Function of a Matrix

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Abstract. Let a be a square matrix with complex entries and f a function holomorphic on an open subset U of the complex plane. It is well known that f can be evaluated on a if the spectrum of a is contained in U. We show that, for a fixed f, the resulting matrix depends holomorphically on a.

The following was explained to me by Jean-Pierre Ferrier.

For any matrix a in $A := M_n(\mathbb{C})$, write $\Lambda(a)$ for the set of eigenvalues of a. Let U be an open subset of $\mathbb{C}$, and let U' be the subset of A, which is open by Rouché's Theorem, defined by the condition $\Lambda(a) \subset U$. Let a be in U', let X be an indeterminate, and let $\mathcal{O}(U)$ be the $\mathbb{C}$-algebra of holomorphic functions on U. Equip $\mathcal{O}(U)$ and $\mathbb{C}[a]$ with the $\mathbb{C}[X]$-algebra structures associated respectively with the element $z \mapsto z$ of $\mathcal{O}(U)$ and the element a of $\mathbb{C}[a]$.

Theorem. (i) *There is a unique $\mathbb{C}[X]$-algebra morphism from $\mathcal{O}(U)$ to $\mathbb{C}[a]$. We denote this morphism by $f \mapsto f(a)$.*

(ii) *There is an $r > 0$ and a neighborhood N of a in A such that*

$$f(b) = \frac{1}{2\pi i} \sum_{\lambda \in \Lambda(a)} \int_{|z-\lambda|=r} \frac{f(z)}{z-b}\, dz$$

for all f in $\mathcal{O}(U)$ and all b in N. In particular the map $b \mapsto f(b)$ from U' to A is holomorphic.

Proof. By the Chinese Remainder Theorem, $\mathbb{C}[a]$ is isomorphic to the product of $\mathbb{C}[X]$-algebras of the form $\mathbb{C}[X]/(X-\lambda)^m$, with $\lambda \in \mathbb{C}$. So we can assume that $\mathbb{C}[a]$ is of this form, and (i) is clear. To prove (ii) we can keep on assuming $\mathbb{C}[a] \simeq \mathbb{C}[X]/(X-\lambda)^m$. On replacing a with $a - \lambda$, we can even assume $a^n = 0$. Choose $r > 0$ so that U contains the closed disk of radius r centered at 0, let N be the set of those b in A whose eigenvalues λ satisfy $|\lambda| < r/2$, and let b be in N. Replacing a with b in the above argument, we can assume $b^n = 0$. Now (ii) follows from Cauchy's Integral Formula and the equalities

$$f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}\, b^k, \quad \frac{1}{z-b} = \sum_{k=0}^{n-1} \frac{b^k}{z^{k+1}} \quad .$$

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