Function of a Matrix

Pierre-Yves Gaillard

**Abstract.** Let \( a \) be a square matrix with complex entries and \( f \) a function holomorphic on an open subset \( U \) of the complex plane. It is well known that \( f \) can be evaluated on \( a \) if the spectrum of \( a \) is contained in \( U \). We show that, for a fixed \( f \), the resulting matrix depends holomorphically on \( a \).

The following was explained to me by Jean-Pierre Ferrier.

For any matrix \( a \) in \( A := M_n(\mathbb{C}) \), write \( \Lambda(a) \) for the set of eigenvalues of \( a \), and \( \mathcal{O}(\Lambda(a)) \) for the algebra of those functions which are holomorphic on [some open neighborhood of] \( \Lambda(a) \).

Let \( U \) be an open subset of \( \mathbb{C} \), and let \( U' \) be the subset of \( A \) defined by the condition \( \Lambda(a) \subset U \). In view of the Rouché’s Theorem \( U' \) is open. Let \( a \) be in \( A \) and \( X \) be an indeterminate.

**Theorem.**

(i) There is a unique \( \mathbb{C}[X] \)-algebra morphism from \( \mathcal{O}(\Lambda(a)) \) to \( \mathbb{C}[a] \). We denote this morphism by \( f \mapsto f(a) \).

(ii) There is an \( r > 0 \) and a neighborhood \( N \) of \( a \) in \( A \) such that

\[
 f(b) = \frac{1}{2\pi i} \sum_{\lambda \in \Lambda(a)} \int_{|z-\lambda|=r} \frac{f(z)}{z-b} \, dz
\]

for all \( f \) in \( \mathcal{O}(U) \) and all \( b \) in \( N \). In particular the map \( b \mapsto f(b) \) from \( U' \) to \( A \) is holomorphic.

**Proof.** By the Chinese Remainder Theorem, \( \mathbb{C}[a] \) is isomorphic to the product of \( \mathbb{C}[X] \)-algebras of the form \( \mathbb{C}[X]/(X-\lambda)^m \), with \( \lambda \in \mathbb{C} \). So we can assume that \( \mathbb{C}[a] \) is of this form, and (i) is clear. In view of the above argument, we can assume that \( a \) is nilpotent. Then (ii) results from the Lemma and the following form of Cauchy’s Integral Formula:

\[
 \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \int_{|z|=r} f(z) \frac{1}{z^{k+1}} \, dz,
\]

where \( f \) is holomorphic around 0 and \( r \) is a small enough positive number.