Stress-energy tensor beyond the Belinfante and Rosenfeld formula

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Abstract

The physical importance of the stress-energy tensor is twofold: at the one hand, it is a fundamental quantity appearing on the equations of mechanics; at the other hand, this tensor is the source of the gravitational field.

Due to this importance, two different procedures have been developed to find this tensor for a given physical system. The first of the systematic procedures gives the canonical tensor, but this tensor is not usually symmetric and it is repaired, via the Belifante and Rosenfeld formula, to give the Hilbert tensor associated to the second procedure.

After showing the physical deficiencies of the canonical and Hilbert tensors, we introduce a new and generalized tensor $\Theta^{\mu\nu}$ without such deficiencies. This $\Theta^{\mu\nu}$ is (**i**) symmetric, (**ii**) conserved, (**iii**) in agreement with the energy and momentum of a system of charges interacting via NILI potentials $\Lambda^{\mu}(R(t))$, and (**iv**) properly generalizes the Belifante and Rosenfeld formula, with the Hilbert tensor being a special case of $\Theta^{\mu\nu}$.

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1 Introduction

The physical importance of the stress-energy tensor, thereafter ECMST [1], is twofold: at the one hand, the ECMST is a fundamental quantity appearing on the equations of mechanics [2]; at the other hand, the ECMST is the source of the gravitational field [3, 4].

Due to this importance, different procedures have been developed to find the ECMST for a given physical system. The first of the systematic procedures emphasizes the role of the ECMST in the equations of motion and obtains a ECMST from the analysis –via Noether's theorem– of the conserved currents associated with the spacetime translations of classical field theories on Minkowski spacetime. This is the canonical ECMST, whose density [1] is

$$\tau^{\mu\nu} \equiv \mathcal{L}\delta^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial\left(\dfrac{\partial \phi^{\zeta}}{\partial x^{\mu}}\right)} \frac{\partial \phi^{\zeta}}{\partial x_{\nu}}, \tag{1}$$

with $\mathcal{L}$ the field Lagrangian density and ϕ^{ζ} the field components. Greek indices run over values 0, 1, 2, 3.

In general, this canonical ECMST density is not symmetric, $\tau^{\mu\nu} \neq \tau^{\nu\mu}$, and cannot be used as source of the gravitational field. This deficiency is particularly relevant for the field theory of gravity, because not only matter but the own gravitons contribute to the total gravitational field through a nonlinear coupling [3, 4].

A second systematic procedure gives a ECMST density by varying the spacetime metric $g_{\mu\nu}$ in the relativistic action. This is the Hilbert ECMST density [1]

$$t^{\mu\nu} \equiv -2\frac{\delta \mathcal{L}}{\delta g_{\mu\nu}}. \tag{2}$$

By definition, the Hilbert ECMST density is symmetric, $t^{\mu\nu} = t^{\nu\mu}$, and the proper source in the Hilbert and Einstein equations $G^{\mu\nu} = (8\pi G/c^4)t^{\mu\nu}$. Nevertheless, $t^{\mu\nu}$ does not provide the ECMST density of the own gravitational field [3, 4].

The Belifante and Rosenfeld formula [5–8] connects both tensor densities

$$t^{\mu\nu} = \tau^{\mu\nu} + D_{\rho}K^{\rho\mu\nu}, \tag{3}$$

for some quantities K due to classical spin and with D_{ρ} denoting either an ordinary ∂_{ρ} or a covariant partial derivative ∇_{ρ}, depending on the version of the formula. The Belifante and Rosenfeld formula is presented in the literature as a systematic «*repair*» [8] of the canonical ECMST density. Recently, Gotay & Marsden have obtained a «*generalized Belinfante-Rosenfeld formula*» [8] valid for arbitrary field theories.

In the next section, we will show additional physical deficiencies of the canonical and Hilbert tensors. A new and generalized ECMST $\Theta^{\mu\nu}$ is defined and analyzed in the section 3. Finally we show, in the section 4, how the Hilbert $T^{\mu\nu}$ must be considered a special case of the new $\Theta^{\mu\nu}$.

2 Further limitations of the existent ECMSTs

Apart from the problems outlined above, the two available procedures to find the ECMST fail for a broader kind of physical systems. Consider the action proposed recently by CHUBYKALO & SMIRNOV-RUEDA in their improvement of the classical field theory of electromagnetism [9,10].

The interaction part of their new action contains NILI potentials $\Lambda^{\mu} = \Lambda^{\mu}(R(t))$, which are «*irreducible*» [9,10] to the field-theoretic $A^{\mu} = A^{\mu}(r,t)$. It is easy to check that the canonical procedure fails to find the ECMST for this system, because in the canonical approach [11]:

> «*the energy-momentum tensor of the whole system is the sum of the energy-momentum tensors for the electromagnetic field and for the particles, where in the latter the particles are assumed to not interact with one another.*»

As a consequence of the assumption of non-interacting particles, the canonical ECMST density $\tau^{\mu\nu} = \tau_{\mathrm{m}}^{\mu\nu} + \tau_{\mathrm{f}}^{\mu\nu}$ for their action contains a matter and a field term, but is lacking an interaction component density $\tau_{\mathrm{int}}^{\mu\nu}$ that depends of the NILI potentials Λ^{μ}. The canonical ECMST for this system is not conserved; moreover, the canonical ECMST is in disagreement with the equations of motion, because the generalized forces are function of the NILI potentials [9,10].

The ordinary Hilbert procedure fails to find the ECMST for this system for essentially the same reason quoted above. In the ordinary Hilbert procedure, the variation of the metric $\delta g_{\mu\nu}$ is not considered in the interaction part of the whole action with ordinary field-theoretic potentials A^{μ}. The trick consists on absorbing the metric into the covariant potentials $A_{\nu} = g_{\mu\nu} A^{\mu}$ and then taking the A_{ν} as constants [12]. *This is equivalent to the canonical assumption of non-interacting particles*. The Hilbert ECMST density $t^{\mu\nu} = t_{\mathrm{m}}^{\mu\nu} + t_{\mathrm{f}}^{\mu\nu}$ for their action is also lacking an interaction component density $t_{\mathrm{int}}^{\mu\nu}$ that depends of the NILI potentials [9,10]. The Hilbert ECMST for this system also disagrees with the conservation laws and with the equations of motion.

It is evident that we need a new definition of the ECMST working for this broader kind of physical systems. This definition is given in the following section.

3 Definition of a new and generalized ECMST

We start with the Hamiltonian formulation of mechanics for a system of $\mathcal{N}$ particles. Using the Hamilton and Jacobi equation in classical mechanics [13] or the Schrödinger equation in quantum mechanics [14], we can derive the action

$$S = \int \left(\sum_{k} p_{[k]} v_{[k]} - H \right) dt, \tag{4}$$

which gives the classical or quantum paths generated by the Hamilton equations or by the Schrödinger equation, respectively.

As WEINBERG emphasizes, the integrand on the quantum action is not the Lagrangian, as many believe, because the momenta $p_{[k]}$ in (4) «*are independent variables*» [14]. His remark is also valid when (4) is the classical action. Effectively, the velocities in (4) are dependent variables, $v_{[k]} = v_{[k]}(p_{[k]}, q_{[k]}) = \partial H / \partial p_{[k]}$, for a given classical Hamiltonian $H(p_{[k]}, q_{[k]})$.

The integrand on the action (4) can be rewritten using

$$H - \sum_k p_{[k]} v_{[k]} = \eta_{\mu\nu} \sum_k p^\mu_{[k]} v^\nu_{[k]} = \eta_{\mu\nu} \sum_k v^\mu_{[k]} p^\nu_{[k]} = \eta_{\mu\nu} \frac{1}{2} \sum_k \left(p^\mu_{[k]} v^\nu_{[k]} + v^\mu_{[k]} p^\nu_{[k]} \right) \tag{5}$$

and this suggests the following general definition of the ECMST

$$\Theta^{\mu\nu} \equiv \frac{1}{2} \sum_k \left(p^\mu_{[k]} v^\nu_{[k]} + v^\mu_{[k]} p^\nu_{[k]} \right). \tag{6}$$

This tensor is symmetric by definition. It is conserved because both p^α and v^β are conserved quantities. The physical meaning of this tensor follows from the action (4): this new $\Theta^{\mu\nu}$ is the tensor whose trace $\eta_{\mu\nu}\Theta^{\mu\nu}$ gives minus the temporal rate of change of the action $\mathrm{d}S/\mathrm{d}t$.

For many dynamical systems, the functional relation $v_{[k]} = v_{[k]}(p_{[k]}, q_{[k]})$ can be inverted to obtain $p_{[k]} = p_{[k]}(q_{[k]}, v_{[k]})$ in whose case the action (4) reduces to $S = \int L dt$ for a Lagrangian $L = L(q_{[k]}, v_{[k]})$. Consider the following Lagrangian for a system of charges interacting via NILI potentials [9, 10]

$$L^{\mathrm{NILI}} = L^{\mathrm{NILI}}_{\mathrm{m}} + L^{\mathrm{NILI}}_{\mathrm{int}} = -\sum_k m_k c^2 \sqrt{\frac{v^\mu_{[k]} \eta_{\mu\nu} v^\nu_{[k]}}{c^2}} - \frac{e_k}{c} v^\mu_{[k]} \eta_{\mu\nu} \Lambda^\nu_{[k]}. \tag{7}$$

The application of the general definition (6) to the Lagrangian (7) gives the following ECMST

$$\Theta^{\mu\nu} = \sum_k m_k \Gamma_{[k]} v^\mu_{[k]} v^\nu_{[k]} + \frac{1}{2} \frac{e_k}{c} \left(v^\mu_{[k]} A^\nu_{[k]} + A^\mu_{[k]} v^\nu_{[k]} \right), \tag{8}$$

with $\Gamma_{[k]}$ the time-dilation factor. When particles are at rest, the energy component is in complete agreement with the energy of this system of charges in stationary regimes [9, 10].

$$\Theta^{00} = \sum_k m_k c^2 + e_k A^0_{[k]}. \tag{9}$$

As emphasized in the previous section, the ordinary Hilbert procedure does not consider the variation of the metric in the interaction part of the whole action. Effectively, if we rewrite the interacting Lagrangian as $L^{\mathrm{NILI}}_{\mathrm{int}} = -\sum_k (e_k/c) v^\mu_{[k]} \Lambda_{\mu[k]}$ and take the functional derivative of the Lagrangian (7) with respect to the flat metric, $T^{\mu\nu} \equiv -2(\delta L^{\mathrm{NILI}}/\delta \eta_{\mu\nu})$, *whereas the NILI potentials $\Lambda_{\mu[k]}$ are set constant* [12], we will obtain the ordinary Hilbert ECMST

$$T^{\mu\nu} = \sum_k m_k \Gamma_{[k]} v^\mu_{[k]} v^\nu_{[k]}, \tag{10}$$

whose energy component for particles at rest, $T^{00} = \sum_k m_k c^2$, is lacking the charge-charge NILI interactions in (9). For the sake of comparison with the new ECMST (6), it is interesting to consider what happens when we relax the assumption of non-interacting particles in the ordinary Hilbert procedure.

If we take the functional derivative of the whole Lagrangian (7) with respect to the flat metric again, but now allowing the NILI potentials $\Lambda_{\mu[k]}$ to vary, we obtain the tensor

$$Z^{\mu\nu} = \sum_k m_k \Gamma_{[k]} v^{\mu}_{[k]} v^{\nu}_{[k]} + \frac{e_k}{c}\left(v^{\mu}_{[k]} \Lambda^{\nu}_{[k]} + \Lambda^{\mu}_{[k]} v^{\nu}_{[k]} \right). \tag{11}$$

When we relax the assumption of non-interacting particles, the energy component for particles at rest, $Z^{00} = \sum_k m_k c^2 + 2 e_k \Lambda^0_{[k]}$, is counting twice the charge-charge NILI interactions.

It is interesting that the correct ECMST (8) is somewhat in the middle between the ordinary Hilbert tensor (10) and the tensor (11). Of course, adding a field term L_{f} to (7) does not eliminate the difficulties with (10) and (11), because the field term depends of the field potentials $A^{\mu}(r,t)$ and cannot compensate the energy, the comomentum, and the stress generated by the NILI potentials $\Lambda^{\mu}(R(t))$. In the next section, we will show why the ordinary Hilbert tensor (10) is valid for the free part of the Lagrangian (7) but is not valid for $L^{\mathrm{NILI}}_{\mathrm{int}}$.

4 The Hilbert ECMST as a special case from $\Theta^{\mu\nu}$

For those systems with Lagrangian action $S = \int L dt$, the general definition (6) of the ECMST and the relation between its trace and the action implies

$$L = -\eta_{\mu\nu} \Theta^{\mu\nu}. \tag{12}$$

Taking the functional derivative with respect to $\eta_{\mu\nu}$, we obtain

$$\frac{\delta L}{\delta \eta_{\mu\nu}} = -\left(1 + \eta_{\mu\nu} \frac{\delta}{\delta \eta_{\mu\nu}} \right) \Theta^{\mu\nu}. \tag{13}$$

This relation can, in general, be inverted to obtain

$$\Theta^{\mu\nu} = -\left(1 + \eta_{\mu\nu} \frac{\delta}{\delta \eta_{\mu\nu}} \right)^{-1} \frac{\delta L}{\delta \eta_{\mu\nu}}. \tag{14}$$

Using the identity $(1+B)^{-1} = (1 - B + B^2 - B^3 + B^4 + \cdots)$ for systems whose Lagrangian verifies the constraint

$$\eta_{\mu\nu} \frac{\delta^2 L}{\delta {\eta_{\mu\nu}}^2} = -\frac{1}{2} \frac{\delta L}{\delta \eta_{\mu\nu}}, \tag{15}$$

the Lagrangian relation (14) reduces to the Hilbert relation

$$\left(1 + \eta_{\mu\nu} \frac{\delta}{\delta \eta_{\mu\nu}} \right)^{-1} \frac{\delta L}{\delta \eta_{\mu\nu}} = \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \right) \frac{\delta L}{\delta \eta_{\mu\nu}} = 2 \frac{\delta L}{\delta \eta_{\mu\nu}}. \tag{16}$$

Therefore, the ordinary Hilbert tensor $T^{\mu\nu} = -2(\delta L / \delta \eta_{\mu\nu})$ can be considered a special case of the general tensor $\Theta^{\mu\nu}$.

As shown in the previous section, the new $\Theta^{\mu\nu}$ defined by (6) gives the correct ECMST for a system with Lagrangian (7). Notice that the kinetic Lagrangian $L^{\mathrm{NILI}}_{\mathrm{m}}$ in (7) satisfies the constraint (15), which implies that (16) can be applied to $L_{\mathrm{m}}^{\mathrm{NILI}}$. Nevertheless, the interaction Lagrangian $L_{\mathrm{int}}^{\mathrm{NILI}}$ does not satisfy (15), but $\eta_{\mu\nu}(\delta^2 L_{\mathrm{int}}^{\mathrm{NILI}}/\delta\eta_{\mu\nu}{}^2) = 0$. Combining both constraints (14) reduces to

$$\Theta^{\mu\nu} = -2\frac{\delta L_{\mathrm{m}}^{\mathrm{NILI}}}{\delta\eta_{\mu\nu}} - \frac{\delta L_{\mathrm{int}}^{\mathrm{NILI}}}{\delta\eta_{\mu\nu}}. \tag{17}$$

This result explains why the Hilbert prescription can be applied only to the kinetic part $(-2\delta L_{\mathrm{m}}^{\mathrm{NILI}}/\delta\eta_{\mu\nu})$ of the whole Lagrangian (7), whereas the incorrect tensor $Z^{\mu\nu} = -2\delta L/\delta\eta_{\mu\nu}$ counts twice the charge-charge NILI interactions. This confirms the remark made at the end of the previous section: *the correct ECMST $\Theta^{\mu\nu}$ is somewhat in the middle between the ordinary Hilbert tensor $T^{\mu\nu}$ and the tensor $Z^{\mu\nu}$.*

Finally, it must be emphasized that the usual expression for the Hilbert ECMST density (2) can be obtained from (16) by taking densities on both sides of (16) and applying a geometrization procedure [4].

Summarizing, the new definition $\Theta^{\mu\nu} \equiv (1/2)\sum_k (p_{[k]}^{\mu} v_{[k]}^{\nu} + v_{[k]}^{\mu} p_{[k]}^{\nu})$ provides a generalized ECMST which is (**i**) symmetric, (**ii**) conserved, (**iii**) in agreement with the energy and momentum of a system of charges interacting via NILI potentials, and (**iv**) properly generalizes the Belifante and Rosenfeld formula, with the Hilbert tensor $T^{\mu\nu}$ being a special case of $\Theta^{\mu\nu}$.

References and notes

[1] Sometimes named the stress-energy-momentum or energy-momentum tensor. We prefer to name this tensor according to its elements –energy, comomentum, and stress–, correct dimensions –momentum has not units of energy–, and in ordered form –energy is given by the zero zero element–. It is also usual in the relativistic literature to confound the tensor with its density.

[2] The theory of relativity; second edition **1977:** *Oxford University Press; Delhi*. Møller, C.

[3] Field Theory of Gravitation: Desire and Reality **1996:** *Gravitation 2, 69–81*. Baryshev, Yurij V.

[4] General relativity as geometrical approximation to a field theory of gravity **2012:** *viXra:1203.0042*. González Álvarez, Juan Ramón.

[5] On the spin angular momentum of mesons **1939:** *Physica 6(7–12), 887–898*. Belinfante, F. J.

[6] On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields **1940:** *Physica 7(5), 449–474*. Belinfante, F. J.

[7] Sur le tenseur d'impulsion-énergie **1940:** *Mém. Acad. Roy. Belg. Sci., 18(6), 1–30*. Rosenfeld, L.

[8] Stress-Energy-Momentum Tensors and the Belinfante-Rosenfeld Formula **1992:** *Contemp. Mathem. 132, 367–392*. Gotay, Mark J.; Marsden, Jerrold E.

[9] Action-at-a-distance as a full-value solution of Maxwell equations: The basis and application of the separated-potentials method **1996:** *Phys. Rev. E 53(5), 5373–5381*. Chubykalo, Andrew E.; Smirnov-Rueda, Roman.

Erratum: Action-at-a-distance as a full-value solution of Maxwell equations: The basis and application of the separated-potentials method [Phys. Rev. E 53, 5373 (1996)] **1997:** *Phys. Rev. E 55(3), 3793–3793*. Chubykalo, Andrew E.; Smirnov-Rueda, Roman.

[10] Reply to "Comment on 'Action-at-a-distance as a full-value solution of Maxwell equations: The basis and application of the separated-potentials method'" **1998:** *Phys. Rev. E 57(3), 3683–3686*. Chubykalo, Andrew E.; Smirnov-Rueda, Roman.

[11] The Classical Theory of Fields; Fourth Revised English Edition **2000:** *Butterworth-Heinemann; Amsterdam*. Landau L. D.; Lifshitz E.M.

[12] Gravitation and Cosmology, Principles and applications of the general theory of relativity **2005:** *John Wiley & Sons (Asia) Pte. Ltd.; Singapore*. Weinberg, Steven.

[13] Classical mechanics; Third edition **2001:** *Addison Wesley; San Francisco*. Goldstein, Herbert; Poole, Charles; Safko, John.

[14] The quantum theory of fields; Volume I **1996:** *Cambridge University Press; Cambridge*. Weinberg, Steven.