

A FUNCTIONAL DETERMINANT EXPRESSION FOR THE RIEMANN XI-FUNCTION

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- **ABSTRACT:** We give an interpretation of the Riemann Xi-function $\xi(s)$ as the quotient of two functional determinants of an Hermitian Hamiltonian $H = H^\dagger$. To get the potential of this Hamiltonian we use the WKB method to approximate and evaluate the spectral Theta function $\Theta(t) = \sum_n \exp(-t\gamma_n^2)$ over the Riemann zeros on the critical strip $0 < \text{Re}(s) < 1$. Using the WKB method we manage to get the potential inside the Hamiltonian H , also we evaluate the functional determinant $\det(H + z^2)$ by means of Zeta regularization, we discuss the similarity of our method to the method applied to get the Zeros of the Selberg Zeta function
- **Keywords:** = Riemann Hypothesis, Functional determinant, WKB semiclassical Approximation, Trace formula, Bolte's law, Quantum chaos.

1. Riemann Zeta function and Selberg Zeta function

Let be a Riemann Surface with constant negative curvature and the modular group $PSL(2, R)$, Selberg [14] studied the problem of the Laplacian

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Psi_n(x, y) = E_n \Psi_n(x, y) \quad E_n = \frac{1}{4} + k_n^2 \quad (1)$$

These momenta k_n are the non-trivial zeros of the Selberg Zeta function, which can be defined by an Euler product over the Geodesic of the surface in an analogy with the Riemann Zeta function

$$\zeta(s) = \prod_n \frac{1}{(1 - p_n^{-s})} \quad Z(s) = \prod_P \prod_{k=0}^{\infty} (1 - N(P)^{-(s+k)}) \quad (2)$$

Selberg also studied a Trace formula which relates the Zeros (momenta of the Laplacian Δ) on the critical line $Z\left(\frac{1}{2} + ik_n\right) = 0$ and the length of the Geodesic of the Surface in the form

$$\sum_n h(k_n) = \frac{\mu(D)}{4\pi} \int_0^\infty dk k h(k) \tanh(\pi k) + \sum_{P \in p.p.o} \frac{\ln N(P)}{N(P)^{1/2} - N(P)^{1/2}} g(\ln N(P)) \quad (3)$$

Here, p.p.o means that we are taking the sum over the length of the Geodesic, $h(k)$ is a test function and $g(k)$ is the Fourier cosine transform of $h(k)$

$$g(k) = \frac{1}{2\pi} \int_0^\infty dx h(x) \cos(kx) \quad \mu(D) \text{ is the area of the fundamental domain describing}$$

the Riemann surface . In case we had a surface with the length of the Geodesic $\ln N(P) = \ln p$ for 'p' on the second side of the equation a prime number, then the Selberg Trace is very similar to the Riemann-weil sum formula [12]

$$\sum_\gamma h(\gamma) = 2h\left(\frac{i}{2}\right) - g(0) \ln \pi - 2 \sum_{n=1}^\infty \frac{\Lambda(n)}{\sqrt{n}} g(\ln n) + \frac{1}{2\pi} \int_{-\infty}^\infty ds h(s) \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{is}{2}\right) \quad (4)$$

This formula (4) related a sum over the imaginary part of the Riemann zeros to another sum over the primes, here $\Lambda(n) \begin{cases} \ln p & n = p^k \\ 0 & \text{otherwise} \end{cases}$ with 'k' a positive

integer is the Mangoldt function, in case $\ln N(P) = \ln p$ both zeta function of Selberg and Riemann are related by $\frac{1}{Z(s)} = \prod_{n=0}^\infty \zeta(n+s)$ and their logarithmic

derivative is quite similar if we set the function $\Lambda_{geodesic}(P) = \frac{\ln N(P)}{1 - N(P)^{-1}}$

$$\frac{\zeta'}{\zeta}(s) = - \sum_{n=1}^\infty \Lambda(n) n^{-s} \quad \frac{Z'}{Z}(s) = \sum_{P \in p.p.o} \Lambda_{geodesic}(P) N(P)^{-s} \quad (5)$$

In both cases the Riemann and Selberg zeta functions obey a similar functional equation which relates the value at s and 1-s

$$Z(1-s) = \exp\left(-\frac{\mu(D)}{4\pi} \int_0^{s-1/2} v \tan(\pi v) dv + c\right) Z(s) \quad \zeta(1-s) = X(s) \zeta(s) \quad (6)$$

The constant of integration 'c' is determined by setting $s = 1/2$, and

$$X(s) = 2(2\pi)^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \text{ for the case of the Riemann zeta function.}$$

With the aid of the Selberg Trace formula (3) , we can evaluate the Eigenvalue staircase for the Laplacian $\Delta = -y^2(\partial_x^2 + \partial_y^2)$

$$N\left(E = \frac{1}{4} + p^2\right) = \sum_{E_n \leq E} 1 = \sum_n \frac{\mu(D)}{4\pi} \int_0^p dk kh(k) \tanh(\pi k) + \frac{1}{\pi} \arg Z\left(\frac{1}{2} + ip\right) \quad (7)$$

Here $p = \sqrt{E - \frac{1}{4}}$, we can immediately see that the smooth part of (7) satisfy

Weyl's law in dimension 2 $N_{smooth}(E) \approx \frac{\mu(D)}{4\pi} E$, the oscillatory part of (7) satisfy

Bole's semiclassical law [4] (page 34, theorem 2.10) $\frac{1}{\pi} \arg Z\left(\frac{\lambda}{2} + i\sqrt{E}\right)$ with

$\lambda = 1$, the branch of the logarithm inside (7) is chosen, so $\arg Z\left(\frac{1}{2}\right) = 0$ in this

case the Selberg Zeta function is the dynamical zeta function of a Quantum system and the Energies are related to the zeros of $Z(s)$.

2. A functional determinant for the Riemann Xi function $\xi(s)$

From the analogies between the Riemann Zeta function and the Selberg Zeta function, we could ask ourselves if there is a Hamiltonian operator (the simplest second order differential operator which has a classical and quantum meaning and it is well studied) in the form

$$H\Psi_n(x) = -\frac{d^2\Psi_n(x)}{dx^2} + V(x)\Psi_n(x) = E_n\Psi_n(x) \quad \Psi_n(0) = 0 = \Psi_n(\infty) \quad E_n = \gamma_n^2 \quad (8)$$

So for the Riemann Xi-function $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(1-s)$ we have

that $\xi\left(\frac{1}{2} + i\sqrt{E_n}\right) = 0 \quad \forall n \in \mathbb{N}$, the potential is given by $V(x) \begin{cases} f(x) & x > 0 \\ \infty & x \leq 0 \end{cases}$, at

$x=0$ there is a infinite wall so the particle inside the well can not penetrate the region $x < 0$. For the case of the Hamiltonian (8) the exact Eigenvalue staircase is [9]

$$N(E) = \sum_n H(E - E_n) = \frac{1}{\pi} \arg \xi\left(\frac{1}{2} + i\sqrt{E}\right) = 1 + \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + i\sqrt{E}\right) + \frac{\vartheta(\sqrt{E})}{\pi} \quad (9)$$

With $H(x) \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$, $\vartheta(T) = \arg \Gamma\left(\frac{1}{4} + i\frac{T}{2}\right) - \frac{T}{2} \ln \pi \approx \frac{T}{2} \ln\left(\frac{T}{2\pi e}\right) - \frac{\pi}{8} + \frac{1}{48T} + \dots$

Also we will prove how the Riemann Xi function $\xi(s)$ is proportional to the functional determinant $\det(H - s(1-s))$, and how the density of states can be

evaluated from the argument of the Xi-function $E = p^2$

$$\frac{1}{2\pi p} \frac{d}{dp} \Im m \log \det(H + i\varepsilon - p) = \rho(E) = \sum_{\gamma_n} \delta(p^2 - \gamma_n^2)$$

As a simple example of how Quantum Mechanics can help to solve problems of finding the roots of functions, let be a particle moving inside an infinite potential well, the energy is given by $E = p^2$ and the one dimensional Schrödinger equation [7] in units $\hbar = 2m = 1$ (\hbar is the reduced Planck's constant with value $\hbar = 1.05.10^{-34} \text{ J.T}^{-1}$)

$$H_0 u_n(x) = -\frac{d^2 u_n(x)}{dx^2} + V(x)u_n(x) = E_n u_n(x) \quad u_n(0) = 0 = u_n(\pi) \quad E_n = n^2 \quad (10)$$

$u_n(x) = A \sin(\pi x)$, in this case the Euler's product formula for the sine function is the quotient between 2 functional determinants

$$\frac{\sin(\pi\sqrt{x})}{\pi\sqrt{x}} = \prod_{n=1}^{\infty} \left(1 - \frac{x}{E_n}\right) = \frac{\det(H_0 - x)}{\det(H_0)} \quad H_0 = H_0^\dagger \quad (11)$$

We can also compute the density of states to get the Poisson sum formula

$$\rho(E) = \sum_{n=1}^{\infty} \delta(E - E_n) = \frac{1}{2p} \left(\sum_n \delta(p - n) + \sum_n \delta(p + n) \right) = \frac{1}{2p} \sum_{n=-\infty}^{\infty} e^{2i\pi np} \quad (11)$$

o *Zeta regularized determinant for $\zeta(s)$:*

Given an Operator P with real Eigenvalues $\{E_n\}$, we can define its Zeta regularized determinant [6] in the form

$$\det(P + k^2) = \exp\left(-\frac{d}{ds} \zeta_P(s, k^2) \Big|_{s=0}\right) \quad (12)$$

Here $\zeta_P(s, k^2) = \text{Tr}\{(P + k^2)^{-s}\} = \sum_n (E_n + k^2)^{-s}$ is the Spectral Zeta function of the operator taken over all the Eigenvalues, the relationship between this spectral zeta function and the Theta function $\Theta(t) = \sum_n \exp(-tE_n)$, $t > 0$ always, is given by

the Mellin transform $\sum_{n=0}^{\infty} \frac{1}{(E_n + k^2)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{dt}{t} e^{-tk^2} \Theta(t) t^{s-1}$. If P is a Hamiltonian

we can obtain the Theta function $\Theta(t) = \sum_n \exp(-tE_n)$ (approximately) by an integral over the Phase space [7]

$$\Theta(t) = \sum_{n=0}^{\infty} \exp(-tE_n) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \int_0^{\infty} dx e^{-tp^2 - tf(x)} = \frac{1}{2\sqrt{\pi t}} \int_0^{\infty} dx e^{-tf(x)} = \Theta_{WKB}(t) \quad (13)$$

The expression (13) depends only on the momentum and the function $f(x)$ defined in (8) to evaluate the Theta function, if we combine (13) and the definition of the Theta function for the Eigenvalues

$$\Theta(t) = \sum_{n=0}^{\infty} \exp(-tE_n) = -s \int_0^{\infty} dt N(t) e^{-st} \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_0^{\infty} dp \exp(-tp^2 - tf(x)) \quad (14)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp \exp(-tp^2 - tf(x)) = \frac{1}{2\sqrt{\pi t}} \int_0^{\infty} dx e^{-tf(x)} = \frac{1}{2\sqrt{\pi t}} \int_0^{\infty} dr e^{-tr} \frac{dV^{-1}(r)}{dr} \quad (15)$$

From expressions (14) and (15) and setting $N(0) = 0$ (after changes of variable)

$$\sqrt{s} \int_0^{\infty} dx N(x) e^{-sx} = \frac{1}{2\sqrt{\pi}} \int_0^{\infty} dx f^{-1}(x) e^{-sx} \quad \rightarrow \quad f^{-1}(x) = 2\sqrt{\pi} \frac{d^{1/2}}{dx^{1/2}} N(x) \quad (16)$$

To prove (16) we have used the properties of the integral representation for the inverse Laplace transform

$$D^\alpha f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds F(s) e^{st} s^\alpha \quad D^\alpha e^{kt} = k^\alpha e^{kt} \quad \forall \alpha \in \mathbb{R} \quad (17)$$

And the fact that if two Laplace transforms are equal then $L\{f(t)\} = L\{g(t)\}$ implies that $f(t) = g(t)$, for the case of the Riemann Zeros

$$N(E) = \frac{1}{\pi} \arg \xi \left(\frac{1}{2} + i\sqrt{E} \right) \quad (\text{Bolte's semiclassical law in one dimension}) \quad \text{so}$$

$$f^{-1}(x) = \frac{2}{\sqrt{\pi}} \frac{d^{1/2}}{dx^{1/2}} \arg \xi \left(\frac{1}{2} + i\sqrt{x} \right), \quad \text{since we want our potential inside (8) to be}$$

positive whenever we take the inverse we must choose the POSITIVE branch of the inverse in order to get $f(x) \geq 0$ on the interval $[0, \infty)$, the half derivative and the half integral for any well behaved function are given in [13]

$$\frac{d^{1/2} f(x)}{dx^{1/2}} = \frac{1}{\Gamma(1/2)} \frac{d}{dx} \int_0^x \frac{df(t)}{\sqrt{x-t}} \quad \frac{d^{-1/2} f(x)}{dx^{1/2}} = \frac{1}{\Gamma(1/2)} \int_0^x dt \frac{f(t)}{\sqrt{x-t}} \quad (18)$$

We have written implicitly the potential inside (8), if the function $f(x)$ is defined by the functional equation

$$f^{-1}(x) = \frac{2}{\sqrt{\pi}} \frac{d^{1/2}}{dx^{1/2}} \arg \xi \left(\frac{1}{2} + i\sqrt{x} \right) = \frac{2}{\pi} \sum_n \frac{H(x - \gamma_n^2)}{\sqrt{x - \gamma_n^2}}, \quad \text{then we may evaluate the}$$

Spectral Zeta function of the Quantum system given in (8), then

$$\frac{\det(H + z^2)}{\det(H)} = \exp\left(-\frac{d}{ds} \zeta_P(s, z^2)|_{s=0} + -\frac{d}{ds} \zeta_P(s, 0)|_{s=0}\right) = \frac{\xi(z+1/2)}{\xi(1/2)} \quad (19)$$

For the potential defined by $f^{-1}(x) = \frac{2}{\sqrt{\pi}} \frac{d^{1/2}}{dx^{1/2}} \arg \xi\left(\frac{1}{2} + i\sqrt{x}\right)$, we can evaluate the Theta kernel using (15) and (16) $\Theta(t) = \sum_n e^{-tE_n} = \frac{1}{2\sqrt{\pi t}} \int_0^\infty dx \frac{df^{-1}(x)}{dx} e^{-tx}$, for this potential the spectral theta function and its derivative are

$$\zeta_H(s, z^2) = \sum_{n=0}^{\infty} \frac{1}{(\gamma_n^2 + z^2)^s} \quad -\frac{d}{ds} \zeta_H(0, z^2) = \sum_{n=0}^{\infty} \ln(\gamma_n^2 + z^2) \quad \zeta\left(\frac{1}{2} + i\gamma_n\right) = 0 \quad (20)$$

Taking exponentials we reach to the infinite product for the Riemann Xi-function as an spectral determinant (functional determinant over the Eigenvalues of H)

$$\frac{\det(H + z^2)}{\det(H)} = \frac{\prod_{n=0}^{\infty} (\gamma_n^2 + z^2)}{\prod_{n=0}^{\infty} \gamma_n^2} = \prod_{n=0}^{\infty} \left(1 + \frac{z^2}{E_n}\right) = \frac{\xi(1/2 + z)}{\xi(1/2)} \quad (21)$$

If we choose the positive branch $f(x) \geq 0$ of the inverse

$$f^{-1}(x) = \frac{2}{\sqrt{\pi}} \frac{d^{1/2}}{dx^{1/2}} \arg \xi\left(\frac{1}{2} + i\sqrt{x}\right)$$

then the potential will be always positive so the

Energies of the Hamiltonian inside (8) will be all positive $E_n = \gamma_n^2 \in R^+$, then all the non-trivial zeros of the Riemann Zeta function will be on the critical line

$$\text{Re}(s) = \frac{1}{2}, \text{ with a simple change of variable } z = s - \frac{1}{2} \text{ we obtain}$$

$$\frac{\xi(s)}{\xi(0)} = \frac{\det\left(H - s(1-s) + \frac{1}{4}\right)}{\det\left(H + \frac{1}{4}\right)} = \frac{\xi(1-s)}{\xi(0)} = \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \quad (22)$$

Equation (22) is the Hadamard product for the Riemann Xi-function in terms of the quotient of 2 functional determinants, since the expected value of the Hamiltonian is positive $\langle \psi_n | H | \psi_n \rangle \geq 0$ and Hermitian, with $f(x) \geq 0$ then all the Energies are positive $E_n = s(1-s) \in R^+$ Riemann Hypothesis should hold.

- *Bohr-Sommerfeld quantization condition and the square of the Riemann zeros:*

The expression $f^{-1}(x) = \frac{2}{\sqrt{\pi}} \frac{d^{1/2}}{dx^{1/2}} \arg \xi \left(\frac{1}{2} + i\sqrt{x} \right)$ could also be obtained from the Bohr-Sommerfeld quantization conditions [7]

$$\int_c pdq = 2\pi \left(n + \frac{1}{2} \right) \quad 2 \int_0^a dx \sqrt{E - f(x)} = p(x) \quad E = f(a) \quad (23)$$

'a' is the classical turning point, $n = N(E)$ is the Eigenvalue staircase, the first integral inside (23) is a line integral taken over the closed orbit of the classical system, equation (23) can be understood as an integral equation for the inverse of the potential in the form

$$2\pi \left(\frac{1}{2} + n(E) \right) = 2 \int_0^{a(E)} \sqrt{E - V(x)} dx = 2 \int_0^E \sqrt{E - x} \frac{df^{-1}}{dx} = \sqrt{\pi} D_x^{-1/2} f(x) \quad (24)$$

If we take the half derivative on both sides of (24) we would get

$$f^{-1}(x) = 2\sqrt{\pi} \frac{d^{1/2}}{dx^{1/2}} \left(\frac{1}{2} + \frac{1}{\pi} \arg \xi \left(\frac{1}{2} + i\sqrt{x} \right) \right)$$

in this case this result is completely equivalent to the one we got by Zeta regularization and by the WKB

approximation of the Theta function $\frac{1}{2\sqrt{\pi t}} \int_0^\infty dx e^{-tf(x)} = \Theta_{WKB}(t)$.

In order to evaluate the inverse of the potential $f^{-1}(x) = \frac{2}{\sqrt{\pi}} \frac{d^{1/2}}{dx^{1/2}} \arg \xi \left(\frac{1}{2} + i\sqrt{x} \right)$

we would need to evaluate $\frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + i\sqrt{x} \right)$, this can be made using the

Riemann-Siegel formula [10] to evaluate the zeta function on the critical line

$$Z(k) = \zeta \left(\frac{1}{2} + ik \right) e^{i\vartheta(k)} = 2 \sum_{n=1}^{U(k)} \frac{\cos(\vartheta(k) - k \ln n)}{\sqrt{n}} + O\left(\frac{1}{k^{1/4}} \right) \quad k \rightarrow \infty \quad (25)$$

The functions inside (25) are $u(k) = \left[\sqrt{\frac{k}{2\pi}} \right]$, $[x]$ is the floor function and

$$\vartheta(T) = \arg \Gamma \left(\frac{1}{4} + i \frac{T}{2} \right) - \frac{T}{2} \ln \pi \approx \frac{T}{2} \ln \left(\frac{T}{2\pi e} \right) - \frac{\pi}{8} + \frac{1}{48T} + \dots$$

From equation (24) the density of states could be evaluated as

$$\frac{1}{2\sqrt{\pi}} \frac{d^{1/2} f^{-1}(x)}{dx^{1/2}} = \rho(x) = \sum_n \delta(x - \gamma_n^2)$$

, the density of states or trace of $Tr\{ \delta(E - f(x)) \}$ depends on the half-derivative of the inverse of the potential for

the Hamiltonian, we will prove in the next section that this density of states reproduces a distributional version of the Riemann-Weil explicit formula

- *Riemann Weil explicit formula as the Trace* $Tr\{\delta(E - f(x))\}$:

The next question is to compute the density of states for the Hamiltonian defined in (8), let be the property of the delta function $p = \sqrt{E}$

$$\delta(E - \gamma^2) = \frac{\delta(p - \gamma) + \delta(p + \gamma)}{2p}, \text{ if we use Shokhotsky's formula for the delta}$$

$$\text{function } \frac{1}{-\pi} \lim_{\varepsilon \rightarrow 0} \Im m \left(\frac{1}{x - a + i\varepsilon} \right) = \delta(x - a), \text{ the density of states } Tr\{\delta(E - f(x))\}$$

$$\begin{aligned} & -\frac{1}{2\pi\sqrt{E}} \frac{d}{dE} \arg \xi \left(\frac{1}{2} + i\varepsilon + i\sqrt{E} \right) = \sum_{\gamma} \delta(E - \gamma^2) = \sum_{\gamma} \delta(p^2 - \gamma^2) = \\ & \frac{1}{\pi} \frac{\zeta'}{\zeta} \left(\frac{1}{2} + ip \right) \frac{1}{2p} + \frac{1}{\pi} \frac{\zeta'}{\zeta} \left(\frac{1}{2} - ip \right) \frac{1}{2p} - \frac{\ln \pi}{2\pi p} + \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + i\frac{p}{2} \right) \frac{1}{4\pi p} + \\ & \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} - i\frac{p}{2} \right) \frac{1}{4\pi p} + \frac{\delta\left(p - \frac{i}{2}\right) + \pi\delta\left(p + \frac{i}{2}\right)}{2p} = \rho(E) \end{aligned} \quad (26)$$

Here $\frac{1}{-\pi} \lim_{\varepsilon \rightarrow 0} \Im m \left(\frac{2}{2x \pm i + 2i\varepsilon} \right) = \delta\left(x \pm \frac{i}{2}\right)$, this factor comes from the logarithmic

derivative of $s(s-1)$ along the critical line $s = \frac{1}{2} + ip$, equation (26) is a

distributional version of the Riemann-Weil trace formula, taking formally the

logarithm of the Euler product for the Riemann Zeta function on the critical line

yields to $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} e^{-ip \ln n} =_{reg} -\frac{\zeta'}{\zeta} \left(\frac{1}{2} + ip \right)$, using two test functions $h(x)$ and $g(x)$

$g(x) = \frac{1}{\pi} \int_0^{\infty} dr \cos(rx) h(r)$ we recover the oscillatory part of the Riemann-Weil

trace formula $-2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} g(\ln n)$.

Unlike the model of Wu and Sprung, we have considered also the oscillatory

part of the Riemann Eigenvalue Staircase $\frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + i\sqrt{E} \right)$, which satisfy

Bole's semiclassical law, Wu and Sprung considered only the smooth part of

teh Eigenvalue staircase in the limit $T \gg 1$ $\frac{T}{2\pi} \ln \left(\frac{T}{2\pi e} \right) \approx N(T)$ in order to get a

Hamiltonian whose Energies are the positive imaginary part of the Riemann Zeros, their starting point is the Harmonic oscillator [15], but unlike the normal quantum mechanical oscillator whose functional determinant gives the Gamma

function $\frac{\sqrt{2\pi}}{\Gamma(s)} = \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)$ the product taken ONLY over the positive imaginary

part of the zeros (even if it converges) $\prod_{n=0}^{\infty} \left(1 + \frac{s}{\gamma_n}\right)$ has no meaning, also the

Wu-Sprung model doesn't obey Weyl's law in one dimension $N_{smooth}(E) = O(E^{d/2})$, in our case, the Hamiltonian (8) with the Smooth part of the Eigenvalue

staircase $N(E) \approx \frac{\sqrt{E}}{2\pi} \log\left(\frac{\sqrt{E}}{2\pi e}\right)$, satisfies a Weyl's law with $d = 1 + \frac{\varepsilon}{2}$ and the

spectral determinant (quotient) $\frac{\Delta(E)}{\Delta(0)} = \prod_{n=0}^{\infty} \left(1 - \frac{E}{E_n}\right)$ $E_n = \gamma_n^2$ is proportional to the

Riemann xi function on the critical line $\xi\left(\frac{1}{2} + i\sqrt{E}\right)$

By analogy with the zeros of the Selberg Zeta function, is better to consider the case with the Energies $E_n = \gamma_n^2$, in this case the Trace of the Resolvent of the Hamiltonian $(E + i\varepsilon - H)^{-1}$ is the Riemann-Weil trace for the Riemann zeros.

- *Analytic expression for $f(x)$:*

From the expression for the fractional derivative of powers

$\frac{d^k x^\lambda}{dx^k} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - k + 1)} x^{\lambda - k}$, we can obtain for the inverse function

$$f^{-1}(x) = \frac{2}{\sqrt{\pi}} \frac{d^{1/2}}{dx^{1/2}} \arg \xi\left(\frac{1}{2} + i\sqrt{x}\right) = \frac{2}{\pi} \sum_{\gamma > 0} \frac{H(x - \gamma^2)}{\sqrt{x - \gamma^2}} \quad (27)$$

Using the Riemann-Weil formula we can rewrite (27) as

$$f^{-1}(x) = \frac{8}{\pi\sqrt{8x+1}} + \frac{1}{\pi^2} \int_{-\sqrt{x}}^{\sqrt{x}} \frac{dr}{\sqrt{x-r^2}} \left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{ir}{2}\right) - \ln \pi \right) - 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} g(x, \ln n) \quad (28)$$

Here $g(u = \ln n, x) = \frac{2}{\pi^2} \int_0^{\sqrt{x}} \frac{\cos(ut)}{\sqrt{x-t^2}} dt$, a final question is could the expression

$f^{-1}(x) = \frac{2}{\sqrt{\pi}} \frac{d^{1/2}}{dx^{1/2}} \arg \xi\left(\frac{1}{2} + i\sqrt{x}\right)$ be inverted to get $f(x)$? , the smooth part of

the Eigenvalue staircase is given by $N(E) \approx \frac{\sqrt{E}}{2\pi} \log\left(\frac{\sqrt{E}}{2\pi e}\right)$, $e = \sum_{n=0}^{\infty} \frac{1}{n!}$, if we

use the expression for the logarithm $\log(x) \approx \frac{x^\varepsilon - 1}{\varepsilon}$ as $\varepsilon \rightarrow 0$ and apply the half derivative expression, then the following holds $\varepsilon \rightarrow 0$

$$f^{-1}_{smooth}(x) \approx \frac{(4\pi^2 e^2)^{-\varepsilon/2} A(\varepsilon) x^{\varepsilon/2} - B}{\sqrt{\pi\varepsilon}} \quad f_{smooth}(x) \approx 4\pi^2 e^2 \left(\frac{\varepsilon\sqrt{\pi}x + B}{A(\varepsilon)} \right)^{\frac{2}{\varepsilon}} \quad (29)$$

$$A(\varepsilon) = \frac{\Gamma\left(\frac{3+\varepsilon}{2}\right)}{\Gamma\left(1+\frac{\varepsilon}{2}\right)} \quad \text{and} \quad B = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}, \quad \text{the second expression inside (29) is}$$

the asymptotic of $f(x)$ as $x \rightarrow \infty$, for this potential, the energies inside (8) are

$$E^{smooth}_n = f(n) = N^{-1}_{smooth}(E) \approx \frac{4\pi^2 n^2}{W^2(ne^{-1})} \quad W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n \quad (30)$$

The function $W(x)$ is the Lambert function (principal branch), more than in the

potential $f^{-1}(x) = \frac{2}{\sqrt{\pi}} \frac{d^{1/2}}{dx^{1/2}} \arg \xi \left(\frac{1}{2} + i\sqrt{x} \right) \in R$ we are interested in the Theta

function $\Theta(t) = \sum_n \exp(-t\gamma_n^2)$, if we use the semiclassical Theta function as an integral over the Phase space and introduce the potential given by

$$f^{-1}(x) = \frac{2}{\sqrt{\pi}} \frac{d^{1/2}}{dx^{1/2}} \arg \xi \left(\frac{1}{2} + i\sqrt{x} \right) \in R \quad \text{one obtains, } \sqrt{a} \int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\pi}$$

$$\begin{aligned} \Theta_{WKB}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \int_0^{\infty} dx e^{-tH(x,p)} = \frac{1}{2\sqrt{\pi}t} \int_0^{\infty} dx e^{-tf(x)} = \sqrt{\frac{t}{\pi}} \int_0^{\infty} dr e^{-tr} f^{-1}(r) = \\ &= \sqrt{\frac{t}{\pi}} \sum_{\gamma} \int_0^{\infty} dr e^{-tr} \frac{H(r-\gamma_n^2)}{\sqrt{r-\gamma_n^2}} = \sqrt{\frac{t}{\pi}} \left(\int_{-\infty}^{\infty} e^{-tx^2} dx \right) \sum_{\gamma} e^{-t\gamma_n^2} = \Theta(t) = \sum_n \exp(-t\gamma_n^2) \end{aligned} \quad (31)$$

In (31) we have obtained the Heat function $\Theta(t) = \sum_n \exp(-t\gamma_n^2)$, from the

potential function $f^{-1}(x) = \frac{2}{\sqrt{\pi}} \frac{d^{1/2}}{dx^{1/2}} \arg \xi \left(\frac{1}{2} + i\sqrt{x} \right) \in R$ of course to be correct we must take the smooth and the oscillatory part of the Eigenvalue staircase

$$f^{-1}(x) = \frac{2}{\sqrt{\pi}} \frac{d^{1/2}}{dx^{1/2}} (N_{smooth}(x) + N_{osc}(x)) \quad \text{otherwise the description will be not}$$

complete as in the Wu-Sprung potential [15], from this Theat Kernel

$$\Theta(t) = \sum_n \exp(-t\gamma_n^2) = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dpe^{-tH(x,p)} \quad \text{here } t>0 \text{ and the Hamiltonian has been}$$

defined in (8) using the Zeta regularization method for the determinant

$$-\partial_s \zeta_H(s, z(z-1)) = \ln \det(H - z(1-z)) \quad \zeta_H(s, z(z-1)) = Tr \left\{ (H + z(z-1))^{-s} \right\} \quad (32)$$

$$\zeta_H(s, z(z-1)) = \sum_n \frac{1}{\left(\frac{1}{4} + z(z-1) + \gamma_n^2\right)^s}, \text{ the zeros of the determinant}$$

$\det(H - z(1-z))$ with H an Hermitian operator are the zeros of the Riemann Xi-function $\xi(z)$

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