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Abstract. The book Categories and Sheaves by Kashiwara and Schapira starts with a few statements which are not proved, a reference being given instead. We spell out the proofs in a short and self-contained way.

Feeling the proofs are easier to read if they are printed on a single page, I left this page almost blank. Please, go to the next page.

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A universe is a set \( \mathcal{U} \) satisfying

(i) \( \emptyset \in \mathcal{U} \),
(ii) \( u \in U \in \mathcal{U} \implies u \in \mathcal{U} \),
(iii) \( U \in \mathcal{U} \implies \{U\} \in \mathcal{U} \),
(iv) \( U \in \mathcal{U} \implies \mathcal{P}(U) \in \mathcal{U} \),
(v) \( I \in \mathcal{U} \) and \( U_i \in \mathcal{U} \) for all \( i \implies \bigcup_{i \in I} U_i \in \mathcal{U} \),
(vi) \( \mathbb{N} \in \mathcal{U} \).

We want to prove:

(vii) \( U \in \mathcal{U} \implies \bigcup_{u \in U} u \in \mathcal{U} \),
(viii) \( U, V \in \mathcal{U} \implies U \times V \in \mathcal{U} \),
(ix) \( U \subset V \in \mathcal{U} \implies U \in \mathcal{U} \),
(x) \( I \in \mathcal{U} \) and \( U_i \in \mathcal{U} \) for all \( i \implies \prod_{i \in I} U_i \in \mathcal{U} \).

[We have kept Kashiwara and Schapira’s numbering of Conditions (i) to (x).]

Obviously, (ii) and (v) imply (vii), whereas (iv) and (ii) imply (ix). Axioms (iii), (vi), and (v) imply

(a) \( U, V \in \mathcal{U} \implies \{U, V\} \in \mathcal{U} \),

and thus

(b) \( U, V \in \mathcal{U} \implies (U, V) := \{\{U\}, \{U, V\}\} \in \mathcal{U} \).

**Proof of (viii).** If \( u \in U \) and \( v \in V \), then \( \{(u, v)\} \in \mathcal{U} \) by (ii), (b), and (iii). Now (v) yields

\[
U \times V = \bigcup_{u \in U} \bigcup_{v \in V} \{(u, v)\} \in \mathcal{U}. \quad \Box
\]

Assume \( U, V \in \mathcal{U} \), and let \( V^U \) be the set of all maps from \( U \) to \( V \). As \( V^U \in \mathcal{P}(U \times V) \), Statements (viii), (iv), and (ii) give

(c) \( U, V \in \mathcal{U} \implies V^U \in \mathcal{U} \).

**Proof of (x).** As

\[
\prod_{i \in I} U_i \in \mathcal{P} \left( \left( \bigcup_{i \in I} U_i \right)^I \right),
\]

(x) follows from (v), (c), and (iv). \( \Box \)