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# The *Unified Theory of Electrical Machines*

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## Introduction

In this paper an entirely new approach is used to solve the old problem known as "The Unified Theory of Electrical Machines" Instead of solving an electric circuit with time dependent resistors and coils. I spent a lot of time in finding the appropriate coordinate system in which the problem becomes very simple. Instead of mathematical reasoning with innumerous number of mathematical equation I used pictorial reasoning

This Unified Theory of Electrical Machines is the shortest and therefore can be used to teach student the principles of electrical machines in a one semester course

The systematic procedure introduced in this paper is not limited to machine theory and can be applied in many physic and engineering problems.

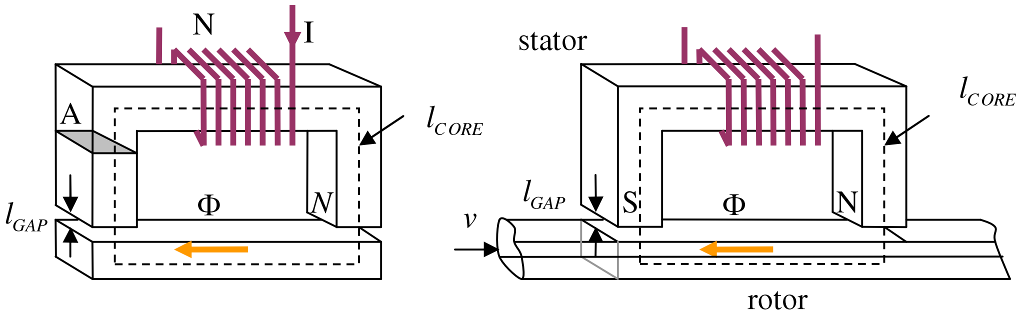
Chapter 1

# The magnetic field and Electromagnets

In order to understand induction motor principle of operation we need to known a bit more about how magnetic field are shaped to fit our needs.

## Magnetic field shaping and forming

Two magnets which are not too far, attract each other, and move toward each other in such a way that the north pole of one magnet moves toward the south pole of the second magnet. The flux in the magnetic material and in the gap between them, become stronger, and the magnetic flux around the magnets become weaker.




Picture 1

The magnetic reluctance of the iron core and the magnetic reluctance in air gap in picture 1 are;

1]

$$R_{CORE} = \frac{l_{CORE}}{\mu_R \mu_0 A}$$

$$R_{GAP} = 2 \cdot \frac{l_{GAP}}{\mu_0 A}$$

and the magnetic flux is ;

2]

$$\Phi = \frac{N \cdot I}{R_{CORE} + R_{GAP}}$$

usually

3]

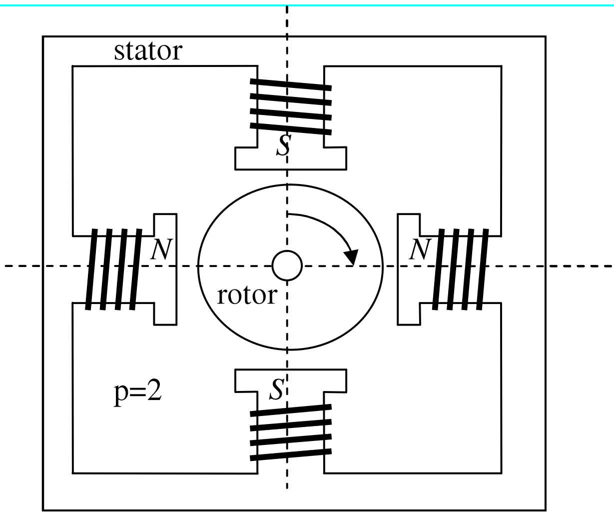
$$R_{GAP} >> R_{CORE}$$

so

4] $$\Phi \approx \frac{N \cdot I}{R_{GAP}}$$

and the flux density is ;

5] $$B = \frac{\Phi}{A}$$




Picture 2

In rotating machines we have a moving part; the rotor and a static part; the stator.
The rotor and stator have the same principles of operation.
On the right side of picture 1, the lower iron core is infinitely long and is moving to the right while $l_{GAP}$ is not changed therefor the flux $\Phi$ and the flux density B are not dependent on rotor movement. In ideal rotating machines, rotor movement does not change flux distribution. **Round machines behave like infinitely long machines**

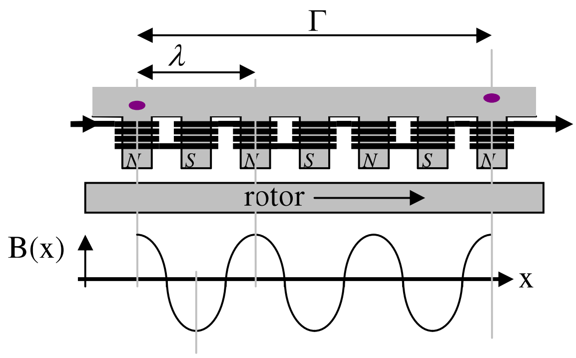
The distance between an N pole and the next N pole in picture 3, is the wave length $\lambda$ of the magnet.
In round machines there are p magnets

6] $$\Gamma = p \cdot \lambda$$

where p is the number of pair of magnets poles in the stator and must be an integer

**p in stator and rotor must be the same.**

In picture 3, $p = 3$ and in picture 2, the round motor has two pair of magnets therefore p=2

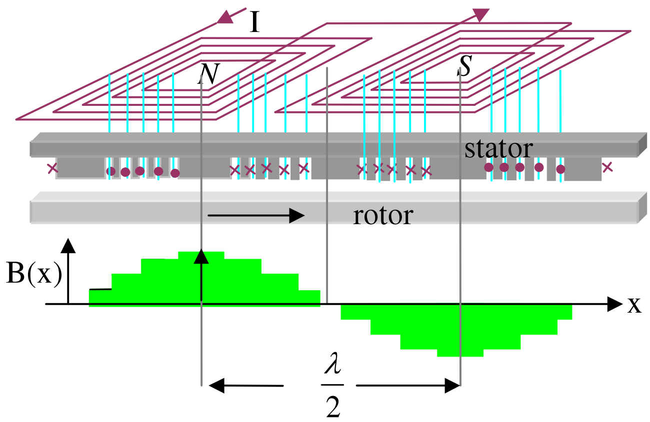



Picture 3

to make flux destiny B(x) sinusoidal the stator is wounded in a sophisticated manner, as shown in picture 4. Windings wires are pushed into slots as seen in picture 5. Two winding option **a** and **b** are equivalent but in machine industry, **a** is preferable.
picture 5-c describe windings in a 3 phase machine, and will be discussed later.

For the winding in picture 4 the flux density is approximately sinusoidal in space (not in time)

7]
$$B(x) = B_0 \cos(kx)$$
$$k = \frac{2\pi}{\lambda}$$




Picture 4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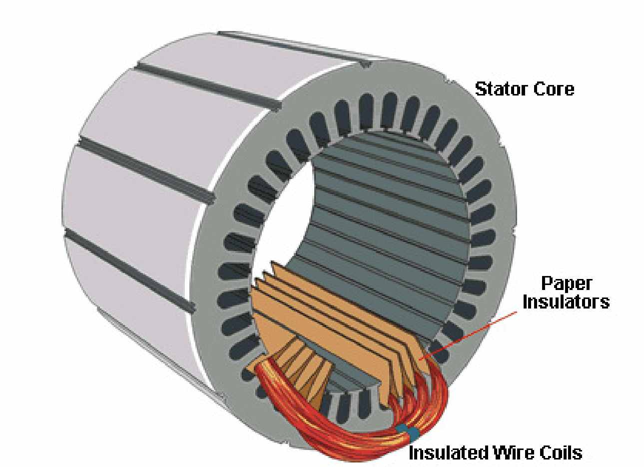
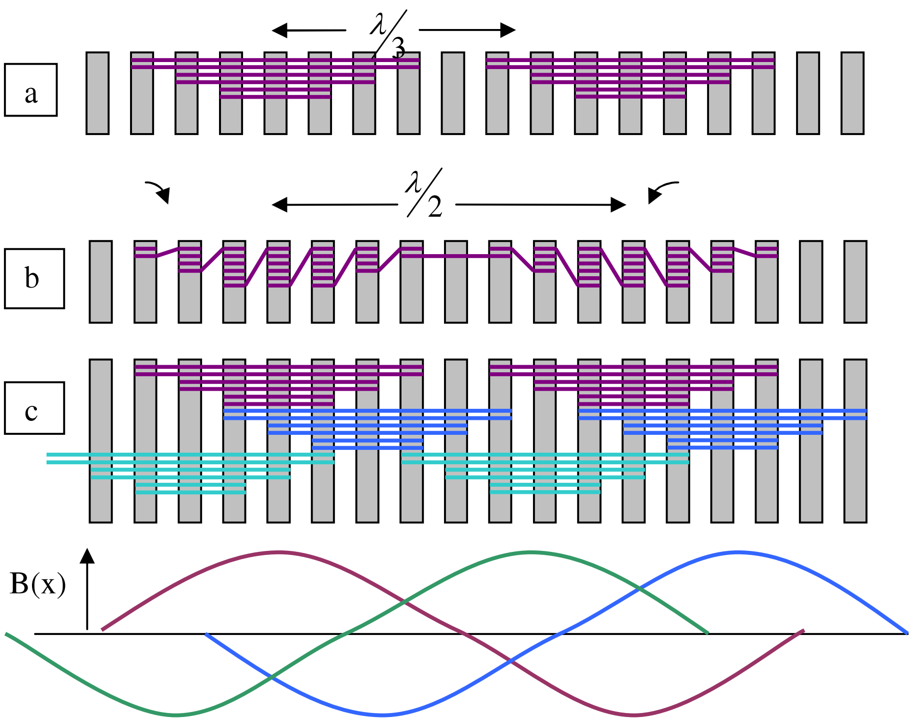

Stator Core
Paper Insulators
Insulated Wire Coils


λ/3
a
λ/2
b
c
B(x)

Picture 5

The flux density B(x) became time dependent B(x,t) if the DC current is replaced with a time dependent current i(t)

8] $$i(t) = I_0 \cdot \cos(\omega t)$$

Suppose that one needs a moving magnetic field that looks like a wave with speed $v$, according to the following identity;

9] $$B(x,t) = B_0 \cos(kx - \omega t) = B_0 \cos\left[k\left(x - \frac{\omega}{k}t\right)\right] = B_0 \cos\left[k\left(x - vt\right)\right]$$

$$v = \frac{\omega}{k}$$

From trigonometry it is well known that;

10] $$\cos(kx - \omega t) = \frac{2}{3}[\cos(\omega t)\cos(kx) + \cos(\omega t + 120^0)\cos(kx + 120^0) + \cos(\omega t + 240^0)\cos(kx + 240^0)]$$

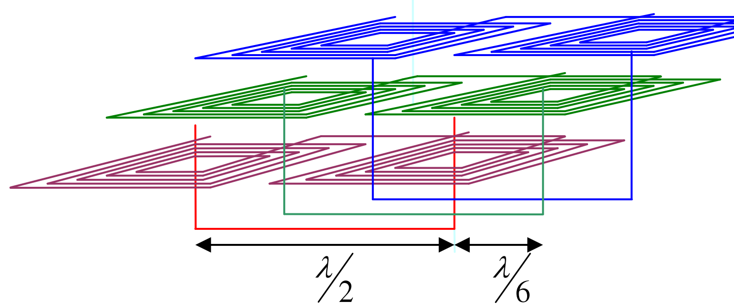
What one needs is an electromagnet that will behave according to that equation
Now, If the current in the windings in picture 1, is sinusiodal than the flux is also sinusoidal

11] $$\Phi(t) \approx \frac{N \cdot i(t)}{R_{GAP}} \approx \frac{N \cdot I_0 \cos(\omega t)}{R_{GAP}}$$

Suppose now that winding is spread in x direction as shown in picture 4 and the current in the coil is sinusoidal
Then

13] $$\Phi(t,x) \approx \frac{N_0 \cos(kx) \cdot I_0 \cos(\omega t)}{R_{GAP}}$$

this is not what we need. In order to make flux to move in the correct direction three winding are spread in stator slots, each winding is $120^0$ apart in the x direction (picture 6) The three windings are red green and blue but R,S,T are more common names.




Picture 6

The current in each winding is delayed as follows

$$14] \quad \begin{aligned} i_R(t) &= I_0\cos(\omega t) \\ i_S(t) &= I_0\cos(\omega t + 120^0) \\ i_T(t) &= I_0\cos(\omega t + 240^0) \end{aligned}$$

the source for those currents is the main 3 phase power system which supplies 3 phase 230VAC at 50Hz
Because of superposition, the magnetic flux will be
15]

$$\Phi(t,x) \approx \frac{N_0 I_0}{R_{GAP}}[\cos(kx)\cdot\cos(\omega t) + \cos(kx+120^0)\cdot\cos(\omega t+120^0) + \cos(kx+240^0)\cdot\cos(\omega t-240^0)]$$

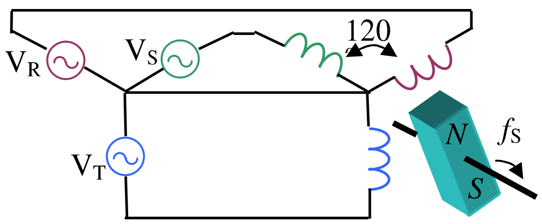
and what we get is a moving field, although the iron core is standing still.
(see equation 10)

$$16] \quad \Phi(t,x) \approx \frac{N_0 I_0}{R_{GAP}}\cdot\cos(kx-\omega t)$$

## Conclusions

To create a rotating magnetic field which is equivalent to a single rotating magnet as shown in picture 7, we need an electromagnet with three windings, fed from the main 3 phase electrical power system. The speed of rotation will be the main power system frequency for a winding with p=1, and $f_S/p$ for a winding with p pair of magnets. Stators and rotor of synchronous and induction machines are creating such rotating fields.
The symbolic notation for a rotating field electromagnet is shown in picture 7

$$V_R = E\angle 0^0 \qquad V_S = E\angle 120^0 \qquad V_T = E\angle 240^0$$



Picture 7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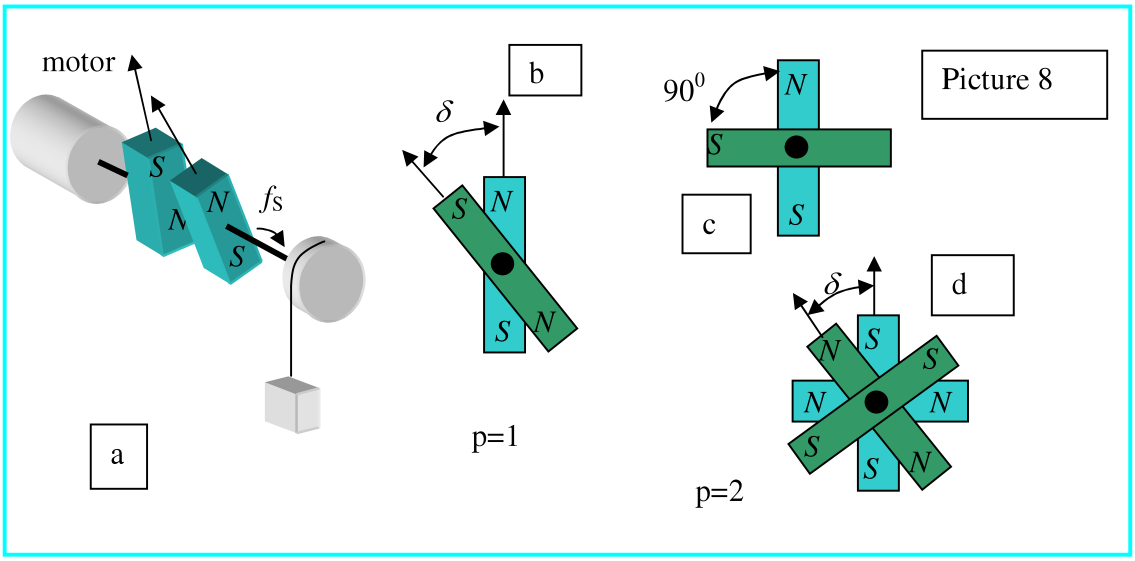
Chapter 2

# Magnetic "Clutch"

In picture 8 two magnets attract each other. One magnet is connected to a motor, the second to the load. If the magnetic force between the magnets is strong enough, the magnet connected to the load will follow the magnet connected to the motor and this is the only case that happens in all kind of electric machines. One can notice an angle $\delta$




Picture 8

between the two magnets. This angle ("Load Angle") depends on load. And must be

less then $90^0$ as seen in "c". Big angle means less force, because the distance between adjacent poles increase. The load angle for magnets with p pair of magnetic poles is $\delta/p$. in picture 8-c a system with p=2 therefore maximum load angle is $45^0$ mechanical but stiil $90^0$ electrical. The load angle of a rotating "Clutch" can be seen with a stroboscope. The stroboscope also known as a **strobe**, is an instrument used to make a cyclically moving object appear to be slow-moving or stationary.

## Synchronous motor and Synchronous generator

Synchronous motor rotates with constant speed. Speed depends solely on main voltage frequency 50Hz. As mentioned earlier an electromagnet wounded with three coils connected to the main electric power create a rotating field. (Picture 9) The rotating field is followed by a real magnet named rotor. If the electromagnet is made of one pair of poles (p=1) the rotor speed will be the main power frequency. Speed become slower if the winding is with greater p

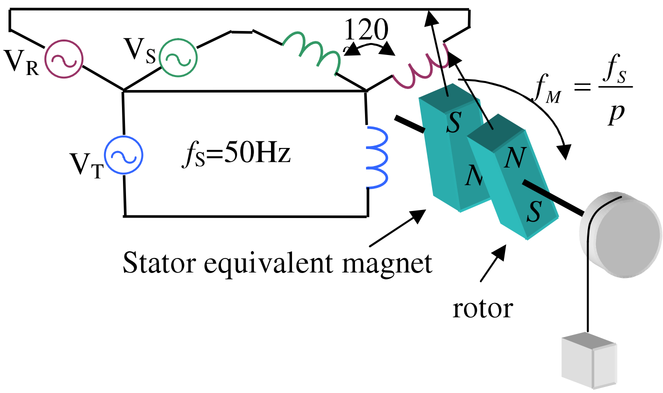
The mechanical speed of the rotor will be

$$f_M = \frac{f_S}{p}$$

Since p is an integer, rotor speed is quantified

Electrical machines are bidirectional. The synchronous motor convert electric energy to mechanical energy the synchronous generator converts mechanical energy to electricity. Picture 9 describe synchronous generator as well.

Synchronous motor and generator mathematical analysis will be discussed later



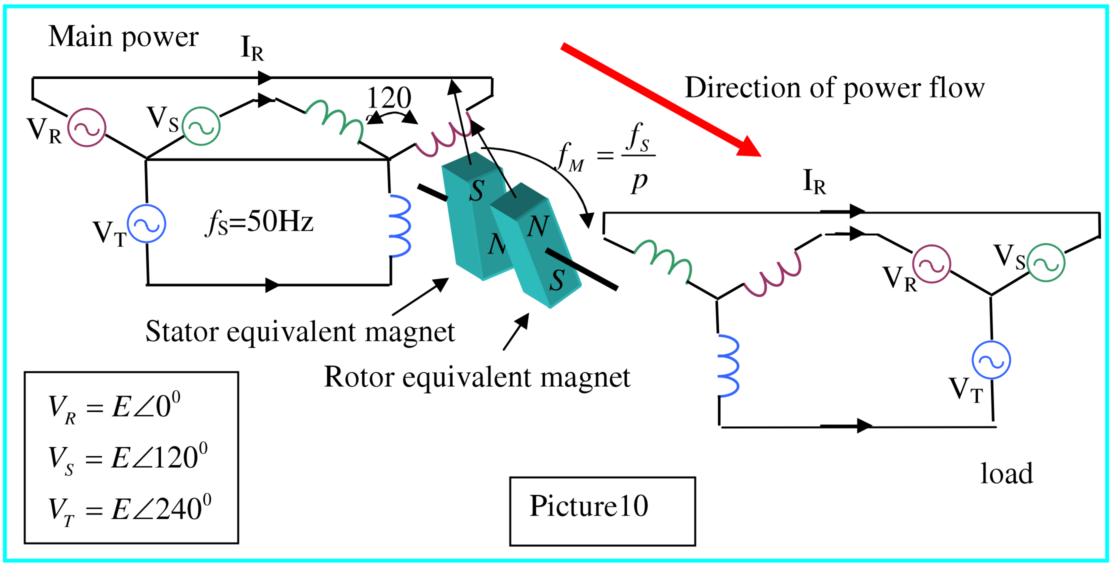

Picture 9

$$V_R = E\angle 0^0$$
$$V_S = E\angle 120^0$$
$$V_T = E\angle 240^0$$

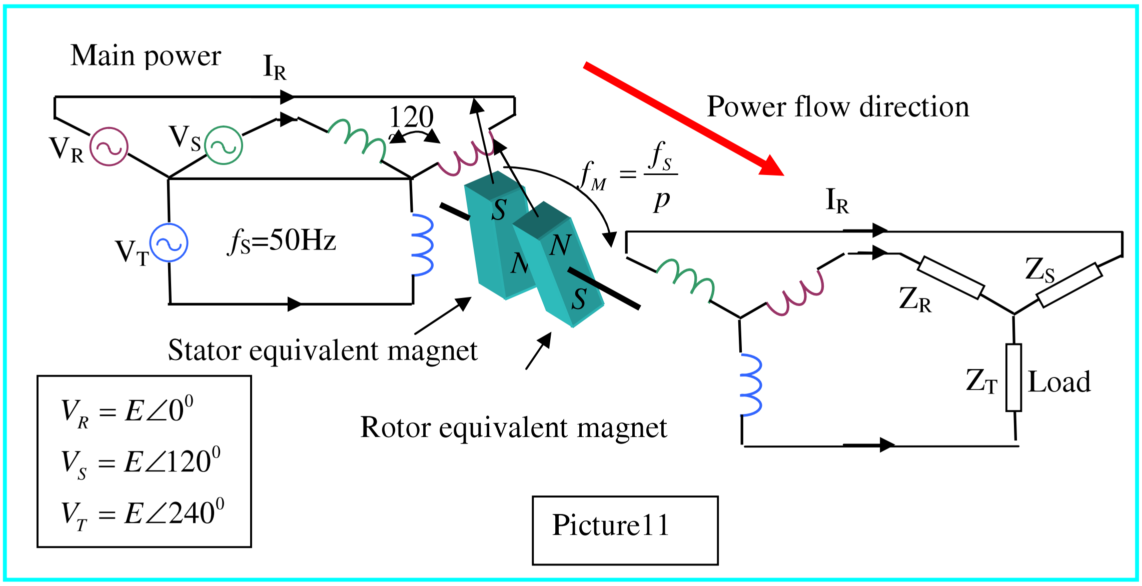
# 3 phase transformer

As claimed before, all electrical machines including transformers are based on two rotating magnets with the same speed ("Clutch" configuration Pic 8). In picture 8, the magnetic "Clutch" is made from two real magnets that trace each other. In a synchronous machine in picture 9, one real magnet is replaced by a rotating field this is the stator. The rotor is a real magnet.

In a 3 phase transformer (picture 10) both real magnet are replaced by rotating fields Since north pole attracts south pole the stator magnet and rotor magnet are in opposite direction , so , direction of current in stator and rotor is also in opposite direction this means that power flows from main power system to the load so that picture 11 is equivalent to picture 10 and is equivalent to pictures 8 and 9




Picture10




Picture11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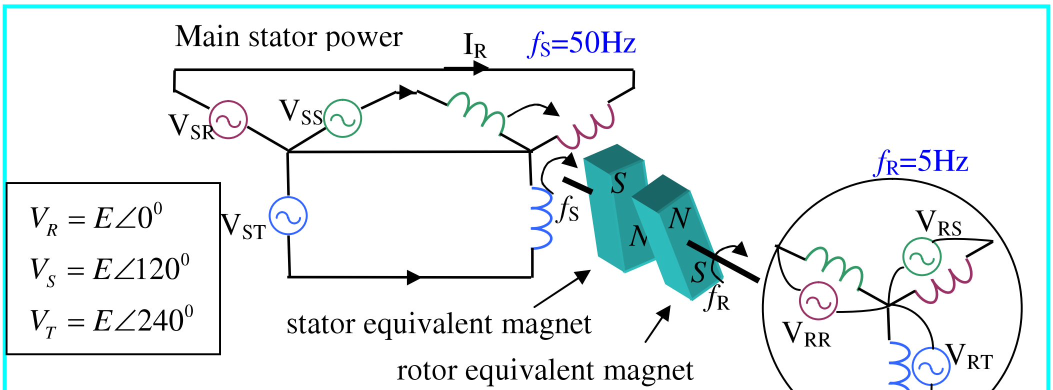
## The induction motor

As claimed before, all electrical machines are based on the **magnetic "Clutch"** configuration. the induction machine is no exception.

In the induction machine, the stator and rotor are both implemented by rotating fields as shown in picture 12. According to this picture, the stator is fed from the main power system (50Hz 230VAC). and therefore rotate at 50Hz

{indexes S,M stands for Stator, R stands for Rotor(electric windings) and M stand for Mechanical part of the rotor and R,S,T belongs to a 3 phase system}




Picture12

Pay attention to Picture 12.c The rotor rotate at 5Hz ($f_R$=5Hz) and the stator equivalent magnet rotate at a different "speed" ($f_S$=50Hz).

It is clear now that both "magnets" are rotating with different speed. I mentiond above that any electric machine must behave as a "Clutch" and the magnets must rotate with the same speed. If the speed of the rotor magnet is not the speed of the stator what we get is useless vibration each time the magnet pole of the stator is passing near the magnet pole of the rotor. We must to make them to rotate at the same speed. To do so, we add a motor to the rotor that will speed up the rotor rotating field from 5Hz to 50Hz this motor will add to the rotor 45Hz and needs power to work properly (picture 13)

In the induction machine, In order to keep Clutch condition demands the following identity is kept

$$f_S = f_R + f_M$$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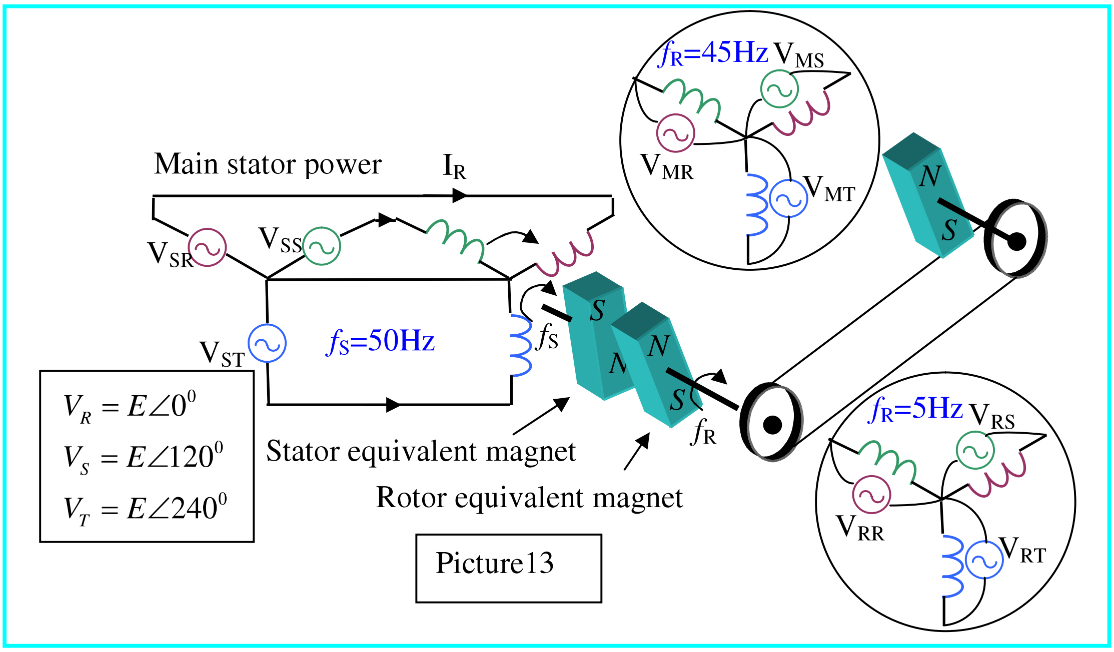
What we are going to do now is to derive the mathematical equation that describes the induction motor under any circumstances.



Picture13

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Chapter 3

## Lenz and Faraday induction laws and the induction machine

In picture 14 , the induction machine is greatly simplified. A bar magnet represents the stator that rotate at speed $f_S$ The one turn coil represent rotor windings, and the coil handle represent the mechanical motor that speed up the rotor. Coil frequency and voltage is measured with a Hertz-meter and a voltmeter.
Slip rings are used to supply current to the rotating rotor

### Experiment 1

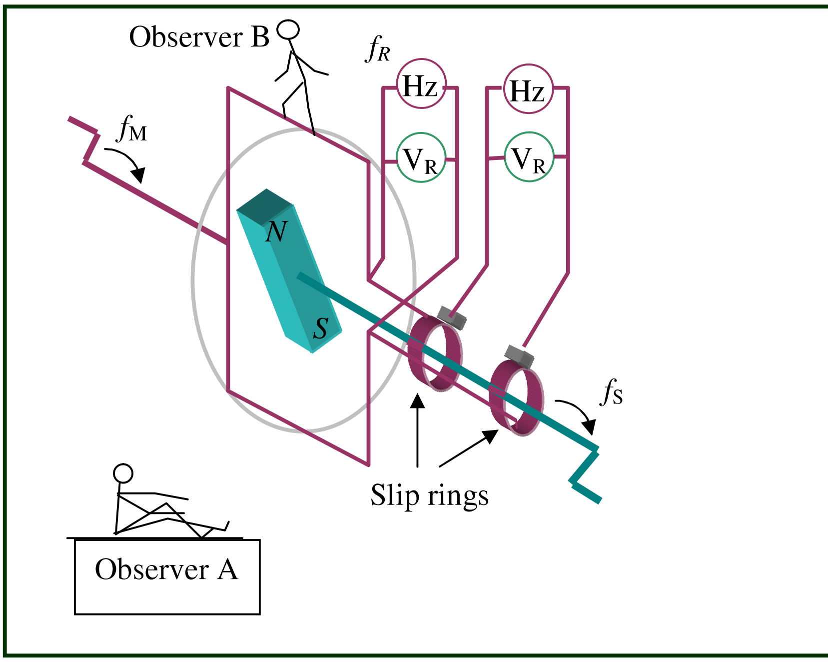
Suppose that the coil and the magnet rotate at the same speed $f_S = f_M$ the relative speed as seen by observer B is zero , the induced voltage measured by the voltmeter according to Lenz law, is zero

### Experiment 2

Suppose that the coil is standing still $f_M$ =0 and the magnet rotate at speed $f_S$ =50Hz. The relative speed as seen by observer B is $f_R=f_S - f_M$=50Hz , the induced voltage measured by the voltmeter for this case is defined as 100% in this case $f_R=f_S$ and the induction machines behave as a transformer

### Experiment 3 etc..

The bar magnet rotate at speed $f_S$ =50Hz and the coil mechanical speed and direction are change to various values . The results of those experiments are summarized in the table below




Picture 14

**(a negative voltage in AC system, means $180^0$ out of phase)**

| | $V_R$ induce voltage in rotor coil | mechanical speed $f_M$ | Frequency in coil $f_R$ | stator magnet $f_S$ |
|---|---|---|---|---|
| relative speed is zero. no induced voltage synchronic-motor | $V_R$=0%=0Volt | $f_M$=50Hz | $f_R=f_S-f_M=0$ | $f_S$=50Hz |
| mechanical rotor stop moving (brake) 50Hz transformer | $V_{BRAKE}$=100% | $f_M$ = 0Hz | $f_R=f_S-f_M=50$ | $f_S$=50Hz |
| | $V_R$=40% | $f_M$ = 30Hz | $f_R=f_S-f_M=20$ | $f_S$=50Hz |
| | $V_R$=10% | $f_M$ = 45Hz | $f_R=f_S-f_M=5$ | $f_S$=50Hz |
| mechanical rotor move in opposite direction and induced high voltage | $V_R$=200% | $f_M$ =-50Hz | $f_R=f_S-f_M=100$ | $f_S$=50Hz |
| mechanical rotor is faster then stator inducing voltage with $180^0$ phase | $V_R$= -10% | $f_M$ = 55Hz | $f_R=f_S-f_M= - 5$ | $f_S$=50Hz |
| | | | | |

## Conclusions

From the table, the induced voltage in the coil is proportional to the electric frequency measured by the Hertz meter

1] $$V_R \sim f_R$$

and when $f_R=f_S$ the machine behave like a transformer and $V_R$=100%
if that transformer is a 1:1 transformer than $V_R= V_S$
we can eliminate the proportion sign and conclude that

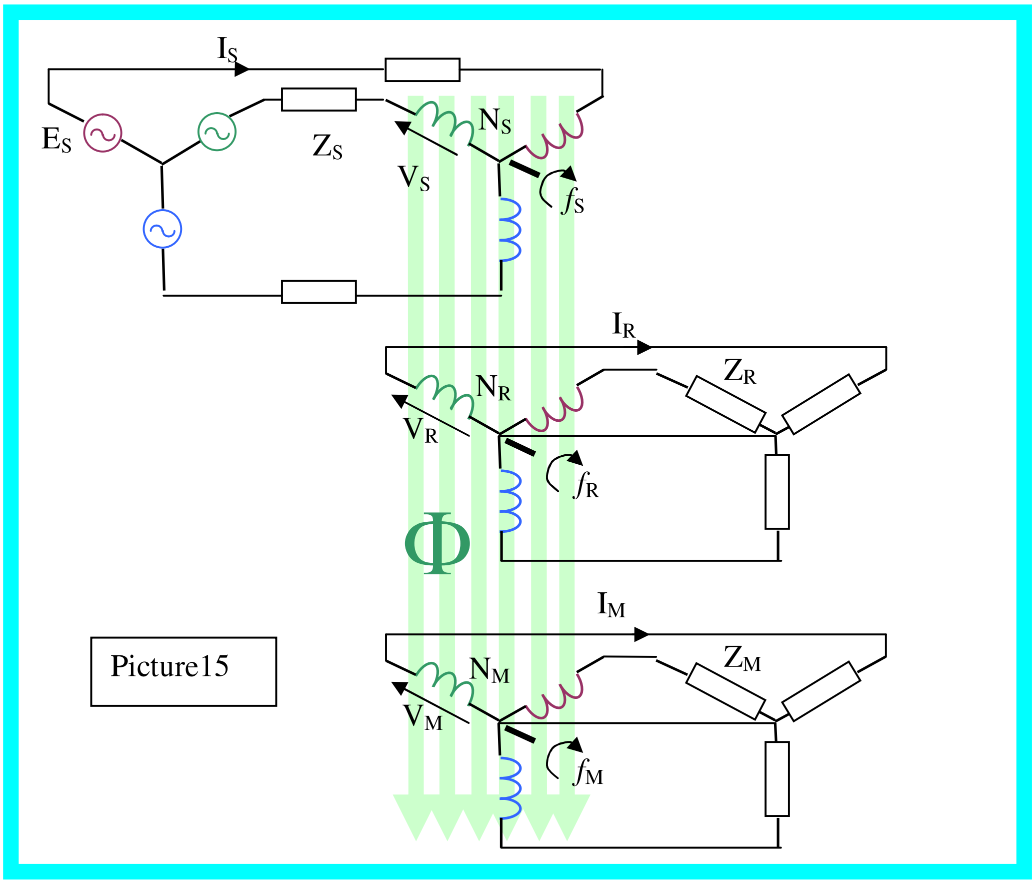
2] $$V_R(f_R) = \frac{f_R}{f_S} V_S(f_S)$$

One can check that when $f_R = f_S$ , $V_R = V_S$ and the transformer behavior is achieved
and when $f_R$=0 , $V_R$= 0 brake condition are achieved
We know also that

3] $$f_R = f_S - f_M$$

**Let assume that there exist a mechanical voltage $V_M$ then this voltage must obey the same rules as other voltages in the induction machine.**

4] $$V_M(f_M) = \frac{f_M}{f_S} V_S(f_S)$$




Picture15

the reason for that is

5] $$V_M(f_M) + V_R(f_R) = \frac{f_M}{f_S} V_S(f_S) + \frac{f_R}{f_S} V_S(f_S) = V_S(f_S)$$

(effective values don't depend on frequency and can be add)

**one can notice that the following equation are invariant**

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6]
$$V_R(f_R)=\frac{f_R}{f_S}V_S(f_S)$$
$$V_R(f_R)=\frac{f_R}{f_M}V_M(f_M)$$
$$V_M(f_M)=\frac{f_M}{f_S}V_S(f_S)$$
$$V_M(f_M)=\frac{f_M}{f_R}V_R(f_R)$$
$$V_S(f_S)=\frac{f_S}{f_M}V_M(f_M)$$
$$V_S(f_S)=\frac{f_S}{f_R}V_R(f_R)$$

Or more easy to remember

**7]**
$$\frac{V_M(f_M)}{f_M}=\frac{V_R(f_R)}{f_R}=\frac{V_S(f_S)}{f_S}$$

---

This is just the proof why in quantum physics energy is proportional to frequency $(E=h\cdot f)$

---

since

**8]**
$$\Phi=\frac{N\cdot I}{R_{GAP}}=\frac{N_S\cdot I_S}{R_{GAP}}=\frac{N_R\cdot I_R}{R_{GAP}}=\frac{N_M\cdot I_M}{R_{GAP}}$$

Let expand eq 5 as follow

9]
$$\frac{V_S}{N_S}\cdot\frac{N_S I_S}{R_{GAP}}=\frac{V_R}{N_R}\cdot\frac{N_R I_R}{R_{GAP}}+\frac{V_M}{N_M}\cdot\frac{N_M I_M}{R_{GAP}}$$

where N is number of winding and I the current in that winding

**10]**
$$\frac{V_S}{N_S}\Phi=\frac{V_R}{N_R}\Phi+\frac{V_M}{N_M}\Phi$$

if we define a normalized voltage as voltage per turn then

11]
$$\frac{V_S}{N_S}=\frac{V_R}{N_R}+\frac{V_M}{N_M}$$

Then Lenz law in the induction machine becomes very simple

12] $$\overline{V}_S = \overline{V}_R + \overline{V}_M$$

if we define a normalized current as number of turns times current then;

13] $$\overline{I} = N_X \cdot I_X$$

and using equation 9

14] $$\overline{I} = \overline{I}_R = \overline{I}_S = \overline{I}_M$$

Let define Slip as "S"

15] $$S = \frac{f_R}{f_S} = \frac{2\pi f_R}{2\pi f_S} = \frac{\omega_R}{\omega_S}$$

from eq 6

16] $$V_R = S \cdot V_S$$

and from eq 3 and eq 6

17] $$V_M = (1 - S) \cdot V_S$$

## Chapter 4
## The equivalent circuit of the induction motor

The equivalent circuit of the induction motor (one phase) is derived from picture 15 as follows;

1] $$Z_R = \frac{V_R}{I_R}$$

Let define a normalized impedance as follows;

2] $$\bar{Z}_R = \frac{\bar{V}_R}{\bar{I}_R} = \frac{V_R}{N_R} \cdot \frac{1}{N_R \cdot I_R} = \frac{Z_R}{N_R^{\,2}}$$

Similarly

3] $$\bar{Z}_M = \frac{Z_M}{N_M^{\,2}}$$

and

4] $$\bar{Z}_S = \frac{Z_S}{N_S^{\,2}}$$

**Normalize impedances are always used with normalized voltage and current**

5] $$\bar{Z} = \frac{\bar{V}}{\bar{I}}$$

Now from picture 15 and equations 12 in the previous chapter

6] $$\bar{E}_S = (\bar{Z}_S + \bar{Z}_R + \bar{Z}_M)\bar{I}_S$$

from equation 5

7] $$\bar{Z}_R = \frac{\bar{V}_R}{\bar{I}_R} = \frac{S \cdot \bar{V}_S}{\bar{I}}$$

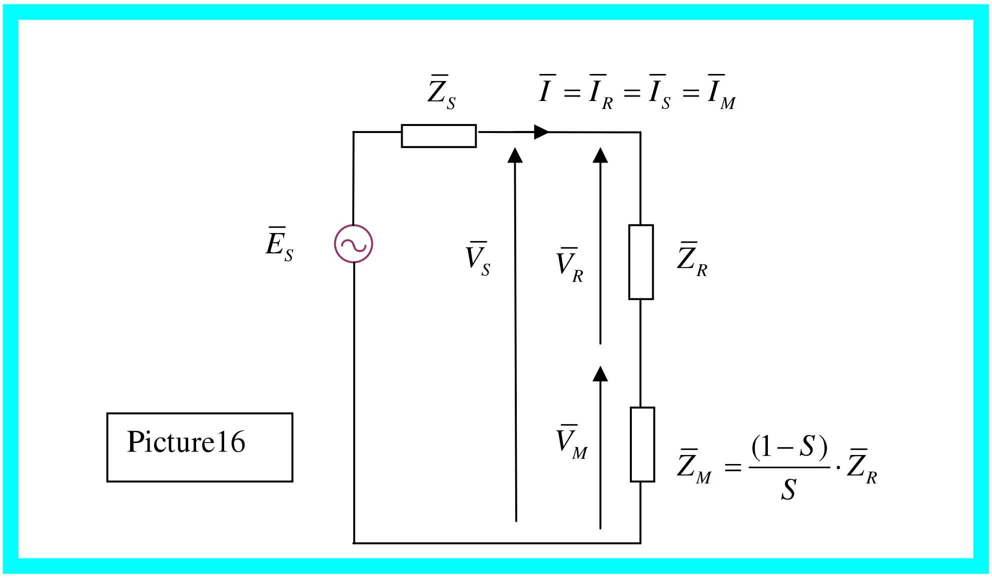
and

8] $$\bar{Z}_M = \frac{\bar{V}_M}{\bar{I}_M} = \frac{(1-S) \cdot \bar{V}_S}{\bar{I}}$$

from the last two equations

9] $$\bar{Z}_M = \frac{(1-S)}{S} \cdot \bar{Z}_R$$

In picture 16, All the equations above are combining to the final equivalent circuit of the induction machine

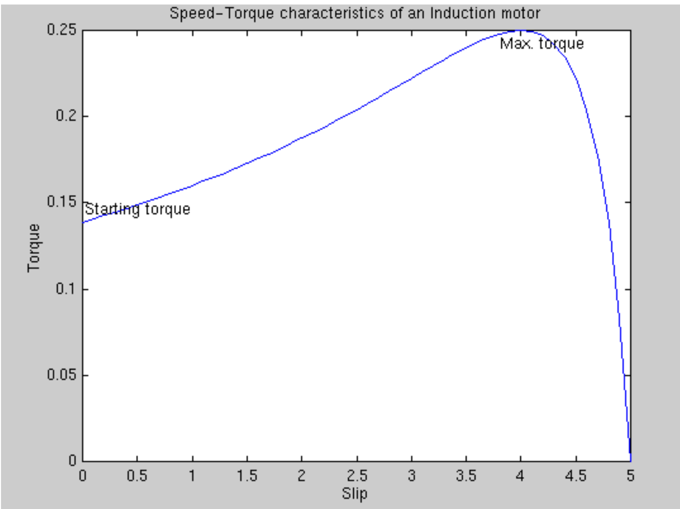



Picture16

Chapter 5

## Torque and power

A typical torque versus speed diagram is shown below:




General relation between torque and power in SI units is given by

1] $$P_M = \omega_M \cdot \mathrm{T} = 2\pi f_M \cdot \mathrm{T} = 2\pi \cdot \frac{f_M}{f_S} \cdot f_S \cdot \mathrm{T} = (1-S)\omega_S \cdot \mathrm{T}$$

$P_M$ is the mechanical power delivered to machine rotating axis. T the mechanical torque (Moment) and $\omega_M$ is the axis rotating speed

Let define stator torque power relation as follows

2] $$P_S = \omega_S \cdot \mathrm{T} = 2\pi f_S \cdot \mathrm{T}$$

$P_S$ is easily found from the equivalent circuit in picture 16

3] $$P_S = 3\left|I_S\right|^2 real\left(Z_R + Z_M\right) = 3\left|I_S\right|^2 real\left(\frac{Z_R}{S}\right)$$

and

4] $$\left|I_S\right|^2 = \frac{\left|E_S\right|^2}{\left|Z_S + \frac{Z_R}{S}\right|^2}$$

the mechanical torque for 3 phase machine is

5] $$\mathrm{T} = \frac{P_S}{\omega_S} = \frac{1}{\omega_S} \cdot \frac{\left|E_S\right|^2}{\left|Z_S + \frac{Z_R}{S}\right|^2} real\left(\frac{Z_R}{S}\right)$$

at start S=1 and start torque is

6] $$\mathrm{T}_{START} = \frac{P_S}{\omega_S} = \frac{1}{\omega_S} \cdot \frac{\left|E_S\right|^2}{\left|Z_S + Z_R\right|^2} real\left(Z_R\right)$$

to obtain maximal torque

7] $$\frac{\partial \mathrm{T}}{\partial S} = 0$$

peak torque is given by:

8] $$\mathrm{T}_{MAX} = \frac{1}{\omega_S} \cdot \frac{3 \cdot \left|E_S\right|^2}{2 \cdot \left(R_S + \sqrt{R_S^{\,2} + \left(X_S + X_R\right)^2}\right)} \approx \frac{3 \cdot \left|E_S\right|^2}{2 \cdot \omega_S \left(X_S + X_R\right)}$$

Returning again to the torque equation, there is a slip value, $S_K$ at which torque is at its maximum value:

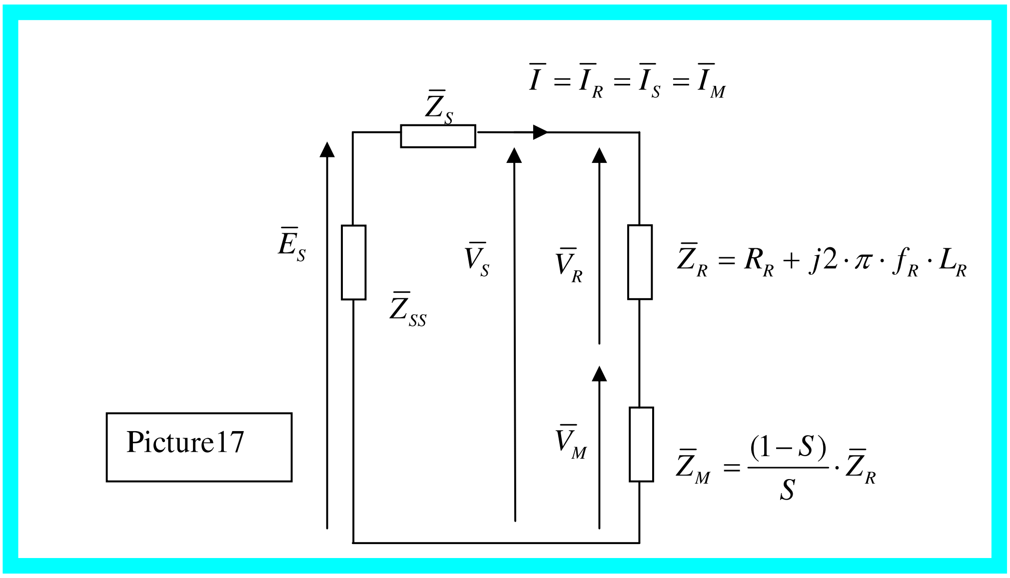
9]
$$S_K = S(T_{MAX}) = \frac{R_R}{\sqrt{R_S^{\,2} + (X_S + X_R)^2}} \approx \frac{R_R}{X_S + X_R}$$

For big machines

10]
$$\frac{T}{T_{MAX}} = \frac{2 \cdot S \cdot S_K}{S_K^{\,2} + S^2} = \frac{2}{\frac{S}{S_K} + \frac{S_K}{S}}$$

## The Induction generator

Under certain conditions the induction motor can be used as a generator




from picture 17

11]
$$0 = (\overline{Z}_{SS} + \overline{Z}_S + \overline{Z}_R + \overline{Z}_M)\overline{I}_S$$

or

12]
$$0 = (\overline{Z}_{SS} + \overline{Z}_S + \frac{1}{S}\overline{Z}_R)\overline{I}_S$$

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for the real part

13] $$0 = \overline{R}_{SS} + \overline{R}_S + \frac{1}{S}\overline{R}_R$$

as we know, resistance is always positive. Fortunately, when the induction machine behave as a generator S is negative
a negative S makes $R_R$/S negative (negative resistance) and the equation can exist
for the imaginary part

14] $$0 = j \cdot \overline{X}_{SS} + j \cdot \overline{X}_S + \frac{1}{S}(j2\pi f_R \overline{L}_R)$$

15] $$0 = j \cdot \overline{X}_{SS} + j \cdot \overline{X}_S + \frac{1}{S}(j2\pi S f_S \overline{L}_R)$$

Now, if S is negative and $X_{SS}$ capacitive, the induction machine acts as a generator. To make a a conventional induction machine to behave like a generator a 3 phase capacitor is connected  parallel to a resistive load connected to stator. Generating process will start only if a magnetic field exist, and capacitors are big enough.  Wind generators in parallel to main power system also work at negative S

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