 and 2 are captured using the bbas $m_1$ and $m_2$ respectively

$$m_{1,2}(A) = \frac{\sum_{B \cap C = A} m_1(B)m_2(C)}{1-c}$$

where $A \neq \emptyset$, $m_{1,2}(\emptyset) = 0$, and $c = \sum_{B \cap C = \emptyset} m_1(B)m_2(C)$

- $c$ is the *degree of conflict* between the two evidence
- The rule is *commutative* $m_{1,2} = m_{2,1}$
- The rule is *associative* $m_{1,(2,3)} = m_{(1,2),3}$
- The *neutral* element is $m_{vac}$, that is $m_{1,vac} = m_{vac,1} = m_1$
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4. **The Second Uncertain Thing**
   - Assumptions
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• A computing system has a number of possible states represented by the space $W$
• Over a time period $t$, the actual state of the system is in the set $A \in 2^W$
• Over the same time period $t$, an attacker is trying to discover this actual state
Initially, the attacker has no evidence about the actual state

$m_{vac} : 2^W \rightarrow [0, 1]$ where $m(W) = 1$ and $m(A) = 0$ for all $A \in 2^W \setminus W$

The attacker obtains evidence from source $src_1$ that the actual state is in the set $A \in 2^W$

$m_1 : 2^W \rightarrow [0, 1]$ where $m(A) = \alpha_1 > 0$ and $m(W) = 1 - \alpha_1$

... 

The attacker obtains the last evidence from source $src_n$ that the actual state is in the set $A \in 2^W$

$m_n : 2^W \rightarrow [0, 1]$ where $m(A) = \alpha_n > 0$ and $m(W) = 1 - \alpha_n$
- Assuming that the sources $src_1, ..., src_n$ are independent
- The attacker will combine to get
  $$m_{1,\ldots,n}(W) = (1 - \alpha_1) \times \ldots \times (1 - \alpha_n) \quad \text{and} \quad m_{1,\ldots,n}(A) = 1 - (1 - \alpha_1) \times \ldots \times (1 - \alpha_n)$$
- The Law of Large Numbers says that $m_{1,\ldots,n}(A)$ will eventually reach 1
- When this happens, the attacker will have full belief that the actual state is in the set $A \in 2^W$, which means that the system is compromised.
- Poisoning the values of $\alpha_1, ..., \alpha_n$ by minimizing them would delay the satisfaction of the Law of Large Numbers, possibly until a next time period $t'$ by the start of which the system becomes in a different actual state
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An agent has an evidence that the actual world is in the set $A \in 2^W$

Later she obtains another evidence that the actual world is in the set $B \in 2^W$

How can she update her knowledge?

$$Bel(B \mid A) = \frac{Bel(B \cup \overline{A}) - Bel(\overline{A})}{1 - Bel(\overline{A})}$$

$$Pl(B \mid A) = \frac{Pl(B \cap A)}{Pl(A)}$$
Meaningful Measure of Uncertainty with Beliefs

A measure $\mathcal{M}$ of the uncertainty of $\text{Bel}$ is **meaningful** if it satisfies the following properties:

1. **Probability Consistency:** If all focal sets are singletons, $\mathcal{M}$ should assume Shannon’s entropy:
   \[
   \mathcal{M}(\text{Bel}) = - \sum_{x \in \mathcal{W}} \text{Bel}(\{x\}) \log_2 \text{Bel}(\{x\})
   \]

2. **Set Consistency:** If $\text{Bel}$ focuses on a single set $A \subseteq \mathcal{W}$, $\mathcal{M}$ should assume Hartley’s entropy: $\mathcal{M}(\text{Bel}) = \log_2 |A|$

3. **Expansibility:** The range of $\mathcal{M}$ is $[0, \log_2 |\mathcal{W}|]$ and $\mathcal{M}$ is measured in bits

4. **Subadditivity:** Let $\text{Bel}_1$, $\text{Bel}_2$, and $\text{Bel}$ be bbas on $\mathcal{W}_1$, $\mathcal{W}_2$, and $\mathcal{W}_1 \times \mathcal{W}_2$, then $\mathcal{M}(\text{Bel}) \leq \mathcal{M}(\text{Bel}_1) + \mathcal{M}(\text{Bel}_2)$

5. **Additivity:** Let $\text{Bel}_1$, $\text{Bel}_2$, and $\text{Bel}$ be bbas on $\mathcal{W}_1$, $\mathcal{W}_2$, and $\mathcal{W}_1 \times \mathcal{W}_2$, and assume that $\text{Bel}_1$ and $\text{Bel}_2$ are **noninteractive**, then $\mathcal{M}(\text{Bel}) = \mathcal{M}(\text{Bel}_1) + \mathcal{M}(\text{Bel}_2)$
The First Uncertain Thing
Build-up Again
The Second Uncertain Thing
Thank You

Generalized Hartley’s Measure with Beliefs

Generalized Hartley’s Measure (U-uncertainty)

- Given a sample space $W$ and a body of evidence $\langle F, m \rangle$ on this space, the generalized Hartley’s measure is given by the formula: 
  $$GH(m) = \sum_{A \in F} m(A) \log_2 |A|$$
- It has the Expansibility Property $GH(m) \in [0, \log_2 |W|]$ and is measured in bits
  - lower bound when all focal sets are singletons
  - upper bound in total ignorance
- It has the Subadditivity Property: 
  $$GH(m) \leq GH(m_1) + GH(m_2)$$
- It has the Additivity Property: 
  $$GH(m) = GH(m_1) + GH(m_2)$$
A number of unsuccessful attempts to generalize Shannon’s measure with beliefs.

1. Measure of Dissonance: \( E(m) = - \sum_{A \in \mathcal{F}} m(A) \log_2 P_l(A) \)
2. Measure of Confusion: \( C(m) = - \sum_{A \in \mathcal{F}} m(A) \log_2 B_{el}(A) \)
3. Measure of Discord:
   \[
   D(m) = - \sum_{A \in \mathcal{F}} m(A) \log_2 \left( 1 - \sum_{B \in \mathcal{F}} m(B) \frac{|B - A|}{|A|} \right)
   \]
4. Measure of Strife:
   \[
   ST(m) = - \sum_{A \in \mathcal{F}} m(A) \log_2 \left( 1 - \sum_{B \in \mathcal{F}} m(B) \frac{|A - B|}{|A|} \right)
   \]

All of these measures do not have the Subadditivity Property, and are thus meaningless.

This frustrated search was replaced with Aggregate Uncertainty.
Aggregate Uncertainty

- $AU(\text{Bel}) = \max_{\mathcal{P}_{\text{Bel}}} \left\{ - \sum_{x \in W} p_x \log_2 p_x \right\}$

- $\mathcal{P}_{\text{Bel}}$ is a set of probability distributions that satisfies:
  - $p_x \in [0, 1]$ for all $x \in W$ and $\sum_{x \in W} p_x = 1$
  - $\text{Bel}(A) \leq \sum_{x \in A} p_x$ for all $A \subseteq W$

- $AU$ is a meaningful measure of uncertainty with beliefs
Efficient Algorithm

1. Find a nonempty set $A \subseteq W$ such that $\frac{Bel(A)}{|A|}$ is maximal
2. For $x \in A$, let $p_x = \frac{Bel(A)}{|A|}$
3. For each $B \subseteq W - A$, let $Bel(B) = Bel(B \cup A) - Bel(A)$
4. Let $W = W - A$
5. If $W \neq \emptyset$ and $Bel(W) > 0$, go to 1
6. If $W \neq \emptyset$ and $Bel(W) = 0$, let $p_x = 0$ for all $x \in W$
7. Compute $AU(Bel) = -\sum_{x \in W} p_x \log_2 p_x$
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A sample space $W = \{x_1, x_2, x_3\}$ of three confidential bank balances.

An attacker would like to learn the highest bank balance by monitoring the execution of the following program:

```c++
int i = 1;
bool f = true;
while (i <= 3) {
    if (x[i] > g) {
        f = false
    }
    i++;
}
cout << f << endl;
```

$g \in \{x_1, x_2, x_3\}$ is the attacker’s guess and $f$ is a flag that tells whether this guess is correct or not.
- \( m : 2^W \rightarrow [0, 1] \) where \( m(\{x_1, x_2\}) = 0.8 \) and \( m(\{x_2, x_3\}) = 0.2 \)

| A               | Bel(A) | \( \frac{Bel(A)}{|A|} \) |
|-----------------|--------|----------------------------|
| \( \{x_1, x_2\} \) | 0.8    | 0.4                        |
| \( \{x_2, x_3\} \) | 0.2    | 0.1                        |

- \( p_{x_1} = p_{x_2} = 0.4 \)
- \( W - A = \{x_1, x_2, x_3\} - \{x_1, x_2\} = \{x_3\} \)
- \( Bel(\{x_3\}) = Bel(\{x_1, x_2, x_3\}) - Bel(\{x_1, x_2\}) = 1 - 0.4 = 0.6 \)
- \( W = W - A = \{x_1, x_2, x_3\} - \{x_1, x_2\} = \{x_3\} \)
Since $W \neq \emptyset$ and $Bel(W) > 0$, we repeat the process

| $A$    | $Bel(A)$ | $\frac{Bel(A)}{|A|}$ |
|--------|----------|-----------------------|
| $\{x_3\}$ | 0.6      | 0.6                   |

- $p_{x_3} = 0.6$
- $W - A = \{x_3\} - \{x_3\} = \{\}$ we stop here!
- $AU(Bel) = -0.4 \log 0.4 - 0.4 \log 0.4 - 0.6 \log 0.6 = 0.528 + 0.528 + 0.442 = 1.498$ bits
The attacker has a **higher degree** of belief that the highest bank balance is in the set \{x_1, x_2\} \implies she feeds the program with \( x_1 \)

- Suppose she gets a *true* flag
- The attacker will **update** her beliefs and get:
  \[
  Bel(\{x_1, x_2\}\|\{x_1\}) = 1.0 \quad \text{and} \quad Bel(\{x_2, x_3\}\|\{x_1\}) = 0.0
  \]
- **AU**(Bel) would be 1.5 bits \( \implies 0.002 \) increase in uncertainty
The attacker has a **higher degree** of belief that the highest bank balance is in the set \( \{x_1, x_2\} \) ~ she feeds the program with \( x_1 \)

- Suppose she gets a **false** flag

- The attacker will **update** her beliefs and get:
  \[
  Bel(\{x_1, x_2\} \| \{x_2, x_3\}) = 0.8 \quad \text{and} \quad Bel(\{x_2, x_3\} \| \{x_2, x_3\}) = 1.0
  \]

- Again (!!!) \( AU(Bel) \) would be 1.5 bits ~ 0.002 increase in uncertainty
Thank You!