A FUNCTIONAL DETERMINANT EXPRESSION FOR THE RIEMANN XI-FUNCTION

Jose Javier Garcia Moreta
Graduate student of Physics at the UPV/EHU (University of Basque country)
In Solid State Physics
Addres: Practicantes Adan y Grijalba 2 5 G
P.O 644 48920 Portugalete Vizcaya (Spain)
Phone: (00) 34 685 77 16 53
E-mail: josegarc2002@yahoo.es

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- ABSTRACT: We give and interpretation of the Riemann Xi-function $\xi(s)$ as the quotient of two functional determinants of an Hermitian Hamiltonian $H = H^{\dagger}$. To get the potential of this Hamiltonian we use the WKB method to approximate and evaluate the spectral Theta function $\Theta(t) = \sum_n \exp(-t\gamma_n^2)$ over the Riemann zeros on the critical strip $0 < \mathrm{Re}(s) < 1$. Using the WKB method we manage to get the potential inside the Hamiltonian H, also we evaluate the functional determinant $\det(H + z^2)$ by means of Zeta regularization, we discuss the similarity of our method to the method applied to get the Zeros of the Selberg Zeta function. In this paper and for simplicity we use units so $2m = 1 = \hbar$
- *Keywords: =* Riemann Hypothesis, Functional determinant, WKB semiclassical Approximation , Trace formula ,Bolte's law, Quantum chaos.

1. Riemann Zeta function and Selberg Zeta function

Let be a Riemann Surface with constant negative curvature and the modular group $PSL(2,R)$, Selberg [14] studied the problem of the Laplacian

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi_n(x,y) = E_n\Psi_n(x,y) \qquad E_n = \frac{1}{4} + k_n^2 \qquad (1)$$

These momenta k_n are the non-trivial zeros of the Selberg Zeta function, which can be defined by an Euler product over the Geodesic of the surface in an analogy with the Riemann Zeta function

$$\zeta(s) = \prod_n \frac{1}{(1-p_n^{-s})} \qquad Z(s) = \prod_P \prod_{k=0}^{\infty}\left(1 - N(P)^{-(s+k)}\right) \qquad (2)$$

Selberg also studied a Trace formula which relates the Zeros (momenta of the Laplacian Δ) on the critical line $Z\left(\frac{1}{2}+ik_n\right)=0$ and the length of the Geodesic of the Surface in the form

$$\sum_n h(k_n)=\frac{\mu(D)}{4\pi}\int_0^\infty dkkh(k)\tanh\left(\pi k\right)+\sum_{P\in p.p.o}\frac{\ln N(P)}{N(P)^{1/2}-N(P)^{1/2}}g(\ln N(P) \qquad (3)$$

Here, p.p.o means that we are taking the sum over the length of the Geodesic, $h(k)$ is a test function and $g(k)$ is the Fourier cosine transform of $h(k)$ $g(k)=\frac{1}{2\pi}\int_0^\infty dxh(x)\cos(kx)$ $\mu(D)$ is the area of the fundamental domain describing the Riemann surface . In case we had a surface with the length of the Geodesic $\ln N(P)=\ln p$ for 'p' on the second side of the equation a prime number,then the Selberg Trace is very similar to the Riemann-weil sum formula [12]

$$\sum_\gamma h(\gamma)=2h\left(\frac{i}{2}\right)-g(0)\ln\pi-2\sum_{n=1}^\infty\frac{\Lambda(n)}{\sqrt{n}}g(\ln n)+\frac{1}{2\pi}\int_{-\infty}^\infty dsh(s)\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{is}{2}\right) \quad (4)$$

This formula (4) related a sum over the imaginary part of the Riemann zeros to another sum over the primes, here $\Lambda(n)\begin{cases}\ln p & n=p^k\\ 0 & \text{otherwise}\end{cases}$ with 'k' a positive integer is the Mangoldt function, in case $\ln N(P)=\ln p$ both zeta function of Selberg and Riemann are related by $\frac{1}{Z(s)}=\prod_{n=0}^\infty\zeta\left(n+s\right)$ and their logarithmic derivative is quite similar if we set the function $\Lambda_{geodesic}(P)=\frac{\ln N(P)}{1-N(P)^{-1}}$

$$\frac{\zeta'}{\zeta}(s)=-\sum_{n=1}^\infty\Lambda(n)n^{-s} \qquad \frac{Z'}{Z}(s)=\sum_{P\in p.p.o}\Lambda_{geodesic}(P)N(P)^{-s} \qquad (5)$$

In both cases the Riemann and Selberg zeta functions obey a similar functional equation which relates the value at s and 1-s

$$Z(1-s)=\exp\left(-\frac{\mu(D)}{4\pi}\int_0^{s-1/2}vtan(\pi v)dv+c\right)Z(s) \qquad \zeta(1-s)=\mathrm{X}(s)\zeta(s) \qquad (6)$$

The constant of integration 'c' is determined by setting $s=1/2$, and $\mathrm{X}(s)=2\left(2\pi\right)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)$ for the case of the Riemann zeta function.
With the aid of the Selberg Trace formula (3) , we can evaluate the Eigenvalue staircase for the Laplacian $\Delta=-y^2\left(\partial_x^2+\partial_y^2\right)$

$$N\left(E=\frac{1}{4}+p^{2}\right)=\sum_{E_n\le E}1=\sum_{n}\frac{\mu(D)}{4\pi}\int_{0}^{p}dkkh(k)\tanh\left(\pi k\right)+\frac{1}{\pi}\arg Z\left(\frac{1}{2}+ip\right) \quad (7)$$

Here $p=\sqrt{E-\frac{1}{4}}$, we can inmediatly see that the smooth part of (7) satisfy Weyl's law in dimension 2 $N_{smooth}(E)\approx\frac{\mu(D)}{4\pi}E$, the oscillatory part of (7) satisfy Bolte's semiclassical law [4] (page 34, theorem 2.10) $\frac{1}{\pi}\arg Z\left(\frac{\lambda}{2}+i\sqrt{E}\right)$ with $\lambda=1$, the branch of the logarithm inside (7) is chosen, so $\arg Z\left(\frac{1}{2}\right)=0$ in this case the Selberg Zeta function is the dynamical zeta function of a Quantum system and the Energies are related to the zeros of $Z(s)$.

2. A functional determinant for the Riemann Xi function $\xi(s)$

From the analogies between the Riemann Zeta function and the Selberg Zeta function, we could ask ourselves if there is a Hamiltonian operator (the simplest second order differential operator which has a classical and quantum meaning and it is well studied)in the form

$$H\Psi_{n}(x)=-\frac{d^{2}\Psi_{n}(x)}{dx^{2}}+V(x)\Psi_{n}(x)=E_{n}\Psi_{n}(x) \qquad \Psi_{n}(0)=0=\Psi_{n}(\infty) \qquad E_{n}=\gamma_{n}^{2} \qquad (8)$$

So for the Riemann Xi-function $\xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\xi(1-s)$ we have that $\xi\left(\frac{1}{2}+i\sqrt{E_{n}}\right)=0 \quad \forall n\in N$, the potential is given by $V(x)\begin{cases}f(x) & x>0\\ \infty & x\le 0\end{cases}$, at x=0 there is a infinite wall so the particle inside the well can not penetrate the region $x<0$. For the case of the Hamiltonian (8) the exact Eigenvalue staircase is [9]

$$N(E)=\sum_{n}H\left(E-E_{n}\right)=\frac{1}{\pi}\arg\xi\left(\frac{1}{2}+i\sqrt{E}\right)=1+\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+i\sqrt{E}\right)+\frac{\vartheta(\sqrt{E})}{\pi} \quad (9)$$

With $H(x)\begin{cases}1 & \text{x>0}\\ 0 & \text{x<0}\end{cases}$, $\vartheta(T)=\arg\Gamma\left(\frac{1}{4}+i\frac{T}{2}\right)-\frac{T}{2}\ln\pi\approx\frac{T}{2}\ln\left(\frac{T}{2\pi e}\right)-\frac{\pi}{8}+\frac{1}{48T}+...$

Also we will prove how the Riemann Xi function $\xi(s)$ is proportional to the functional determinant $\det\left(H-s(1-s)\right)$, and how the density of states can be evaluated from the argument of the Xi-function $E = p^2$

$$\frac{1}{2\pi p}\frac{d}{dp}\Im m \log\det\left(H + i\varepsilon - p\right) = \rho(E) = \sum_{\gamma_n}\delta\left(p^2 - \gamma_n^2\right)$$

As a simple example of how Quantum Mechanics can help to solve problems of finding the roots of functions , let be a particle moving inside an infinite potential well , the energy is given by $E = p^2$ and the one dimensional Schröedinger equation [7] in units $\hbar = 2m = 1$ ($\hbar$ is the reduced Planck's constant with value $\hbar = 1.05.10^{-34}\ J.T^{-1}$)

$$H_0 u_n(x) = -\frac{d^2 u_n(x)}{dx^2} + V(x)u_n(x) = E_n u_n(x) \quad u_n(0) = 0 = u_n(\pi) \quad E_n = n^2 \qquad (10)$$

$u_n(x) = A\sin\left(\pi x\right)$, in this case the Euler's product formula for the sine function is the quotient between 2 functional determinants

$$\frac{\sin\left(\pi\sqrt{x}\right)}{\pi\sqrt{x}} = \prod_{n=1}^{\infty}\left(1 - \frac{x}{E_n}\right) = \frac{\det\left(H_0 - x\right)}{\det\left(H_0\right)} \qquad H_0 = H_0^{\dagger} \qquad (11)$$

We can also compute the density of states to get the Poisson sum formula

$$\rho(E) = \sum_{n=1}^{\infty}\delta\left(E - E_n\right) = \frac{1}{2p}\left(\sum_n \delta\left(p-n\right) + \sum_n \delta\left(p-n\right)\right) = \frac{1}{2p}\sum_{n=-\infty}^{\infty} e^{2i\pi np} \qquad (11)$$

- *Zeta regularized determinant for $\xi(s)$:*

Given an Operator P with real Eigenvalues $\left\{E_n\right\}$, we can define its Zeta regularized determinant [6] in the form

$$\det\left(P + k^2\right) = \exp\left(-\frac{d}{ds}\zeta_P(s,k^2)\big|_{s=0}\right) \qquad (12)$$

Here $\zeta_P(s,k^2) = Tr\left\{(P+k^2)^{-s}\right\} = \sum_n \left(E_n + k^2\right)^{-s}$ is the Spectral Zeta function of the operator taken over all the Eigenvalues, the relationship between this spectral zeta function and the Theta function $\Theta(t) = \sum_n \exp(-tE_n)$,t>0 always , is given by the Mellin transform $\sum_{n=0}^{\infty}\frac{1}{\left(E_n + k^2\right)^s} = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{dt}{t}e^{-tk^2}\Theta(t)t^{s-1}$. If P is a Hamiltonian we can obtain the Theta function $\Theta(t) = \sum_n \exp(-tE_n)$ (approximately) by an integral over the Phase space [7]

$$\Theta(t)=\sum_{n=0}^{\infty}\exp\left(-tE_n\right)\approx\frac{1}{2\pi}\int_{-\infty}^{\infty}dp\int_0^{\infty}dxe^{-tp^2-tf(x)}=\frac{1}{2\sqrt{\pi t}}\int_0^{\infty}dxe^{-tf(x)}=\Theta_{WKB}(t) \qquad (13)$$

The expression (13) depends only on the momentum and the function $f(x)$ defined in (8) to evaluate the Theta function, if we combine (13) and the definition of the Theta function for the Eigenvalues

$$\Theta(t)=\sum_{n=0}^{\infty}\exp\left(-tE_n\right)=-s\int_0^{\infty}dtN(t)e^{-st}\approx\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_0^{\infty}dp\exp(-tp^2-tf(x)) \quad (14)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}dq\int_{-\infty}^{\infty}dp\exp(-tp^2-tf(x))=\frac{1}{2\sqrt{\pi t}}\int_0^{\infty}dxe^{-tf(x)}=\frac{1}{2\sqrt{\pi t}}\int_0^{\infty}dre^{-tr}\frac{dV^{-1}(r)}{dr} \quad (15)$$

From expressions (14) and (15) and setting $N(0)=0$ (after changes of variable)

$$\sqrt{s}\int_0^{\infty}dxN(x)e^{-sx}=\frac{1}{2\sqrt{\pi}}\int_0^{\infty}dxf^{-1}(x)e^{-sx} \qquad \to f^{-1}(x)=2\sqrt{\pi}\frac{d^{1/2}}{dx^{1/2}}N(x) \qquad (16)$$

To prove (16) we have used the properties of the integral representation for the inverse Laplace transform

$$D^{\alpha}f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}dsF(s)e^{st}s^{\alpha} \qquad D^{\alpha}e^{kt}=k^{\alpha}e^{kt} \quad \forall\alpha\in R \quad (17)$$

And the fact that if two Laplace transforms are equal then $L\{f(t)\}=L\{g(t)\}$ implies that $f(t)=g(t)$, for the case of the Riemann Zeros $N(E)=\frac{1}{\pi}\arg\xi\left(\frac{1}{2}+i\sqrt{E}\right)$ (Bolte's semiclassical law in one dimension) so $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$, since we want our potential inside (8) to be positive whenever we take the inverse we must choose the POSITIVE branch of the inverse in order to get $f(x)\ge 0$ on the interval $[0,\infty)$, the half derivative and the half integral for any well behaved function are given in [13]

$$\frac{d^{1/2}f(x)}{dx^{1/2}}=\frac{1}{\Gamma(1/2)}\frac{d}{dx}\int_0^x\frac{dtf(t)}{\sqrt{x-t}} \qquad \frac{d^{-1/2}f(x)}{dx^{1/2}}=\frac{1}{\Gamma(1/2)}\int_0^x dt\frac{f(t)}{\sqrt{x-t}} \quad (18)$$

We have written implicitly the potential inside (8) , if the function $f(x)$ is defined by the functional equation $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)=2\sum_n\frac{H(x-\gamma_n^2)}{\sqrt{x-\gamma_n^2}}$, then we may evaluate the Spectral Zeta function of the Quantum system given in (8), then

$$\frac{\det\left(H+z^2\right)}{det\left(H\right)}=\exp\left(-\frac{d}{ds}\zeta_P(s,z^2)\big|_{s=0}+\frac{d}{ds}\zeta_P(s,0)\big|_{s=0}\right)=\frac{\xi(z+1/2)}{\xi(1/2)} \quad (19)$$

For the potential defined by $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$, we can evaluate the Theta kernel uisng (15) and (16) $\Theta(t)=\sum_n \mathrm{e}^{-tE_n}=\frac{1}{2\sqrt{\pi t}}\int_0^\infty dx\frac{df^{-1}(x)}{dx}e^{-tx}$, for this potential the spectral theta function and its derivative are

$$\zeta_H(s,z^2)=\sum_{n=0}^{\infty}\frac{1}{\left(\gamma_n^{\ 2}+z^2\right)^s} \qquad -\frac{d}{ds}\zeta_H(0,z^2)=\sum_{n=0}^{\infty}\ln\left(\gamma_n^{\ 2}+z^2\right) \qquad \zeta\left(\frac{1}{2}+i\gamma_n\right)=0 \quad (20)$$

Taking exponentials we reach to the infinite product for the Riemann Xi-function as an spectral determinant (functional determinant over the Eigenvalues of H)

$$\frac{\det\left(H+z^2\right)}{\det(H)}=\frac{\prod_{n=0}^{\infty}(\gamma_n^2+z^2)}{\prod_{n=0}^{\infty}\gamma_n^2}=\prod_{n=0}^{\infty}\left(1+\frac{z^2}{E_n}\right)=\frac{\xi\left(1/2+z\right)}{\xi\left(1/2\right)} \quad (21)$$

If we choose the positive branch $f(x)\ge 0$ of the inverse $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$ then the potential will be always positive so the Energies of the Hamiltonian inside (8) will be all positive $E_n=\gamma_n^2\in R^+$, then all the non-trivial zeros of the Riemann Zeta function will be on the critical line $\mathrm{Re}(s)=\frac{1}{2}$, with a simple change of variable $z=s-\frac{1}{2}$ we obtain

$$\frac{\xi(s)}{\xi(0)}=\frac{\det\left(H-s(1-s)+\frac{1}{4}\right)}{\det\left(H+\frac{1}{4}\right)}=\frac{\xi(1-s)}{\xi(0)}=\prod_\rho\left(1-\frac{s}{\rho}\right) \quad (22)$$

Equation (22) is the Hadamard product for the Riemann Xi-function in terms of the quotient of 2 functional determinants, since the expected value of the Hamiltonian is positive $\langle\psi_n|H|\psi_n\rangle\ge 0$ and Hermitian ,with $f(x)\ge 0$ then all the Energies are positive $E_n=s(1-s)\in R^+$ Riemann Hypothesis should hold. If se set $s=\frac{1}{2}+i\sqrt{E}$, then is it clear that the roots of the functional determinant $\det(E-H)$ are the roots of the function $\xi\left(\frac{1}{2}+i\sqrt{E}\right)$

- *Bohr-Sommerfeld quantization condition and the square of the Riemann zeros:*

The expression $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$ could also be obtained from the Bohr-Sommerfeld quantization conditions [7]

$$\int_C pdq=2\pi\left(n+\frac{1}{2}\right) \qquad 2\int_0^a dx\sqrt{E-f(x)}=p(x) \qquad E=f(a) \qquad (23)$$

'a' is the classical turning point, $n=N(E)$ is the Eigenvalue staircase, the first integral inside (23) is a line integral taken over the closed orbit of the classical system, equation (23) can be understood as an integral equation for the inverse of the potential in the form

$$2\pi\left(\frac{1}{2}+n(E)\right)=2\int_0^{a=a(E)}\sqrt{E-V(x)}dx=2\int_0^E\sqrt{E-x}\,\frac{df^{-1}}{dx}=\sqrt{\pi}D_x^{-1/2}f(x) \quad (24)$$

If we take the half derivative on both sides of (24) we would get $f^{-1}(x)=2\sqrt{\pi}\,\frac{d^{1/2}}{dx^{1/2}}\left(\frac{1}{2}+\frac{1}{\pi}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)\right)$ in this case this result is completely equivalent to the one we got by Zeta regularization and by the WKB approximation of the Theta function $\frac{1}{2\sqrt{\pi t}}\int_0^\infty dxe^{-tf(x)}=\Theta_{WKB}(t)$.

In order to evaluate the inverse of the potential $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$ we would need to evaluate $\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+i\sqrt{x}\right)$, this can be made using the Riemann-Siegel formula [10] to evaluate the zeta function on the critical line

$$Z(k)=\zeta\left(\frac{1}{2}+ik\right)e^{i\vartheta(k)}=2\sum_{n=1}^{U(k)}\frac{\cos\left(\vartheta(k)-k\ln n\right)}{\sqrt{n}}+O\left(\frac{1}{k^{1/4}}\right) \quad k\to\infty \quad (25)$$

The functions inside (25) are $u(k)=\left[\sqrt{\frac{k}{2\pi}}\right]$, $[x]$ is the floor function and

$$\vartheta(T)=\arg\Gamma\left(\frac{1}{4}+i\frac{T}{2}\right)-\frac{T}{2}\ln\pi\approx\frac{T}{2}\ln\left(\frac{T}{2\pi e}\right)-\frac{\pi}{8}+\frac{1}{48T}+...$$

From equation (24) the density of states could be evaluated as $\frac{1}{2\sqrt{\pi}}\frac{d^{1/2}f^{-1}(x)}{dx^{1/2}}=\rho(x)=\sum_n\delta\left(x-\gamma_n^2\right)$, the density of states or trace of

$Tr\left\{\delta\left(E-f(x)\right)\right\}$ depends on the half-derivative of the inverse of the potential for the Hamiltonian, we will prove in the next section that this density of states reproduces a distributional version of the Riemann-Weil explicit formula

- *Riemann Weil explicit formula as the Trace* $Tr\left\{\delta\left(E-f(x)\right)\right\}$ *:*

The next question is to compute the density of states for the Hamiltonian desfined in (8) , let be the property of the delta function $p=\sqrt{E}$

$\delta\left(E-\gamma^2\right)=\dfrac{\delta(p-\gamma)+\delta(p+\gamma)}{2p}$, if we use Shokhotsky's formula for the delta function $\dfrac{1}{-\pi}\lim_{\varepsilon\to 0}\Im m\left(\dfrac{1}{x-a+i\varepsilon}\right)=\delta(x-a)$, the density of states $Tr\left\{\delta\left(E-f(x)\right)\right\}$

$$-\frac{1}{2\pi\sqrt{E}}\frac{d}{dE}\arg\xi\left(\frac{1}{2}+i\varepsilon+i\sqrt{E}\right)=\sum_{\gamma}\delta\left(E-\gamma^2\right)=\sum_{\gamma}\delta\left(p^2-\gamma^2\right)=$$
$$\frac{1}{\pi}\frac{\zeta}{\zeta}\left(\frac{1}{2}+ip\right)\frac{1}{2p}+\frac{1}{\pi}\frac{\zeta'}{\zeta}\left(\frac{1}{2}-ip\right)\frac{1}{2p}-\frac{\ln\pi}{2\pi p}+\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+i\frac{p}{2}\right)\frac{1}{4\pi p}+ \qquad (26)$$
$$\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}-i\frac{p}{2}\right)\frac{1}{4\pi p}+\frac{\delta\left(p-\frac{i}{2}\right)+\pi\delta\left(p+\frac{i}{2}\right)}{2p}=\rho(E)$$

Here $\dfrac{1}{-\pi}\lim_{\varepsilon\to 0}\Im m\left(\dfrac{2}{2x\pm i+2i\varepsilon}\right)=\delta\left(x\pm\dfrac{i}{2}\right)$, this factor comes from the logarithmic derivative of $s(s-1)$ along the critical line $s=\dfrac{1}{2}+ip$, equation (26) is a distributional version of the Riemann-Weil trace formula , taking formally the logarithm of the Euler product for the Riemann Zeta function on the critical line yields to $\sum_{n=1}^{\infty}\dfrac{\Lambda(n)}{\sqrt{n}}e^{-ip\ln n}=_{reg}-\dfrac{\zeta'}{\zeta}\left(\dfrac{1}{2}+ip\right)$, using two test functions h(x) and g(x) $g(x)=\dfrac{1}{\pi}\int_0^{\infty}dr\cos(rx)h(r)$ we recover the oscillatory part of the Riemann-Weil trace formula $-2\sum_{n=1}^{\infty}\dfrac{\Lambda(n)}{\sqrt{n}}g(\ln n)$.

In general the formula (26) for the density of states can be obtained by taking the Laplace transform of the Theta function $\Theta(t)=\sum_{n}\exp(-tE_n)$, in our case the WKB Theta function $\dfrac{1}{2\sqrt{\pi t}}\int_0^{\infty}dxe^{-tf(x)}=\Theta_{WKB}(t)$ with the potential

$f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$ is equal to $\Theta(t)=\sum_n \exp(-tE_n)$, if we use the two identities $-\frac{1}{\pi}\lim_{\varepsilon\to 0}\Im m\left(\frac{1}{x+i\varepsilon}\right)=\delta\left(x\right)$ and $s\int_0^\infty dt\exp\left(-st\right)=1$ then

$$\frac{1}{\pi}\Im m\int_0^\infty dt e^{t\left(i\varepsilon+E\right)}\Theta(t)=\rho(E)=\frac{1}{2\pi p}\frac{d}{dp}Arg\xi\left(\frac{1}{2}+ip+\varepsilon\right)=\sum_n\delta\left(p^2-\gamma_n^2\right) \quad (27)$$

With $\varepsilon\to 0$ and $E=p^2$

Unlike the model of Wu and Sprung, we have considered also the oscillatory part of the Riemann Eigenvalue Staircase $\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+i\sqrt{E}\right)$, which satisfy Bolte's semiclassical law , Wu and Sprung [17] considered only the smooth part of teh Eigenvalue staircase in the limit $T>>1$ $\frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right)\approx N(T)$ in order to get a Hamiltonian whose Energies are the positive imaginary part of the Riemann Zeros, their starting point is the Harmonic oscillator [15] , but unlike the normal quantum mechanical oscillator whose functional determinant gives the Gamma function $\frac{\sqrt{2\pi}}{\Gamma(s)}=\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)$ the product taken ONLY over the positive imaginary part of the zeros (even if it converges) $\prod_{n=0}^{\infty}\left(1+\frac{s}{\gamma_n}\right)$ has no meaning, also the Wu-Sprung model doesn't obey Weyl's law in one dimension mainly $N_{smooth}(E)=O\left(E^{d/2}\right)$, in our case , the Hamiltonian (8) with the Smooth part of the Eigenvalue staircase $N(E)\approx\frac{\sqrt{E}}{2\pi}\log\left(\frac{\sqrt{E}}{2\pi e}\right)$, satisfies a Weyl's law with $d=1+\frac{\varepsilon}{2}$ and the spectral determinant (quotient) $\frac{\Delta(E)}{\Delta(0)}=\prod_{n=0}^{\infty}\left(1-\frac{E}{E_n}\right)$ $E_n=\gamma_n^2$ is proportional to the Riemann xi function on the critical line $\xi\left(\frac{1}{2}+i\sqrt{E}\right)$

By analogy with the zeros of the Selberg Zeta function, is better to consider the case with the Energies $E_n=\gamma_n^2$, in this case the Trace of the Resolvent of the Hamiltonian $\left(E+i\varepsilon-H\right)^{-1}$ is the Riemann-Weil trace for the Riemann zeros.

- *Analytic and asymptotic expressions for the potential* $f(x)$:

From the expression for the fractional derivative of powers $\frac{d^k x^\lambda}{dx^k}=\frac{\Gamma\left(\lambda+1\right)}{\Gamma\left(\lambda-k+1\right)}x^{\lambda-k}$, we can obtain for the inverse function

$$f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)=2\sum_{\gamma>0}\frac{H(x-\gamma^2)}{\sqrt{x-\gamma^2}} \quad (28)$$

Using the Riemann-Weil formula we can rewrite (28) as

$$f^{-1}(x)=\frac{4}{\sqrt{4x+1}}+\frac{1}{2\pi}\int_{-\sqrt{x}}^{\sqrt{x}}\frac{dr}{\sqrt{x-r^2}}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right)-\ln\pi\right)-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}J_0\left(\sqrt{x}\ln n\right) \quad (29)$$

Here $g(u=\ln n,x)=\frac{1}{\pi}\int_0^{\sqrt{x}}\frac{\cos(ut)}{\sqrt{x-t^2}}dt=\frac{J_0\left(u\sqrt{x}\right)}{2}$, here the integral can be expressed in terms of the zeroeth order Bessel function.

a final question is could the expression $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$ be inverted to get $f(x)$ at least asymptotitcally in the limit $x\to\infty$? ,the smooth part of the Eigenvalue staircase is given by $N(E)\approx\frac{\sqrt{E}}{2\pi}\log\left(\frac{\sqrt{E}}{2\pi e}\right)$, $e=\sum_{n=0}^{\infty}\frac{1}{n!}$, if we use the expression for the logarithm $\log(x)\approx\frac{x^{\varepsilon}-1}{\varepsilon}$ as $\varepsilon\to 0$ and apply the half derivative expression, then the following holds $\varepsilon\to 0$

$$f^{-1}{}_{smooth}(x)\approx\frac{\left(4\pi^2e^2\right)^{-\varepsilon/2}A(\varepsilon)x^{\varepsilon/2}-B}{\sqrt{\pi}\varepsilon} \qquad f_{smooth}(x)\approx 4\pi^2e^2\left(\frac{\varepsilon\sqrt{\pi}x+B}{A(\varepsilon)}\right)^{\frac{2}{\varepsilon}} \quad (30)$$

$A(\varepsilon)=\frac{\Gamma\left(\frac{3+\varepsilon}{2}\right)}{\Gamma\left(1+\frac{\varepsilon}{2}\right)}$ and $B=\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}$, the second expression inside (30) is the asymptotic of $f(x)$ as $x\to\infty$, for this potential , the energies inside (8) are

$$E^{smooth}{}_n=f(n)=N^{-1}{}_{smooth}(E)\approx\frac{4\pi^2n^2}{W^2\left(ne^{-1}\right)} \qquad W(x)=\sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}x^n \quad (31)$$

The function $W(x)$ is the Lambert function (principal branch), more than in the potential $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)\in R$ we are interested in the Theta function $\Theta(t)=\sum_n\exp(-t\gamma_n^2)$, if we use the semiclassical Theta function as an integral over the Phase space and introduce the potential given by $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)\in R$ one obtains , $\sqrt{a}\int_{-\infty}^{\infty}dxe^{-ax^2}=\sqrt{\pi}$

$$\Theta_{WKB}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dp\int_{0}^{\infty}dxe^{-tH(x,p)}=\frac{1}{2\sqrt{\pi t}}\int_{0}^{\infty}dxe^{-tf(x)}=\sqrt{\frac{t}{\pi}}\int_{0}^{\infty}dre^{-tr}f^{-1}(r)=$$
$$\sqrt{\frac{t}{\pi}}\sum_{\gamma}\int_{0}^{\infty}dre^{-tr}\frac{H(r-\gamma_n^2)}{\sqrt{r-\gamma_n^2}}=\sqrt{\frac{t}{\pi}}\left(\int_{-\infty}^{\infty}e^{-tx^2}dx\right)\sum_{\gamma}e^{-t\gamma_n^2}=\Theta(t)=\sum_{n}\exp(-t\gamma_n^2) \quad (32)$$

In (32) we have obtained the Heat function $\Theta(t)=\sum_{n}\exp(-t\gamma_n^2)$, from the potentiall function $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)\in R$ of course to be correct we must take the smooth and the oscillatory part of the Eigenvalue staircase $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\left(N_{smooth}(x)+N_{osc}(x)\right)$ otherwise the description will be not complete as in the Wu-Sprung potential [17] , from this Theat Kernel $\Theta(t)=\sum_{n}\exp(-t\gamma_n^2)=\int_{-\infty}^{\infty}dp\int_{-\infty}^{\infty}dpe^{-tH(x,p)}$ here t>0 and the Hamiltonian has been defined in (8) using the Zeta regularization method for the determinant

$$-\partial_s\zeta_H(s,z(z-1))=\ln\det\left(H-z(1-z)\right) \quad \zeta_H(s,z(z-1))=Tr\left\{\left(H+z(z-1)\right)^{-s}\right\} \quad (33)$$

$\zeta_H(s,z(z-1))=\sum_{n}\frac{1}{\left(\frac{1}{4}+z(z-1)+\gamma_n^2\right)^s}$, the zeros of the determinant $\det\left(H-z(1-z)\right)$ with H an Hermitian operator are the zeros of $\xi(z)$

- *Numerical evaluation of the functional determinant* :

We need to evaluate the half-derivative inside the inverse of the potential $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$, to do so we can use the Grunwald-Letnikov formula [13] with an step $\varepsilon=0.01$ and $q=\frac{1}{2}$

$$\frac{\Delta^q g(x)}{\varepsilon^q}\approx\frac{d^{1/2}g(x)}{dx^{1/2}}\approx\frac{1}{\varepsilon^q}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(q+1)}{\Gamma(n+1)\Gamma(q-n+1)}g\left(x+(q-n)\varepsilon\right) \quad (34)$$

For the case of the functional determinant of our Hamiltonian operator with the potential $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$ defined as

$$\left(-\frac{d^2}{dx^2}+\frac{1}{4}+f(x)-\lambda\right)y(x,\lambda)=0 \qquad y(0,\lambda)=0=y(L,\lambda) \qquad L\to\infty \quad (35)$$

$\lambda=s(1-s)$, to evaluate the functional determinant by the Gelfand-Yaglom method [18] we need to solve the initial value problem

$$\left(-\frac{d^2}{dx^2}+\frac{1}{4}+f(x)-\lambda\right)y(x,\lambda)=0 \qquad y(0,\lambda)=0 \qquad \frac{dy(0,\lambda)}{dx}=1 \quad (36)$$

Unfortunately exact solutions to (36) can not be found , in the WKB approximation (36) has the solution

$$y(x,\lambda)\approx\frac{1}{\Pi(x)^{1/2}}\left(C_+\exp\int_0^x\Pi(t)dt+C_-\exp-\int_0^x\Pi(t)dt\right) \qquad \Pi(x)=\sqrt{f(x)+\frac{1}{4}-\lambda} \quad (37)$$

The 2 constants $C_\pm$ are chosen so (37) solves the initial value problem (36).

The Gelfand-Yaglom theorem [18] tells us that the functional determinant is related to the solution of the initial value problem (36) in the form

$$\lim_{L\to\infty}\frac{y(L,\lambda)}{y(L,0)}=\frac{\det\left(H+\frac{1}{4}-s(1-s)\right)}{\det\left(H+\frac{1}{4}\right)}=\frac{\xi(s)}{\xi(0)}=\prod_\rho\left(1-\frac{s}{\rho}\right) \qquad \lambda=s(1-s) \qquad (38)$$

The main advantage of the Gelfand-Yaglom method , is that we do not need to evaluate any single eigenvalue in order to obtain the functiona determinant $\det\left(H+\frac{1}{4}-\lambda\right)$, unfortunately this method is only valid for ordinary differential equations

TABLE1 : comparison between the Riemann Zeros (square) from the tables of Odlyzko and the Numerical values of the energies for our Hamiltonian operator (8) with $f^{-1}(x)=\frac{2}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$, to obtain numerically the potential we have used formula (34) to evaluate the fractional derivative and the Riemann-Siegel formula (25) to evaluate $S(T)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right)$

n	Zeros (square)	Energies
0	199.7897	198.7886
1	441.9244	441.9240
2	625.5401	625.5406
3	925.6684	925.6683
4	1084.7142	1084.7139
5	1412.7149	1412.7146
6	1674.3400	1674.3398
7	1877.2289	1877.2287
8	2304.4896	2304.4893
9	6363.8591	6363.8589

We can also test our formula $f^{-1}(x) = 2\sqrt{\pi}\,\frac{d^{1/2}n(x)}{dx^{1/2}}$ with the potentials x^n $n = 1, 2, \infty$ (an infinite potential well is assumed at the point $x = 0$) these are the cases of the linear potential (bouncer) , Harmonic oscillator and infinite potential well

$$f(x) = \frac{(\omega x)^2}{4} \qquad N(E) = \frac{E}{2\omega} \qquad f^{-1}(x) = \frac{2\sqrt{E}}{\omega} \qquad (39)$$

$$f(x) = kx \qquad N(E) = \frac{2E^{3/2}}{3\pi k} \qquad f^{-1}(x) = \frac{x}{k} \qquad (40)$$

$$f(x) = x^n \qquad N(E) = \frac{1}{\sqrt{4\pi}} \cdot \frac{\Gamma\left(\frac{1}{m}+1\right)}{\Gamma\left(\frac{1}{m}+\frac{3}{2}\right)} E^{\frac{1}{m}+\frac{1}{2}} \qquad f^{-1}(x) = x^{\frac{1}{n}} \qquad n \to \infty \qquad (41)$$

We can compare these results with equation (30) for the smooth part of the Riemann zeros potential obtained by our WKB quantization method.

References

[1] Abramowitz, M. and Stegun, I. A. (Eds.). "Riemann Zeta Function and Other Sums of Reciprocal Powers." §23.2 in "*Handbook of Mathematical Functions*". New York: Dover, pp. 807-808, 1972.

[2] Apostol Tom "*Introduction to Analytic Number theory" ED: Springuer-Verlag ",* (1976)

[3] Berry M. Keating J.P " The Riemann Zeros and Eigenvalue Asymptotics " Siam Review archive Vol. 41 Issue 2, June 1999 pages: 236-266

[4] Bolte J: "*Semiclassical trace formulae and eigenvalue statistics in quantum chaos",* Open Sys. & Information Dyn.**6** (1999) 167-226 , avaliable at: http://arxiv.org/abs/chao-dyn/9702003

[5] Cartier P . and Voros A. " Une nouvelle interpretation de la formule des traces de Selberg. In P. Cartier, L. Illusie, N. M. Katz, G. Laumon, Y. Manin, and K. A. Ribet, editors, The Grothendieck Festschrift (vol. II), volume 87 of Progress in Mathematics, pages 1–67. Birkh¨auser, 1990.

[6] Elizalde E. , Odintsov S. ,Romeo A. , Bytsenko A. , Zerbini S "Zeta Regularization Techniques With Applications" ISBN-10: 9810214413 Ed: World Scientific pub (1994)

[7] Griffiths, David J. (2004). "*Introduction to Quantum Mechanics"* Prentice Hall. ISBN 0-13-111892-7.

[8] Grosche " *Path Integrals , Hyperbolic Spaces and Selberg trace formulae"* (1996) ISBN-10: **9810224311**

[9] Gutzwiller, M. "*Chaos in Classical and Quantum Mechanics"*, (1990) Springer-Verlag, New York ISBN=0-387-97173-4.

[10] Hejhal D. "*The Selberg Trace formula and the Riemann Zeta function "* Duke Mathematical Journal 43 (1976)

[11] Hurt N. **"**Quantum Chaos and Mesoscopic Systems: Mathematical Methods in the Quantum Signatures of Chaos" (2010) **ISBN-13:** 978-9048148110

[12] Marklof J. " *Selberg Trace formula: An introduction'* PDF avaliable at http://arxiv.org/abs/math/0407288

[13] Nishimoto, K. "*Fractional Calculus".* New Haven, CT: University of New Haven Press, 1989.

[14] Odlyzko A. " Tables of Zeros of Riemann Zeta functions" ,webpage http://www.dtc.umn.edu/~odlyzko/zeta_tables/

[15] Selberg, Atle (1956), "Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series", *J. Indian Math. Soc. (N.S.)* **20**: 47–87, MR0088511

[16] Voros A. " *Exercises in exact quantization"* e-print avaliable at http://arxiv.org/pdf/math-ph/0005029v2.pdf

[17] Wu H. and D. W. L. Sprung "Riemann zeros and a fractal potential" Phys. Rev. E 48, 2595–2598 (1993)

[18] Yaglom, A.M Gelfand I .M "Integration in functional spaces and its applications in quantum physics," J. Math. Phys. 1, 48–69 1960.