New Expression of the Factorial of $n$ ($n!$, $n \in N$)

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Abstract

New Expression of the factorial of $n$ ($n!$, $n \in N$) is given in this article. The general expression of it has been proved with help of the Principle of Mathematical Induction. It is found in the form

$$1 + \sum_{i=1}^{n} a_i + \sum_{i,j=1}^{n} a_i a_j + \sum_{i,j,k=1}^{n} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_n,$$

(1)

where $a_i = i - 1$ for $i = 1, 2, \cdots, n$. More convenient expression of this form is provided in Appendix.

Keywords: Factorial, new expression of factorial

1 Introduction

In mathematics, the factorial of a non-negative integer $n$ is denoted by $n!$. It is defined by the product of all positive integers less than or equal to $n$. Thus $n! = 1 \times 2 \times 3 \times \cdots \times n$. For example, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$ etc; while the value of $0!$ is 1 according to the convention for an empty product [1]. The most basic occurrence of factorial function is the fact that there are $n!$ ways to arrange $n$ distinct objects into a sequence (i.e., number of permutations of the objects). To Indian scholars this fact was well known at least as early as the 12th century [2]. Although the factorial function has its roots in combinatorics, the factorial operation is encountered in many different areas of mathematics such as permutations, algebra, calculus, probability theory and number theory.

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2 Theorem

The following expression holds for factorial of \( n \) (\( n! \), \( n \in \mathbb{N} \)):

\[
n! = 1 + \sum_{i=1}^{n} a_i + \sum_{\substack{i,j=1 \\ (i<j)}}^{n} a_i a_j + \sum_{\substack{i,j,k=1 \\ (i<j<k)}}^{n} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_n, \tag{2}
\]

where \( a_i = i - 1 \) for \( i = 1, 2, \cdots, n \).

3 Proof

Case: \( n = 1 \)

\( a_1 = 0, \sum_{i=1}^{n} a_i = a_1 = a_1 a_2 \cdots a_n = 0 \) for this case. The value of right hand side (RHS) of (2) can then be obtained as 1, while we know \( n! = 1 \) for \( n = 1 \). These show that the formula is valid for \( n = 1 \).

Case: \( n = 2 \)

To prove the formula for this case, the values \( a_1 = 0, a_2 = 1, \sum_{i=1}^{n} a_i = a_1 + a_2 = 1, \sum_{\substack{i,j=1 \\ (i<j)}}^{n} a_i a_j = a_1 a_2 = a_1 a_2 \cdots a_n = 0 \) have been used at RHS of (2). The computed value is then given as 2 that is the exact value of \( n! \) for \( n = 2 \). Thus the formula is valid for \( n = 2 \).

Case: \( n = 3 \)

We have \( a_1 = 0, a_2 = 1, a_3 = 2, \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 = 3, \sum_{\substack{i,j=1 \\ (i<j)}}^{n} a_i a_j = a_1 a_2 + a_2 a_3 + a_3 a_1 = 2, \)

\[
\sum_{\substack{i,j,k=1 \\ (i\neq j\neq k)}}^{n} a_i a_j a_k = a_1 a_2 a_3 = a_1 a_2 \cdots a_n = 0. \tag{2}
\]

The value of RHS of (2) can be evaluated as 6 and we have \( n! = 6 \) for \( n = 3 \). So the formula has been verified for this case.
Case: inductive step \( n = m \)

Keeping the formula general, the help of the Principle of Mathematical Induction will be considered to prove the theorem for all natural values of \( n \). It has been assumed that the formula is true for an arbitrary natural number \( n = m \). Then

\[
m! = 1 + \sum_{i=1}^{m} a_i + \sum_{i,j=1}^{m} a_i a_j + \sum_{i,j,k=1}^{m} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_m,
\]

(3)

Case: \( n = m + 1 \)

To prove the formula for arbitrary natural number \( n \), we have to prove the formula for \( n = m + 1 \) when it is true for \( n = 1 \) and is assumed true for an arbitrary \( n = m \). Now from RHS of (2), we have

\[
1 + \sum_{i=1}^{m+1} a_i + \sum_{i,j=1}^{m+1} a_i a_j + \sum_{i,j,k=1}^{m+1} a_i a_j a_k + \cdots + \sum_{i,j\ldots,s_t=1}^{m+1} a_i a_j \cdots a_{s_t} + a_1 a_2 \cdots a_{m+1},
\]

where \( s_t \) stands for \( m \);

\[
= 1 + \left( \sum_{i=1}^{m} a_i + a_{m+1} \right) + \left( \sum_{i,j=1}^{m} a_i a_j + a_{m+1} \sum_{i=1}^{m} a_i \right) + \left( \sum_{i,j,k=1}^{m} a_i a_j a_k + a_{m+1} \sum_{i=1}^{m} a_i a_j \right) + \cdots + \left( a_1 a_2 \cdots a_{m+1} \sum_{i,j\ldots,s_r=1}^{m} a_i a_j \cdots a_{s_r} \right) + a_1 a_2 \cdots a_{m+1},
\]

where \( s_r = m - 1 \);

\[
= \left( 1 + \sum_{i=1}^{m} a_i + \sum_{i,j=1}^{m} a_i a_j + \sum_{i,j,k=1}^{m} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_m \right) + a_{m+1} \left( 1 + \sum_{i=1}^{m} a_i + \sum_{i,j=1}^{m} a_i a_j + \sum_{i,j,k=1}^{m} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_m \right)
\]

3
\[ = m! + m \times m! = (m + 1)!, \quad (4) \]

which is the desired value of \( n! \) for \( n = m + 1 \). Hence the new expression of \( n! \) \((n \in N)\) has been proved by the Principle of Mathematical Induction.

**Appendix:**

Since \( a_1 = 0 \) in the new expression (2) of \( n! \) for all \( n \in N \), the formula can be represented as

\[
n! = 1 + \sum_{i=1}^{n-1} a_i + \sum_{i,j=1}^{n-1} a_i a_j + \sum_{i,j,k=1}^{n-1} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_{n-1}, \quad (5)
\]

where \( a_i = i \) for \( i = 1, 2, \cdots, n - 1. \)

**References**
