Function of a Matrix

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Abstract. Let a be a square matrix with complex entries and f a function holomorphic on an open subset U of the complex plane. It is well known that f can be evaluated on a if the spectrum of a is contained in U. We show that, for a fixed f, the resulting matrix depends holomorphically on a.

The following was explained to me by Jean-Pierre Ferrier.

For any matrix a in $A := M_n(\mathbb{C})$, write $\Lambda(a)$ for the set of eigenvalues of a, and $\mathcal{O}(\Lambda(a))$ for the algebra of those functions which are holomorphic on [some open neighborhood of] $\Lambda(a)$.

Let U be an open subset of $\mathbb{C}$, and let U' be the subset of A defined by the condition $\Lambda(a) \subset U$. In view of the Rouché's Theorem U' is open. Let a be in A and X be an indeterminate.

Theorem. (i) *There is a unique $\mathbb{C}[X]$-algebra morphism from $\mathcal{O}(\Lambda(a))$ to $\mathbb{C}[a]$. We denote this morphism by $f \mapsto f(a)$.*

(ii) *There is an $r > 0$ and a neighborhood N of a in A such that*

$$f(b) = \frac{1}{2\pi i} \sum_{\lambda \in \Lambda(a)} \int_{|z-\lambda|=r} \frac{f(z)}{z-b}\, dz$$

for all f in $\mathcal{O}(U)$ and all b in N. In particular the map $b \mapsto f(b)$ from U' to A is holomorphic.

Proof. By the Chinese Remainder Theorem, $\mathbb{C}[a]$ is isomorphic to the product of $\mathbb{C}[X]$-algebras of the form $\mathbb{C}[X]/(X-\lambda)^m$, with $\lambda \in \mathbb{C}$. So we can assume that $\mathbb{C}[a]$ is of this form, and (i) is clear. In view of the above argument, we can assume that a is nilpotent. Then (ii) results from the Lemma and the following form of Cauchy's Integral Formula:

$$\frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{k+1}}\, dz,$$

where f is holomorphic around 0 and r is a small enough positive number.

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