Noncommutative geometry of AdS coordinates on a D-brane

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In this short paper the noncommutative geometry and quantization of branes and the AdS is discussed. The question in part addresses an open problem left by this author in [1] on how branes are generated by stringy physics. The breaking of an open type I string into two strings generates a nascent brane at the new endpoints with inflationary cosmologies. This was left as a conjecture at the end of this paper on the role of quantum critical points in the onset of inflationary cosmology. The noncommutative geometry of the clock and lapse functions for the AdS-brane are derived as is the number of degrees of freedom which appear. The role of the AdS spacetime, or in particular its boundary, in cosmology is discussed in an elementary regularization scheme of the cosmological constant on the boundary. This is compared to schemes of conformal compactification of the AdS spacetime and the Heisenberg group.

1 Geometry of QCD theory of D-brane with AdS coordinates

In the recent article [1] a quantum phase transition model for the onset of inflation is proposed. The quantum critical behavior is proposed as a change in the physics of a type I string attached to two D-branes. As the D-branes separate under Casimir vacuum pressure the string stretches and breaks, with a new D-brane holding the endpoints of the two strings. The emergence of a D-brane is a quantum to classical transition. D-branes are classical objects which emerge in the limit of large $\mathcal{N}$ modes or degrees of freedom on the brane. In this letter the physics of this phase transition is examined in the light of holographic bounds.

The open string is a near Planck scale version of a meson. The endpoints are quark-like particles with an analogue of a gluon flux tube connecting them, which serves as the string. The system is a type of two-quark QCD system. The endpoints or quarks exist in a family of N_f quark fields and we represent this theory as $SU(2N_f)_r \times SU(2N_f)_\ell$. The general Lagrangian for a QCD system such as this [2] is

$$\mathcal{L} \;=\; -\frac{1}{4\pi g^2}F^a_{\mu\nu}F^{a\mu\nu} \;+\; i\bar{\psi}\sigma^\mu\Big(\partial_\nu \;+\; \frac{i}{2}A^a_\mu\tau^\mu\Big)\psi \;-\; \frac{1}{2}m_q\psi^T\tau_2\mathbf{\Omega}\psi \;+\; HC$$

for ψ the two component spinor

$$\psi \;=\; \begin{pmatrix} q_\ell \\ \sigma_2\tau_2 q_r \end{pmatrix}$$

and A^a_μ the gluon field strength with chromo-index $a \;=\; 1,\;2,\;3$. The matrix $\mathbf{\Omega}$ is the $2N \times 2N$ skew symmetric matrix

$$\mathbf{\Omega} \;=\; \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$$

The massless limit with $m_q \;=\; 0$ the $SU(2N_f)$ symmetry is replaced with $U(2N_f)$ for $Sp(n) \;=\; U(2n) \cap Sp(2n,\;\mathbb{C})$.

The Hermitian generator of $SU(2N_f)$ of dimension $2N_f \;-\; 1$ are normalized as $Tr(T^aT^b) \;=\; \delta^{ab}/2$ exist in two sets. The first set pertain to the symplectic group $Sp(2N_f) \;\subset\; SU(2N_f)$, denoted as X^a, $a \;=\; 1,\;\dots,\;2N_f \;-\; N_f$, and the remainder Y^a pertain to the quotient group $SU(2N_f)/Sp(2N_f)$ for $a \;=\; 1,\;\dots,\;2N_f \;+\; N_f \;-\; 1$. The quotient group generators "left over" from the group reduction are the Goldstone bosons in the $2N_f \times 2N_f$ matrix

$$\mathbf{Z} \;=\; e^{in_ax^a/\sqrt{N}}\mathbf{\Omega}$$

The algebraic elements of $Sp(2N_f)$ group obey

$$\mathbf{X}^T\mathbf{\Omega} \;+\; \mathbf{\Omega}\mathbf{X} \;=\; 0$$

and the quotient group obey

$$\mathbf{Y}^T\mathbf{\Omega} \;-\; \mathbf{\Omega}\mathbf{Y} \;=\; 0.$$

An important example is the group $SU(4)$, since $SU(4) \;\sim\; SO(4,\;2)$ is the isometry group of $AdS_4 \;\sim\; SO(4,\;2)/SO(4,\;1)$. The generators X^a and Y^a of $Sp(2,\;2) \;\sim\; Sp(4)$ and the quotient subgroup in $SU(4)$ can be written as

$$X^a \;=\; \frac{1}{2\sqrt{2}}\begin{pmatrix} \sigma^a & 0 \\ 0 & -\sigma^{aT} \end{pmatrix}_{a=1,\dots 4},\;\; X^a \;=\; \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & x^a \\ x^{a\dagger} & 0 \end{pmatrix}_{a=5,\dots 10},$$

$$Y^a \;=\; \frac{1}{2\sqrt{2}}\begin{pmatrix} \sigma^a & 0 \\ 0 & \sigma^{aT} \end{pmatrix}_{a=1,\dots 3} \;\; Y^a \;=\; \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & y^a \\ -y^{a\dagger} & 0 \end{pmatrix}_{a=4,5}$$

X^a can be seen from the $Sp(4)$ identity $\mathbf{X}^T\mathbf{\Omega} + \Omega\mathbf{X} = 0$. σ^a are the standard Pauli matrices for $a < 4$, and for $a = 4$ this is a unit matrix. The group is then $SU(2)\times U(1)$. For $a = 5,\ \dots,\ 10$ the elements are

$$x^5 = 1,\ x^7 = \sigma^3,\ x^9 = \sigma^1,\ x^{a+1} = ix^a$$

The Y^a elements are seen from for $\mathbf{Y}^T\mathbf{\Omega} - \Omega\mathbf{Y} = 0$ and the elements $y^4 = \sigma^2$, $y^5 = i\sigma^2$.

Now decompose the matrix $\mathbf{Z} = \mathbf{U} + \mathbf{V}$ with

$$U^a = \frac{1}{2\sqrt{2}}e^{-2\sqrt{2}}\begin{pmatrix} 0 & e^{\sigma^a/\sqrt{n}} \\ 0 & 0 \end{pmatrix},\ V^a = \frac{1}{2\sqrt{2}}e^{-2\sqrt{2}}\begin{pmatrix} 0 & 0 \\ -e^{-\sigma^{aT}/\sqrt{n}} & 0 \end{pmatrix}$$

with the result that the multiplication of the two matrices is

$$U^aV^b = \frac{1}{8}e^{-2\sqrt{2}}\begin{pmatrix} -e^{\sigma^a/\sqrt{n}}e^{-\sigma^{bT}/\sqrt{n}} & 0 \\ 0 & 0 \end{pmatrix},\ V^bU^a = \frac{1}{8}e^{-2\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 0 & -e^{-\sigma^{bT}/\sqrt{n}}e^{\sigma^a/\sqrt{n}} \end{pmatrix}$$

The product $e^{\sigma^a/\sqrt{n}}e^{-\sigma^{bT}/\sqrt{n}} \simeq e^{(\sigma^a - \sigma^{bT})/\sqrt{n}}e^{[\sigma^a,\ \sigma^{bT}]/2n}$ and the commutator is

$$U^aV^b - V^bU^a = \frac{1}{8}e^{-4\sqrt{2}}e^{(\sigma^a - \sigma^{bT})/\sqrt{n}}\begin{pmatrix} -e^{[\sigma^a,\ \sigma^{bT}]/2n} & 0 \\ 0 & e^{-[\sigma^a,\ \sigma^{bT}]/2n} \end{pmatrix}$$

The transpose of the Pauli matrix $\sigma^{2T} = -\sigma^2$ with the rest remaining the same means the commutator in the matrix is$[\sigma^a,\ \sigma^{bT}] = 2i\epsilon^{abc}\sigma^{cT}$ and we have for $a = +$ and $b = -$ that

$$U^+V^- - V^-U^+ = \frac{1}{8}e^{-4\sqrt{2}}e^{(\sigma^+ - \sigma^{-T})/\sqrt{n}}\begin{pmatrix} -e^{i\sigma^z/n} & 0 \\ 0 & e^{-i\sigma^z/n} \end{pmatrix}$$

or approximately

$$U^+V^-e^{-i\sigma^z/n} - V^-U^+e^{i\sigma^z/n} = \frac{1}{8}e^{-4\sqrt{2}}e^{(\sigma^+ - \sigma^{-T})/\sqrt{n}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

which leads to

$$U^+V^-e^{-i\sigma^z/n} - V^-U^+e^{i\sigma^z/n} = 0$$

If we reset $1/n \rightarrow \pi/n$ and evaluate this matrix on the eigenvector $|+\rangle$ of σ^z we can write this result as $U^+V^- - V^-U^+e^{2\pi i/n} = 0$. This construction of noncommutative geometry is a $\star$-product extension of a symplectic geometry.

2 Clock-shift operators under Lorentz boost and degrees of freedom

This means the manifold is a "fuzzy" space with noncommutative geometry. The operators $U^\pm,\ V^\pm$ are spinor versions of the clock and shift functions on an $n = 2$ dimensional Hilbert space [3]. The structure is then a reduced version of the large N version of the noncommutative coordinates of a D-brane. This theory may be extended into a Witt algebra, or a Virasoro algebra. The Pauli matrices are elements of $SU(2) \sim SO(3)$. The Lorentzian form of this theory is $SU(1,\ 1) \sim SO(2,\ 1)$. The projective $2 + 1$ Lorentz group $PSL(2,\ 1)$ is isomorphic to the $SL(2,\ \mathbb{R})$ defined for the operators $L_1,\ L_{-1}$ and L_0 by

$$[L_0,\ L_{-1}] = L_{-1},\ [L_0,\ L_1] = -L_1,\ [L_1,\ L_{-1}] = 2L_0.$$

where these operators are expanded in modes as according to the Laurent expansion

$$L^n = \oint \frac{dz}{2\pi iz} z^{n+2} T(z),\ T(z) = -\sum_{n=\infty}^{\infty} \frac{L^n}{z^{n+2}}.$$

The $SL(2,\ \mathbb{R})$ algebra may be embedded into a Virasoro algebra [4]

$$[L_m,\ L_n] = (m - n)L^{m+n} + c(m)\delta_{ij}.$$

The anomaly term is $c(m) = D(m^3 - m)/12$, for $D = 26$. The states of the system are then given by $L_m = \sum_n \alpha_{m-n}\alpha_m$, and if the group is restricted to $SL(2,\ \mathbb{R})$ the mode operators which form this algebra sum accordingly. The L_0 portion of the $SL(2,\ \mathbb{R})$ is the operator $L_0 = \sum_n \alpha_{-n}\alpha_n$ which is the Hamiltonian for the bosonic string.

We have extended this construction to a larger Hilbert space, where if $c(n) = 0$ for all n this is the Witt algebra for $L_n = -z^{n+1}\partial/\partial z$. For the Witt algebra over a finite field the largest L_N value would correspond to the upper frequency limit on the Hilbert space of N dimensions. the finite Witt algebra is some imposed by a time resolution in the observation of the D-brane. The D-brane is composed of cells of minimal uncertainty with $[p,\ x] = \hbar$. The uncertainty in the momentum is given by a resolution time $\delta t = \epsilon$. We may then remove any energy-momentum greater than $1/\epsilon$. On the infinite momentum frame the energy is $E = (p_\perp^2 + m^2)/2P$, for P the longitudinal momentum. This conversely means the longitudinal momentum must be $P < m^2\epsilon$. A D-brane with N degrees of freedom is then determined by the longitudinal boost of that brane relative to another brane. A degrees of freedom on the brane are increased by Lorentz boosting the brane, such as doubling the momentum means $P < 2m^2\epsilon$, and the number of energy states on the brane has increased. The boost in the brane increases the resolution time by the dilation of time so a new set of degrees of freedom appear in the energy region $m^2\epsilon < P < 2m^2\epsilon$. The Witt algebra over a finite field is then extended. The Witt algebra over a field $k[z]$ of characteristic $N > 0$, is the Lie algebra of derivations of the ring $k[z]/z^{N+2}$ The Witt algebra is spanned by L_n for $-1 \leq n \leq N$. The boost $P \rightarrow 2P$ redefines the ring to $k[z]/z^{2N+2}$, and the increase in the number of modes or degrees of freedom is a manifestation of the Lorentz boost factor. This is a form of generating Feynman's wee-partons [5].

A QCD-like string with the field A^a_μ interacting on branes with a QCD-like chromocharge and open ends as fermion fields ψ, or quarks on these branes, is broken as the two branes separate. The separating endpoints are connected to a nascent brane with a few degrees of freedom. This nascent brane is a S^3 corresponding to a FLRW metric, which expands to its turn around or maximum expansion point. The violation of the Bekenstein bound at the turn around point forces this surface to becomes $\mathbb{R}^3$. Equivalently the S^3 becomes enormously Lorentz boosted relative to the end points and their Dirichlet boundary conditions on the brane and on the infinite momentum frame appears as a stretched horizon. The metric on this surface is an anti de Sitter spacetime. The transverse modes of the string become enormously Lorentz boosted relative to the nascent brane, which is a form of stretched horizon as measured by an observer near either original endpoints of the string. The transverse modes of the string increase and the string covers the nascent brane increasing the number of modes observed on it. The appearance of the stretched horizon covered by the string means each region of the surface with a Planck unit of area $G\hbar/c^3$ contain a mode in the limit $N \rightarrow \infty$. The two operators U and V are then elements of an enveloping algebra of complementary observables with a minimal uncertainty $\hbar$.

With the construction with Pauli matrices we have the $SU(2)$ commutation relationship for angular momentum $[L_i,\ L_j] = i\epsilon\hbar L_k$. Now choose a coordinate system on the sphere with L_z through the origin. Then $L_z \simeq\tilde{}$ constant and we may write $L_x = px$ and $L_y = py$ so that $[x,\ y] = i\hbar\theta$. The same applies for the momentum space. The conversion to this construction with σ^{2T} then maps this sphere into the hyperbolic coordinates considered. This gives a meaning to the noncommutativity of the coordinates of the manifold.

$$e^{i\nabla_i}e^{i\nabla_j} = e^{i\nabla_i + i\nabla_j + \frac{1}{2}R_{ijkl}y^iy^k}.$$

The Riemann curvature pertains to the AdS Riemann curvature tensor components. In addition the curvature here is in $O(\hbar/N)$ and is then a quantized effect. The world volume swept out by the D3-brane is defined by the AdS_4 curvatures in $t,\ \chi, \theta,\ \phi$ coordinates

$$R_{t\chi t\chi} = cos^2(t),\ R_{t\theta t\theta} = cos^2(t)sinh^2(\chi),\ R_{t\phi t\phi} = cos^2(t)sinh^2(\chi)sin^2(\theta)$$

$$R_{\chi\theta\chi\theta} = -cos^2(t)sinh^2(\chi) + cos(t)sin(t)sinh^2(\chi)$$

$$R_{\chi\phi\chi\phi} = -cos^2(t)sinh^2(\chi)sin^2(\theta) + cos^2(t)sin^2(t)sinh^2(\chi)sin^2(\theta)$$

$$R_{\theta\phi\theta\phi} = cos^2(t)sinh^2(\chi)sin^2(\theta) + cos^2(t)sin^2(t)sinh^4(\chi)sin^2(\theta) - cos^2(t)sinh^2(\chi)sin^2(\theta)cosh^2(\chi)$$

The last three of these curvatures are the curvature of the spacetime on the D3-brane, while the first three above are the curvature of the D3-brane in the world volume it sweeps through.

In this setting the U and V operators with a commutation given by $UV = e^{2\pi/N}VU$ and the deviation from commutation is determined by this curvature in units of $\hbar/N$. Equivalently we may think of the variables $\chi \rightarrow \chi/\sqrt{N}$, where in the limit $N \rightarrow \infty$ the curvatures approach zero. The number of degrees of freedom in the system is "large N," not infinite. Consequently the anti de Sitter spacetime on the D-brane "matures" into a state with curvature present only over considerable distances on the brane. A realization of clock and shift operators is a noncommutative geometry on the brane, and large N corresponds to a high boost of the brane and a classical limit.

3 Is the observable cosmology anti de Sitter, or the boundary of AdS_n?

Hartle, Hawking and Hertog [5] have suggested the observable universe may indeed by anti de Sitter. With the Wheeler DeWitt equation they derive an expanding wave function in an AdS spacetime with the energy constraint

$$\left(\frac{a'}{N}\right)^2 - 1 - \frac{a^2}{\ell^2} = 0,$$

where the sign change is such that $\ell^{-2} = -\Lambda/3$ for the AdS. The negative cosmological constant makes the relationship between quantum physics and gravity far easier to understand as the AdS/CFT correspondence and holography. The quantum wave functional of the Wheeler-DeWitt equation expands with the scale factor a, which it is argued would correspond to an expanding universe.

The AdS_4 metric

$$ds^2 = -dt^2 + cos^2(t)d\chi^2 + cos^2(t)sinh^2(\chi)d\theta^2 + cos^2(t)sinh^2(\chi)sin^2(\theta)d\phi^2$$

for $d\phi = 0$ and $cos(t) = cos(at)$ with $a = 0$ reduces to a three dimensional space with the metric $ds^2 = -dt^2 + d\chi^2 + sinh^2(\chi)d\theta^2$. We set the T and X coordinates so that $\chi = tanh^{-1}(T/X)$ and $T = x\ sinh(\chi)$, $X = x\ cosh(\chi)$, for $x = \sqrt{X^2 - T^2}$. This gives the Poincaré half-plane on a times slice with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2},\ y > 0$$

It is easily shown that $\Gamma^x_{xy} = \Gamma^y_{yy} = -1/y$, and $\Gamma^y_{xx} = 1/y$. The nonzero curvature components in these coordinates are

$$R^x_{xyx} = -\frac{1}{y^2},\ R^x_{xxx} = R^y_{xyx} = -\frac{1}{y^2}$$

With the Ricci curvatures $R_{11} = R_{22} = -1/y^2$, Ricci scalar curvature $R = -2$ and Gaussian curvature $K = -1$. This by way of elementary illustration indicates the curvature is negative throughout the space. The curvature approaches zero as $y \rightarrow \infty$and diverges as $y \rightarrow 0$.

The geodesics of the Poincaré half plane are circles which perpendicularly intersects at $y = 0$ metric. The line element is then $ds \simeq dy/y$ with a small increment $s \simeq ln(y)$. This diverges at zero, where near this point we write the logarithm for $y = 1 - x$ so $ln(1 - x) = -\sum_n x^n/n$. For $x = \epsilon - 1$, so that $x^n \simeq 1 + n\epsilon$ the Taylor series is

$$ln(1 - x) \simeq -\sum_n \frac{1 + n\epsilon}{n} = -\sum_n \left(\frac{1}{n} + \epsilon\right)$$

This constructs the discrete form of the logarithmic divergence as $\epsilon \to 0$. Now substitute $n \to ne^{(n-1)\epsilon}$ with the implied limit $\epsilon \to 0$ so that

$$s \simeq ln(y) = -\sum_n \frac{e^{-(n-1)\epsilon}}{n} = -\int \sum_n e^{-(n-1)\epsilon} d\epsilon.$$

This is a geometric series,

$$\sum_m e^{-m\epsilon} = e^{-\epsilon}(1 + e^{-\epsilon} + e^{-2\epsilon} + \dots) = \frac{e^{-\epsilon}}{1 - e^{-\epsilon}}$$

The integral of this is $ln(1 - e^{-\epsilon}) - \epsilon$ or with the logarithm Taylor series $\simeq -2\epsilon$. Hence the line element is regular.

We now turn our attention to curvature. The line element in a small neighborhood of size ϵ, a variation of the line element is

$$s = s_0 + \epsilon\frac{ds}{d\epsilon} + \epsilon^2 \frac{1}{2}\frac{d^2 s}{d\epsilon^2} + \dots$$

It is clear that the second order term contains curvature information from

$$\frac{d^2 s}{d\epsilon^2} = \nabla_i \nabla_j g_{kl} \frac{dx^i}{d\epsilon}\frac{dx^j}{d\epsilon} dx^k dx^l = \frac{1}{2}[\nabla_i, \nabla_j] g_{kl} \frac{dx^i}{d\epsilon}\frac{dx^j}{d\epsilon} dx^k dx^l$$

where the commutator is the components of a curvature two-form. The essential information to describe the behavior of curvature near the boundary is the second derivative of $ln(y)$ or equivalently

$$\frac{d^2 s}{d\epsilon^2} = -\frac{d}{d\epsilon}\sum_n e^{-(n-1)\epsilon} = -\frac{d}{d\epsilon}\frac{e^{-\epsilon}}{1 - e^{-\epsilon}}$$

The term differentiated is expanded in a Taylor series for small ϵ so $e^{-\epsilon} = 1 - \epsilon + \epsilon^2/2$ and this summation, with use of the binomial theorem and eliminating powers $O(\epsilon^3)$ and higher, is

$$\sum_m e^{-m\epsilon} = -\frac{1}{\epsilon}(1 - \epsilon/2 - \epsilon^2/12).$$

Now take the derivative with elementary calculus rules change the sign as above and we have

$$\frac{d^2 s}{d\epsilon^2} \;=\; -\frac{1}{\epsilon^2} \;+\; \frac{1}{12}.$$

As $\epsilon \;\rightarrow\; 0$ the first term blows up. This UV divergence can be absorbed into the definition of the momentum and regularized away. The remaining term is the value of the curvature on the boundary, which is finite and positive.

The physical interpretation is the quantum vacuum energy. The vacuum energy of quantum fields in spacetime is $E \;=\; D\omega_0 \sum_n n/2$, in a box normalization, and with D the dimension of the harmonic oscillators. The sum $1 \;+\; 2 \;+\; 3 \;+\; \ldots + n \;+\; \ldots$ is equal to

$$\sum_n n \;=\; \frac{d}{d\epsilon}\frac{e^{-\epsilon}}{1 \;-\; e^{-\epsilon}}$$

which means the vacuum energy near the boundary is equal to

$$E \;=\; D\omega_0\left(\frac{1}{2\epsilon^2} \;-\; \frac{1}{24}\right)$$

The regularized vacuum energy requires that $D \;=\; 24$. This is the spatial dimension of the oscillator, which is in a light cone (light front) frame, and so the number of spatial dimension is 25 and the spacetime dimension is 26. The vacuum energy contributes a curvature term that is $\Lambda \;\sim\; \omega_0$, which is positive. The sign is changed by the negative curvature of the AdS_4, which in the evaluation of the vacuum state on ∂AdS_4 changes the sign so $\Lambda \;\propto\; -12\omega_0 \sum_n n$.

The boundary of the anti-de Sitter spacetime is more general than this elementary case with the Poincaré disk. The boundary of the AdS_5, $\partial AdS_5 \;=\; E^4_c$, is a conformally flat spacetime [7]. In line with the even dimensional construction we may for instance consider AdS_6, $\partial AdS_6 \;=\; E^5_c$,with a compactification of one dimension of E^5_c into $E^4_c \times S^1$. The conformal transformation $g_{\mu\nu} \;\rightarrow\; \Omega^2 g_{\mu\nu}$ the flat spacetime element $ds^2 \;=\; du^2 \;-\; \sum_i dx^i dx^i$ is a time dependent transformation for $du/dt \;=\; \Omega^{-1}$ and a de Sitter spacetime for $\Omega^2 \;=\; e^{\sqrt{\Lambda/3}\; t}$. This is approximately the spacetime for our physical universe. It is then argued that the negative cosmological constant in the AdS_5 spacetime may manifest itself as a positive cosmological constant on the boundary.

4 Geometric quantization of brane-world?

The AdS_{n+1} group of isometries $O(n, 2)$ contains a Möbius subgroup, or modular transformations, so this discrete group does not necessarily act effectively on AdS_{n+1}. This means that the discrete group Γ is not necessarily convergent on the boundary space M_n. Such a convergence means there exists a sequence $g_i \;\in\; \Gamma$ which admits a north-south dynamics of poles $p^{\pm}$ on a sphere, which in the hyperbolic case defines the past and future portions of a light cone [8]. The limit set of a discrete group is a closed Γ-invariant subset that defines a

$\Lambda_\Gamma \subset M_n$ so the complement Ω_Γ acts properly on M_n. This Γ-invariant closed subset of $\Lambda_\Gamma \subset L_n$ is the space of lightlike geodesic in M_n. The action of Γ on $\Omega_\Gamma \cup AdS_{n+1}$ is contained in M_n. The open set Λ_Γ is the maximal set that the Γ acts properly on $\Omega_\Gamma \cup AdS_{n+1}$. The other is the discrete group Γ is Zariski dense in $O(n,\ 2)$.

The lightlike geodesics in M_n are copies of $\mathbb{R}P^1$, which at a given point p define a set that is the light cone $C(p)$ [8]. The point p is the projective action of $\pi(v)$ for v a vector in a local patch $\mathbb{R}^{n,2}$ and so $C(p)$ is then $\pi(P \cap C^{n,2})$, for P normal to v, and $C^{n,2}$ the region on $\mathbb{R}^{n,2}$ where the interval vanishes. The space of lightlike geodesics is a set of invariants and then due to a stabilizer on $O(n, 2)$, so the space of lightlike curves L_n is identified with the quotient $O(n, 2)/P$, where P is a subgroup defined the quotient between a subgroup with a Zariski topology, or a Borel subgroup, and the main group $G = O(n,\ 2)$. This quotient G/P is a projective algebraic variety, or flag manifold and P is a parabolic subgroup. The natural embedding of a group $H \rightarrow G$ composed with the projective variety $G \rightarrow G/P$ is an isomorphism between the H and G/P. This is then a semi-direct product $G = P \rtimes H$. For the G any $GL(n)$ the parabolic group is a subgroup of upper triangular matrices, called Borel groups [9].

The connection between the symplectic group symplectic group $Sp(2N_f) \subset SU(2N_f) \sim SO(2,\ 2)$ and the parabolic group of upper triangular matrices is a geometric quantization [10]. The symplectic manifold $(M,\ \Omega)$, where Ω is the skew symmetric matrix defines a prequantization as a representation of elements of the Poisson algebra $f \in C^\infty(M)$ as sections of a Kahler line bundle L, with $\pi : L \rightarrow M$. The prequantization line bundle contains the one form $\omega = df + 2\pi i\alpha$, for α on the line bundle, such that the curvature $R = D \wedge D$, for $D = d + \omega$ under $\pi : R = i\Omega$. Let $T = T(M)$ be the tangent bundle to M for elements $u,\ v \in T$. The Poisson bracket $\{u,\ v\} \in \Gamma(T)$ exists on sections of T, and the quantum algebra Q_M of M is the operators formed from functions f such that their Hamiltonian vector fields $x_f\ x^a = \Omega^{ab}\partial_b H$, and $[x_f,\ T] \subset T$. Q_M forms a pre-Hilbert space of half-forms, tensor density fields with weight $s = 1/2$, with $f_{op} = f + i\hbar^{1/2}\mathcal{L}_T x_f$, with $\alpha = i\hbar^{1/2}\mathcal{L}_T x_f$. This is a form of the $\star$-product that is an extension of the function on a symplectic manifold into a quantum algebra.

This gives two routes to quantization. The first is with a geometric quantization approach with the enveloping algebra on U, V and $Sp(2N_f)$ for a D-brane, the other is with the conformal completion of the AdS_n. In the latter case the parabolic subgroup of Borel groups or Heisenberg groups. The parabolic group defines light cones, which are an invariant of spacetime. The invariance of spacetime is proper time, where in this construction the proper time is zero. In the geometric quantization approach the coordinates of phase space are employed, and Hamiltonian vector fields $u^a = d\gamma^a/dt$ are defined according to a coordinate definition of time. Quantum fields in spacetime are defined according to local operators that commute on a spatial surface of simultaneity. Hence QFT is defined according to coordinate time. In one case the quantization $\star$-product is constructed according to light cones, or proper intervals, which is more commensurate with the structure of general relativity. In the braney approach the quantization is tied to coordinate geometry and is in line with quantum field theory. These rely entirely on different definitions of time. The open question is then how are these related, whether they ultimately give identical results, or whether these two schemes are aspects of a more general quantum gravity scheme.

5 Concluding statements

The braney dynamics with strings is a form of QCD, where the endpoints of a string are "quarks" with a color identified with the brane it is anchored to. The QCD dynamics of the brane with $SU(2,\ 2)$ symmetry, or $SO(4,\ 2)$, is governed by the noncommutative geometry of the $\star$-product. The development of a brane from the bifurcation of a string is a form of infinite momentum boost which increases the number of degrees of freedom on the brane. As the number of modes increases the brane becomes a classical-like object. The $SU(2,\ 2)\ \sim\ SO(4,\ 2)$ is the isometry group for the AdS_4, and acts as a QCD-like gauge field. The decomposition of $SU(2,\ 2)$ into $Sp(4)\ \sim\ Sp(2,\ 2)$ form the symplectic basis for the noncommutative geometry of the brane, or AdS-brane.

The observable universe is likely connected to AdS spacetime. Hartle, Hawking and Hertog argue the physical universe may indeed by anti-de Sitter. As with the Poincaré half plane, or the Poincaré disk, the geodesics are great arcs which leave the boundary with enormous curvature and high energy, traverse the space and return to the boundary. An observer in an anti-de Sitter spacetime would observe distant objects to be highly blue shifted. Any object observed at a great distance would emit radiation which is blue shifted towards the observer. It is for this reason the anti-de Sitter spacetime was considered in quantum gravity, for this property makes it the perfect box to hold a black hole. The boundary has a repellant gravitational influence. The argument is made for why the observable universe is a conformally flat spacetime on the boundary of the AdS_nspacetime.

The noncommutative geometry of the brane, or geometric quantization, then shares some relationship with the conformal completion of the AdS_n spacetime and the Borel group upper triangular matrix form of the Heisenberg group. The two approaches then share some relationship which is as yet not clear. It could be the two forms of quantization are not equivalent and then must embed in some more general form of quantum gravity.

6 References

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