Flavors of Clifford Algebra

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Abstract

Extensions of Clifford algebra are presented. Applications in physics, especially with regard to the flavor structure of Standard Model, are discussed. Modified gravity is also suggested.

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1 Introduction

Clifford algebra, also known as geometric algebra or space-time algebra, has found a wide variety of applications in physics[1, 2]. Attempts[3, 4, 5] have been made to identify species of fermions as ideals (idempotent projections of the original spinor) and derive Standard Model gauge symmetries from various dimensions of Clifford algebras.

Our previous work[6], which proposes a model of gravity and Yang-Mills interactions in term of Clifford algebra, is based on three premises. Firstly, all idempotent projections of the original spinor should be realized as fermions of physical world. In other words, no spinor projection should be casually discarded. Hence, finding the right Clifford algebra turns out to be a simple process of counting numbers of fermion species. There are 16 Weyl fermions (including right-handed neutrino) with $16 \times 4 = 64$ real components in the first generation. Clifford algebra $\mathcal{C}\ell_{0,6}$, with $2^6 = 64$ degrees of freedom, seems to be a natural choice.

The second premise is that rotations should be generalized. As well known in Clifford algebra approaches, a rotation is realized by a rotor, which is an exponential of bivectors. It rotates a vector into another vector. However, a rotor could be defined to be an exponential of any multivectors. It could rotate a vector into a multivector, generalizing definition of rotations. Hence, one can entertain large symmetry groups with lower dimensional Clifford algebras, whereas the same symmetry groups would otherwise require higher dimensions within the conventional framework.

The third premise is the generalization of MacDowell/Mansouri[7] gravity in a natural way to include Yang-Mills interactions as well. The key is to take a page from effective field theory, where an infinite number of terms allowed by symmetry requirements should be included in the action and the first few terms of the action are relevant in low-energy limit. Gravity and Yang-Mills actions are formulated as different order terms in a generalized action. The first order terms include Einstein-Cartan action with cosmological constant. The trace (scalar part of Clifford algebra) of every Yang-Mills-field-related first order term is zero. Yang-Mills fields appear in second order terms only. The coefficients of second order terms are much smaller than those of first order terms. Thus Yang-Mills coupling constants appear to be much larger than that of gravity.

The current paper, targeting the issue of flavor structure, is a continuation of our previous work. There are 16*3 Weyl fermions with $64 * 3$ real components for 3 generations. The orthogonal Clifford algebra, with 2^N degrees of freedom, would not work. We turn to extensions of Clifford algebra. Along the way, we also consider possible modifications to gravity.

This paper is structured as follows: Section 2 introduces orthogonal Clifford algebra and various extensions, with or without supersymmetry. In section 3, a direct product of binary and ternary Clifford algebras $\mathcal{C}\ell_{0,6} * \mathcal{C}\ell_T$ is defined. Applications in physics, especially with regard to the flavor structure of Standard Model, are discussed. In section 4, we take gravity into consideration. Modified gravity is suggested. In the last section we draw our conclusions.

2 Clifford Algebras

2.1 6D Orthogonal Clifford Algebra

We begin with a review of orthogonal Clifford algebra $C\ell_{0,6}$. It is defined by anticommutators of orthonormal vector basis $(\gamma_j, \Gamma_j; j = 1, 2, 3)$

$$\{\gamma_j, \gamma_k\} \equiv \frac{1}{2}(\gamma_j\gamma_k + \gamma_k\gamma_j) = -\delta_{jk}, \tag{1}$$

$$\{\Gamma_j, \Gamma_k\} = -\delta_{jk}, \tag{2}$$

$$\{\gamma_j, \Gamma_k\} = 0, \tag{3}$$

where $j, k = 1, 2, 3$. All basis vectors are space-like. There are $\binom{6}{k}$ independent k-vectors. The complete basis for $C\ell_{0,6}$ is given by the set of all k-vectors. Any multivector can be expressed as a linear combination of $2^6 = 64$ basis elements.

Two trivectors

$$\gamma_0 = \Gamma_1\Gamma_2\Gamma_3, \tag{4}$$

$$\Gamma_0 = \gamma_1\gamma_2\gamma_3 \tag{5}$$

square to 1, so they are time-like. Orthonormal vector-trivector basis $\{\gamma_a, a = 0, 1, 2, 3\}$ defines space-time Clifford algebra $C\ell_{1,3}$. The unit pseudoscalar

$$i = \Gamma_1\Gamma_2\Gamma_3\gamma_1\gamma_2\gamma_3 = \gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_0\Gamma_0 \tag{6}$$

squares to -1, anticommutes with odd-grade elements, and commutes with even-grade elements.

Reversion of a multivector $M \in C\ell_{0,6}$, denoted $\tilde{M}$, reverses the order in any product of vectors (there is an additional minus sign if both vectors are with Grassmann odd coefficients). There are algebraic properties $(MN)\tilde{} = \tilde{N}\tilde{M}$ and $\langle MN\rangle = \langle NM\rangle$ for any multivectors M and N, where $\langle\cdots\rangle$ means scalar part of enclosed expression. The magnitude of a multivector M is defined as

$$|M| \equiv \sqrt{\langle M^\dagger M\rangle}, \tag{7}$$

where $M^\dagger \equiv i\tilde{M}(-i)$ is the Hermitian conjugate.

Algebraic spinor $\psi \in C\ell_{0,6}$ is a multivector (with Grassmann odd coefficients) which obeys transformation law

$$\psi \quad \rightarrow \quad \mathbb{R}\psi\underline{\mathbb{R}}, \tag{8}$$

where $\mathbb{R}$ and $\underline{\mathbb{R}} \in C\ell_{0,6}$ are independent left and right-sided gauge transformations (with Grassmann even coefficients). Spinor bilinear

$$\left\langle \tilde{\psi}\gamma_0\psi \right\rangle \tag{9}$$

is invariant if

$$\tilde{\mathbb{R}}\gamma_0\mathbb{R} = \gamma_0, \tag{10}$$
$$\underline{\mathbb{R}\tilde{\mathbb{R}}} = 1, \tag{11}$$

where we restrict our discussion to gauge transformations continuously connected to identity. General solution of these equations has the form

$$\mathbb{R} = e^{\frac{1}{2}\Theta}, \tag{12}$$
$$\underline{\mathbb{R}} = e^{\frac{1}{2}\underline{\Theta}}, \tag{13}$$

where $\Theta \sim so(4,4)$ is a linear combination of 28 gauge transformation generators

$$(\gamma_a, \gamma_a\gamma_b, \gamma_0\Gamma_j, \Gamma_0\Gamma_j, i\Gamma_j, \Gamma_0\gamma_j\Gamma_k; j,k = 1,2,3, a,b = 0,1,2,3, a > b), \tag{14}$$

and $\underline{\Theta} \sim sp(8)$ is a linear combination of 36 gauge transformation generators of all bivectors, trivectors, and 6-vector

$$(\gamma_0\Gamma_j, \Gamma_0\gamma_j, \gamma_j\Gamma_k, \gamma_0, \gamma_0\gamma_j\Gamma_k, \Gamma_0, \Gamma_0\gamma_j\Gamma_k, i; j,k = 1,2,3). \tag{15}$$

The de Sitter algebra $(\gamma_a, \gamma_a\gamma_b) \sim so(1,4)$ and weak interaction $su(2)$ algebra $\{\gamma_0\Gamma_j\}$ are commuting subalgebras of left-sided gauge transformations.

2.2 2D Symplectic Clifford Algebra

2D symplectic Clifford algebra is defined by anticommutators of symplectic vector basis $(\zeta_1 c, \zeta_2\bar{c})$

$$\{\zeta_1 c, \zeta_2\bar{c}\} = \zeta_1\zeta_2, \tag{16}$$
$$\{\zeta_1 c, \zeta_2 c\} = \{\zeta_1\bar{c}, \zeta_2\bar{c}\} = 0, \tag{17}$$

where ζ_1 and ζ_2 are Grassmann odd numbers. With multiplications of multiple $\zeta_m c$ and $\zeta_n\bar{c}$, there are infinite numbers of independent k-vectors.

We propose an extension of algebraic spinor for symplectic Clifford algebra, which is a linear combination of all k-vectors (superspinor). Grassmann odd part of a superspinor behaves like fermions, while Grassmann even part (superpartner) behaves like bosons. To avoid ambiguity, $\langle\cdots\rangle$ is defined by the scalar part of a normally ordered multivector.

One can study invariance properties of a superspinor by applying rotation $e^{\frac{1}{2}\Theta}$. There could be both grassmann even and odd components in Θ. One example of Θ is gauge transformation superalgebra $osp(1|2)$ with Bose (Grassmann even) part $(c^2, \bar{c}^2, \frac{1}{2}(c\bar{c} + \bar{c}c))$ and Fermi (Grassmann odd) part $(c, \bar{c})$.

2.3 6D Mixed Clifford Algebra

In an attempt to connect with physical world, one can consider a Clifford Algebra mixing Dirac ($\gamma_a, a = 0, 1, 2, 3$) and symplectic vectors ($\zeta_1 c, \zeta_2 \bar{c}$) defined by anticommutators of mixing terms

$$\{\zeta_1 c, \gamma_a\} = 0, \tag{18}$$

while other anticommutators remain the same as Dirac algebra and 2D symplectic Clifford algebra. An algebraic superspinor can be constructed along the same line as in the previous subsection. One example of rotation $e^{\frac{1}{2}\Theta}$ involves gauge transformation superalgebra $osp(5|2)$ with Bose part $(c^2, \bar{c}^2, \frac{1}{2}(c\bar{c} + \bar{c}c), \gamma_a, \gamma_a\gamma_b)$ and Fermi part $(c, \bar{c}, c\gamma_a, \bar{c}\gamma_a)$.

In the same fashion, one can construct other mixed Clifford algebras with N orthonormal vectors and $2M$ symplectic vectors, and study their gauge transformation superalgebras.

2.4 Ternary Clifford Algebra

Superspinors suggest a plethora of fermion species and their superpartners, which have not been experimentally observed so far. We turn to other extensions of Clifford algebra with finite numbers of independent k-vectors. They involve n -ary communication relationships[8, 9, 10] rather than the usual binary ones.

Let's propose a Clifford algebra defined by anticommutators of a binary vector γ and a ternary vector η

$$\{\gamma, \gamma\} = 1, \tag{19}$$

$$\{\eta, \eta, \eta\} \equiv \eta^3 = 1, \tag{20}$$

$$\{\eta, \eta, \gamma\} \equiv \frac{1}{3}(\eta^2\gamma + \eta\gamma\eta + \gamma\eta^2) = 0, \tag{21}$$

$$\{\eta, \gamma, \gamma\} \equiv \frac{1}{3}(\eta\gamma^2 + \gamma\eta\gamma + \gamma^2\eta) = 0. \tag{22}$$

The ternary communication relationships between γ and η introduce some unusual complications. Instead, we are going to focus on a direct product of binary and ternary Clifford algebras, which is the subject of following sections.

3 Flavors

3.1 $C\ell_{0,6} * C\ell_T$

With the purpose of studying 3 generations of Standard Model fermions, let's consider a direct product of $C\ell_{0,6} * C\ell_T$, which is defined by

$$\{\eta, \eta, \eta\} \equiv \eta^3 = 1, \tag{23}$$
$$[\eta, \gamma_j] \equiv \eta\gamma_j - \gamma_j\eta = 0, \tag{24}$$
$$[\eta, \Gamma_j] = 0, \tag{25}$$

while the rest anticommutators remain the same as $C\ell_{0,6}$. The complete basis for $C\ell_{0,6} * C\ell_T$ is given by the set of all k-vectors. An algebraic spinor can be expressed as a linear combination of $2^6 * 3 = 192$ basis elements.

Spinors with right/left chirality correspond to even/odd multivectors

$$\psi = \psi_+ + \psi_-, \tag{26}$$
$$\psi_\pm = \frac{1}{2}(\psi \mp i\psi i). \tag{27}$$

A projection operator squares to itself. Idempotents are a set of projection operators

$$G_1 = \frac{1}{3}(1 + e^{\theta+\theta'}\eta + e^{-\theta-\theta'}\eta^2), \tag{28}$$
$$G_2 = \frac{1}{3}(1 + e^{\theta}\eta + e^{-\theta}\eta^2), \tag{29}$$
$$G_3 = \frac{1}{3}(1 + e^{\theta-\theta'}\eta + e^{-\theta+\theta'}\eta^2), \tag{30}$$
$$P_0 = \frac{1}{4}(1 + iJ_1 + iJ_2 + iJ_3) = \frac{1}{4}(1 + iJ), \tag{31}$$
$$P_1 = \frac{1}{4}(1 + iJ_1 - iJ_2 - iJ_3), \tag{32}$$
$$P_2 = \frac{1}{4}(1 - iJ_1 + iJ_2 - iJ_3), \tag{33}$$
$$P_3 = \frac{1}{4}(1 - iJ_1 - iJ_2 + iJ_3), \tag{34}$$
$$P_q = P_1 + P_2 + P_3 = \frac{1}{4}(3 - iJ), \tag{35}$$
$$P_\pm = \frac{1}{2}(1 \pm \Gamma_0\Gamma_3), \tag{36}$$

where $J_j = \gamma_j\Gamma_j, J = J_1 + J_2 + J_3, \theta = \frac{2\pi}{3}I, \theta' = \frac{2\pi}{3}i, I = \frac{1}{2}(i + J), I^2 = -1, G_1 + G_2 + G_3 = 1, P_0 + P_1 + P_2 + P_3 = P_0 + P_q = 1, P_+ + P_- = 1$. Here G_j and P_j are flavor and color projection operators, respectively. Reversions of flavor projectors are defined as not changing signs of θ and θ' inside G_j, so that $\tilde{G}_j = G_j$. The bivectors $J_j = \gamma_j \Gamma_j$ appearing in the color projectors P_j suggest an interesting duality between 3 space dimensions and 3 colors of quarks.

Now we are ready to identify idempotent projections of spinor

$$\psi = (P_+ + P_-)(\psi_+ + \psi_-)(P_0 + P_1 + P_2 + P_3)(G_1 + G_2 + G_3) \tag{37}$$

with j'th generation left-handed leptons, red, green, and blue quarks

$$\left\{ \begin{array}{l} \nu_{l,j} = P_+\psi_- P_0 G_j, \\ e_{l,j} = P_-\psi_- P_0 G_j, \\ u_{l,j} = P_+\psi_- P_1 G_j + P_+\psi_- P_2 G_j + P_+\psi_- P_3 G_j = P_+\psi_- P_q G_j, \\ d_{l,j} = P_-\psi_- P_1 G_j + P_-\psi_- P_2 G_j + P_-\psi_- P_3 G_j = P_-\psi_- P_q G_j, \end{array} \right. \tag{38}$$

and right-handed leptons, red, green, and blue quarks

$$\left\{ \begin{array}{l} \nu_{r,j} = P_-\psi_+ P_0 G_j, \\ e_{r,j} = P_+\psi_+ P_0 G_j, \\ u_{r,j} = P_-\psi_+ P_1 G_j + P_-\psi_+ P_2 G_j + P_-\psi_+ P_3 G_j = P_-\psi_+ P_q G_j, \\ d_{r,j} = P_+\psi_+ P_1 G_j + P_+\psi_+ P_2 G_j + P_+\psi_+ P_3 G_j = P_+\psi_+ P_q G_j. \end{array} \right. \tag{39}$$

It is understood that $e_{l,2}$ and $e_{l,3}$ are regarded as μ_l and τ_l, respectively. Naming conventions for other fermions, which are not listed here, follow similar pattern.

3.2 Symmetries

At unification energy scale, symmetries could be like the ones (i.e. $SO(4,4) * SP(8)$) explored in the earlier section[1]. In this section, we are interested in symmetry transformations $\psi \to e^{\frac{1}{2}\Theta} \psi e^{\frac{1}{2}\underline{\Theta}}$ at intermediate electroweak energy scale (after symmetry breaking at a higher energy scale). They include Lorentz $SO(1,3)$ gauge transformations Θ_{LO} (which are part of de Sitter transformations $\Theta_{DS} = (\gamma_a, \gamma_a\gamma_b)$)

$$\gamma_a \gamma_b \in \Theta, \tag{40}$$

color $SU(3)$ gauge transformations $\underline{\Theta_C}$

$$\left(\begin{array}{l} \frac{1}{2}(\Gamma_1\Gamma_2 + \gamma_1\gamma_2), \frac{1}{2}(\Gamma_1\gamma_2 - \gamma_1\Gamma_2), \frac{1}{2}(\Gamma_1\gamma_1 - \Gamma_2\gamma_2), \\ \frac{1}{2}(\Gamma_3\Gamma_1 + \gamma_3\gamma_1), \frac{1}{2}(\Gamma_3\gamma_1 - \gamma_3\Gamma_1), \\ \frac{1}{2}(\Gamma_2\Gamma_3 + \gamma_2\gamma_3), \frac{1}{2}(\Gamma_2\gamma_3 - \gamma_2\Gamma_3), \\ \frac{1}{2\sqrt{3}}(\Gamma_1\gamma_1 + \Gamma_2\gamma_2 - 2\Gamma_3\gamma_3) \end{array} \right) \in \underline{\Theta}, \tag{41}$$

electroweak $SU(2)_L * U(1)_L$ double-sided gauge transformations Θ_L and $\underline{\Theta_L}$ acting on left-handed fermions

$$(\Gamma_2\Gamma_3, \Gamma_3\Gamma_1, \Gamma_1\Gamma_2) \in \Theta, \quad J/3 \in \underline{\Theta}, \tag{42}$$

[1]If ternary vector η and bivector η^2 are also gauged, the size of symmetries would be tripled.

and electromagnetic $U(1)_R$ synchronized double-sided gauge transformations Θ_R and $\underline{\Theta_R}$ acting on right-handed fermions

$$\epsilon\Gamma_1\Gamma_2 \in \Theta, \quad \epsilon J/3 \in \underline{\Theta}, \tag{43}$$

where a shared rotation angle ϵ enforces synchronization of the double-sided gauge transformations.

Electric charges are calculated as $0, -1, \frac{2}{3}$, and $-\frac{1}{3}$ for neutrino, electron, up quarks, and down quarks, respectively[6]. Because the product of lepton projector P_0 with any generator in color algebra is zero, leptons are invariant under color gauge transformations.

3.3 Gauge Fields, Higgs Fields, and Fermion Actions

Gauge fields are Clifford-valued 1-forms (Clifforms[11, 12]) on 4-dimensional space-time manifold[2] (x_μ, $\mu = 0, 1, 2, 3$)

$$\begin{aligned}
\omega, (1/l)e &\in \Theta_{DS}, && (44)\\
A_C &\in \underline{\Theta_C}, && (45)\\
(W_1^L, W_2^L, W_3^L), B^L &\in \Theta_L, \underline{\Theta_L}, && (46)\\
W_3^R, B^R &\in \Theta_R, \underline{\Theta_R}, && (47)
\end{aligned}$$

where ω is spin connection, e is vierbein, A_C is strong interaction, and the rest are electroweak interactions. Here $W_3^R = \frac{1}{2}A_\mu\Gamma_1\Gamma_2 dx^\mu$, $B^R = \frac{1}{6}A_\mu J dx^\mu$, and A_μ represents electromagnetic gauge field. We adopt the summation convention for repeated space-time indices. In the following, outer products between differential forms are implicitly assumed.

For breaking of gauge symmetries, Higgs fields[3] Φ, ϕ and $\underline{\phi}$ are introduced. They are multivectors obeying transformation rules

$$\begin{aligned}
\Phi &\rightarrow e^{\frac{1}{2}\Theta_{DS}} \; \Phi \; e^{-\frac{1}{2}\Theta_{DS}}, && (48)\\
\phi &\rightarrow e^{\frac{1}{2}\Theta_L} \; \phi \; e^{-\frac{1}{2}\Theta_R}, && (49)\\
\underline{\phi} &\rightarrow e^{-\frac{1}{2}(\underline{\Theta_R}+\underline{\Theta_C})} \; \underline{\phi} \; e^{\frac{1}{2}(\underline{\Theta_L}+\underline{\Theta_C})}. && (50)
\end{aligned}$$

Gauge-covariant derivatives of fermion fields and Higgs fields can be defined according to their gauge transformation rules[6]. At electroweak energy scale, de Sitter symmetry is already broken down to Lorentz symmetry. With replacement of Higgs field Φ by its vacuum expectation value (VEV) $\bar{\Phi} = Vi$, $D\Phi$ reduces to $\frac{2Vi}{l}e$. The vestigial vierbein

[2] Additional dimensions will be considered in the sequel.

[3] We call them Higgs fields in the general sense that they are 0-forms with symmetry-breaking VEVs. The VEVs should be invariant under Lorentz gauge transformations.

field e is acting like space-time frame field, which is essential in building actions at all energy scales. The 4-dimensional space-time manifold is initially without metric. It's the vierbein field which gives notion to metric $g_{\mu\nu} = \langle e_\mu e_\nu \rangle$.

The gauge- and diffeomorphism-invariant action for spinor field is now written down as

$$S_{Kinetic} = \int \left\langle \tilde{\psi}\gamma_0 (D\Phi)^7 D\psi \right\rangle / \left\langle -i(D\Phi)^4 \right\rangle, \tag{51}$$

where it's understood that the 4-form within $(D\Phi)^7$ should be canceled by the 4-form of $-i(D\Phi)^4$ before any further outer multiplication (or one can simply replace $(D\Phi)^7$ with $i(D\Phi)^3$ and remove $-i(D\Phi)^4$ in the denominator). Otherwise it is strictly zero, since no n -form with $n > 4$ is allowed on 4-dimensional space-time manifold. The same reasoning applies to all the following actions. The ubiquitous $-i(D\Phi)^4$ could be regarded as related to Planck's constant.

For the case of flat space-time, spinor action can be obtained by substituting Higgs field Φ and vierbein $(1/l)e$ with their VEVs $\bar{\Phi} = Vi$ and $(1/l)\bar{e} = (1/l)\delta^a_\mu \gamma_a dx^\mu$, respectively. Factor $-i(D\Phi)^4$ reduces to a constant (Clifford scalar) 4-form, so that we recover the regular spinor action.

We can write Dirac-type mass terms as

$$S_{Mass,Lepton,jk} = \int \left\langle (\widetilde{\nu_{l,j} + e_{l,j}})\gamma_0 (D\Phi)^8 \phi (y_{\nu,jk}\nu_{r,k} + y_{e,jk}e_{r,k})\underline{\phi} \right\rangle / \left\langle -i(D\Phi)^4 \right\rangle, \tag{52}$$

$$S_{Mass,Quark,jk} = \int \left\langle (\widetilde{u_{l,j} + d_{l,j}})\gamma_0 (D\Phi)^8 \phi (y_{u,jk}u_{r,k} + y_{d,jk}d_{r,k})\underline{\phi} \right\rangle / \left\langle -i(D\Phi)^4 \right\rangle, \tag{53}$$

and Majorana-type mass terms for right-handed neutrinos as

$$S_{Majorana,jk} = \int \left\langle \widetilde{\nu_{r,j}}\gamma_0 (D\Phi)^8 (Y_{jk} e^{\epsilon_{jk}\Gamma_1\Gamma_2}\Gamma_2\Gamma_3)\nu_{r,k} M_{jk} \right\rangle / \left\langle -i(D\Phi)^4 \right\rangle, \tag{54}$$

where $y_{\nu,jk}, y_{e,jk}, y_{u,jk}, y_{d,jk}$, and Y_{jk} are Yukawa coupling constants, $e^{\epsilon_{jk}\Gamma_1\Gamma_2}$ are phase factors, M_{jk} are trivectors of unit magnitudes. Majorana-type mass terms are the results of symmetry breaking above electroweak scale. They are usually assumed as being much heavier than Dirac-type masses. Thus very small effective masses are generated for neutrinos, known as seesaw mechanism.

3.4 Symmetry Breaking

We leave the study of dynamics and self-interactions of electroweak Higgs sector to future research. We just assume that, after electroweak symmetry breaking, Higgs fields ϕ and $\underline{\phi}$ acquire VEVs as

$$\bar{\phi} = v\Gamma_1\Gamma_2, \tag{55}$$

$$\bar{\underline{\phi}} = \underline{v}i, \tag{56}$$

where v and $\underline{v}$ are magnitudes of VEVs. Since ϕ and $\underline{\phi}$ appear in pairs in Dirac-type mass terms, in the following we set $\underline{v} = 1$ by rescaling v. Left-handed electromagnetic interaction remains massless and is synchronized with the right-handed counterpart. Interactions of W_1^L, W_2^L and Z (characterized by $W_3^L = \frac{1}{2} Z_\mu \Gamma_1 \Gamma_2 dx^\mu, B^L = -\frac{1}{6} Z_\mu J dx^\mu$) acquire masses.

3.5 Flavor Mixing

With electroweak Higgs fields replaced by their VEVs, Dirac-type masses become diagonalized in terms of flavors, since $G_j i G_k = i\delta_{jk} G_j$. The only flavor mixing terms are Majorana-type masses of right-handed neutrinos between 2nd and 3rd generations.

When electroweak Higgs fields fluctuate around their VEVs (specifically when $\Delta\underline{\phi} = \underline{\phi} - \bar{\underline{\phi}}$ does not commute with flavor projection operators G_j), Dirac-type masses may not be diagonalized in terms of flavors of quarks (between 1st and 2nd generations) and leptons (between 2nd and 3rd generations). Higher order corrections introduce further mixing between generations. One may potentially couple above effects with appropriate choices of Yukawa constants to explain the observed CKM and PMNS matrices with quite different patterns.

3.6 Different Flavor Projectors

Hypothetically, other sets of flavor projectors can be assigned to fermion species. For example, we can define idempotents

$$G_1^q = \frac{1}{3}(1 + e^{\theta_q}\eta + e^{-\theta_q}\eta^2), \tag{57}$$

$$G_2^q = \frac{1}{3}(1 + e^{-\theta_q}\eta + e^{\theta_q}\eta^2), \tag{58}$$

$$G_3^q = \frac{1}{3}(1 + \eta + \eta^2), \tag{59}$$

where $\theta_q = \frac{2\pi}{3}\Gamma_1\Gamma_2$ and $G_1^q + G_2^q + G_3^q = 1$. These quark flavor projection operators are applied to the left side of quarks, while lepton flavor projection operators remain the same as in previous sections.

Because of the bivector $\Gamma_1\Gamma_2$ in flavor projection operators G_j^q, there is quark flavor mixing between 1st and 2nd generations via direct weak interactions (W_1^L, W_2^L). This kind of flavor projection operators may not be a viable option, since the mixing is too strong for quarks.

4 Gravity

4.1 Gauge Actions

We begin by introducing gravity curvature 2-form

$$F_G = d(\frac{1}{l}e + \omega) + (\frac{1}{l}e + \omega)^2 = R + \frac{1}{l}T + \frac{1}{l^2}e^2, \tag{60}$$

where spin connection curvature R and torsion T are defined by $R = d\omega + \omega^2$ and $T = de + \omega e + e\omega$, respectively. Yang-Mills curvature 2-form is denoted as F_{YM}.

The generalized formal curvature 2-form is defined as

$$\mathcal{F} = \frac{1}{g}F_G + \frac{1}{g'}F_{YM}, \tag{61}$$

where g, g', are dimensionless coupling constants of order 1 (or close to order 1). Different coupling constants can be assigned to different Yang-Mills interactions. The elements, which are covariant under left/right-sided gauge transformations Θ and $\underline{\Theta}$, are formally assigned to two sets of Clifford algebras. Elements from different sets formally commute with each other. In the following, $\langle \cdots \rangle$ means scalar part of both sets.

The gauge field action is written as

$$S_{Gauge} = \int \left\langle ((D\Phi)^6 \mathcal{F}) \right\rangle / \left\langle -i(D\Phi)^4 \right\rangle + \int \left\langle ((D\Phi)^2 \mathcal{F})((D\Phi)^2 \mathcal{F}) \right\rangle / \left\langle -i(D\Phi)^4 \right\rangle. \tag{62}$$

With Higgs field Φ acquiring large VEV $\bar{\Phi} = Vi$, the most significant terms are Einstein-Cartan action

$$S_{Einstein-Cartan} \sim \int \frac{V^2}{l^2} \left\langle e^2 R i \right\rangle, \tag{63}$$

cosmological constant term

$$S_{Cosmological-Constant} \sim \int \frac{V^2}{l^4} \left\langle e^4 i \right\rangle, \tag{64}$$

and Yang-Mills action

$$S_{Yang-Mills} \sim \int \left\langle (e^2 F_{YM})^2 \right\rangle / \left\langle (ie^4) \right\rangle. \tag{65}$$

There is no explicit Hodge dual in Yang-Mills action. Vierbein plays the role of Hodge dual, when it acquires non-zero VEV $\bar{e} = \delta^a_\mu \gamma_a dx^\mu$ in the case of flat space-time. There is no CP-violating theta term

$$S_{Theta} \sim \int \theta_{QCD} \left\langle (e^2 F_{YM})^2 i \right\rangle / \left\langle (ie^4) \right\rangle. \tag{66}$$

4.2 Hierarchies

For comparison's sake, let's also rewrite fermion kinetic action

$$S_{Kinetic} \sim \int \left\langle \tilde{\psi}\gamma_0 e^3 D\psi \right\rangle, \tag{67}$$

a typical Dirac-type mass term

$$S_{Mass} \sim \int \frac{yvV}{l} \left\langle \tilde{\psi}\gamma_0 e^4 \psi i \right\rangle, \tag{68}$$

and a typical Majorana-type mass term

$$S_{Majorana} \sim \int \frac{YV}{l} \left\langle \widetilde{\nu_r}\gamma_0 e^4 (\Gamma_2\Gamma_3)\nu_r\gamma_0 \right\rangle, \tag{69}$$

where spinor fields are rescaled by a factor of $(V/l)^{3/2}$, v is the VEV magnitude of electroweak Higgs fields, y and Y are typical Yukawa coupling constants for Dirac and Majorana -type mass terms, respectively.

The mass hierarchies are

$$m_{pl} \sim G^{-\frac{1}{2}} \sim \frac{V}{l}, \qquad \text{(Planck mass)}$$

$$m_f \sim \frac{yvV}{l} \sim (yv)m_{pl} \sim 10^{-20} m_{pl}, \qquad \text{(typical fermion mass)}$$

$$m_\nu \sim (m_f^2)/(\frac{YV}{l}) \sim \frac{(yv)^2}{Y}\frac{V}{l} \sim \frac{(yv)^2}{Y} m_{pl} \sim 10^{-30} m_{pl}, \qquad \text{(neutirno mass)}$$

$$\Lambda^{\frac{1}{2}} \sim H \sim \frac{1}{l} \sim \frac{1}{V} m_{pl} \sim 10^{-60} m_{pl}, \qquad \text{(cosmological constant)}$$

where G is Newton constant, H is Hubble constant, the typical fermion mass m_f is set to Λ_{QCD} (Dirac-type masses range over many orders of magnitude for different fermions), and m_ν is neutrino seesaw mass. Along with Yukawa coupling constants, the parameters involve three VEV magnitudes $(V, v, 1/l)$, where $1/l$ is of mass dimension one. The dimensionless parameters are estimated as $V \sim 10^{60}$, $yv \sim 10^{-20}$, and $Y \sim 10^{-10}$.

4.3 8 Dimensions

There are proposals[13, 14, 15, 16] trying to account for origins and/or relationships amongst mass hierarchies. They usually involve extra dimensions of the underlying space-time manifold.

In the process of building fermion and gauge actions in earlier sections, we see the pattern of pseudo 8-forms divided by 4-forms. It seems clumsy. The benefit is that standalone Higgs field Φ and 6-vector i can be excluded from pseudo 8-form part of the actions ($D\Phi$ is allowed). As a consequence, CP-violating theta term is suppressed.

It strongly suggests that the underlying manifold may be 8-dimensional. The 4 additional dimensions behave like mirrors (with some constant scaling factor) of 4 space-time dimensions we live in, so that we don't notice their existence. Fermion and gauge actions are real 8-forms, with 8-dimensional diffeomorphism invariance. It may deform gravity as seen in 4 dimensions, which is the subject of the next subsection.

4.4 Deformed Gravity

Let's consider a deformation of 4-dimensional Schwarzschild metric, with units $G = 1$ and $H = 1$. The spherically symmetric interval corresponding to Schwarzschild metric reads

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega, \tag{70}$$

where $f(r) = 1 - 2m/r$. Here the widely used signature of space-time is adopted.

The interval can be transformed into (2D conformal) format

$$ds^2 = \chi(\rho)^2(-dt^2 + d\rho^2) + (f(r)^{-1} - g(r))dr^2 + r^2 d\Omega, \tag{71}$$

where $g(r)$ is an arbitrary function, $(d\rho(r)/dr)^2 = g(r)/f(r)$, and $\chi(\rho)^2 = f(r(\rho))$. After a de Sitter-like deformation, the new interval reads

$$ds'^2 = -(d(\chi(\rho)t))^2 + (d(\chi(\rho)\rho))^2 + (d(\chi(\rho)w))^2 + (f(r)^{-1} - g(r))dr^2 + r^2 d\Omega, \tag{72}$$

where $-t^2 + \rho^2 + w^2 = L^2$, L is a constant. With the help of Kruskal transformations[4] (in the quadrant of $\rho < L$ and $w > 0$)

$$t = (L^2 - \rho^2)^{\frac{1}{2}} sinh(t'/L), \tag{73}$$

$$w = (L^2 - \rho^2)^{\frac{1}{2}} cosh(t'/L), \tag{74}$$

the deformed interval is calculated as

$$ds'^2 = -\chi(\rho)^2(1 - \rho^2/L^2)dt'^2 + ((d\chi(\rho)/d\rho)^2 L^2 + \chi(\rho)^2 L^2/(L^2 - \rho^2))d\rho^2 \tag{75}$$

$$+ (f(r)^{-1} - g(r))dr^2 + r^2 d\Omega. \tag{76}$$

With appropriate choices of L and $g(r)$ (for example, $L \sim r_0 \sim m^{\frac{1}{2}}$, $g(r) \sim r_0 df(r)/dr$, and a change of space-time signature for L), one can obtain deformed $f(r)_D - 1$ as

$$\chi(\rho(r))^2(1 - \rho(r)^2/L^2) - 1 \sim m^{\frac{1}{2}}(ln(r/r_0))^2, \tag{77}$$

for $r > r_0 \sim m^{\frac{1}{2}}$. The acceleration for a mass in the modified gravitational field is

$$|\vec{a}| \sim \frac{m^{\frac{1}{2}}}{r} ln(r/r_0). \tag{78}$$

Compared with modification of Newtonian dynamics (MOND[17], which is postulated to explain the non-Keplerian behavior of rotation curves of galaxies without the presence of dark matter), there is an additional logarithmic correction. It may help resolve the dark matter issue within and beyond the galaxy scale.

[4] A different choice of signature would entail trigonometric functions.

5 Conclusion

We follow the notion that all idempotent projections of an algebraic spinor should be realized as fermions of physical world. There are 16*3 Weyl fermions with $64 * 3$ real components for 3 generations. The orthogonal Clifford algebra, with 2^N degrees of freedom, would not work.

Amongst various extensions of Clifford algebras, with or without supersymmetry, a direct product of binary and ternary Clifford algebras $C\ell_{0,6} * C\ell_T$ seems to be physically relevant. By applying flavor and color projectors, one can identify idempotent projections of an algebraic spinor with 3 generations of leptons and quarks.

With electroweak Higgs fields replaced by their VEVs, the only flavor mixing terms are Majorana-type masses of right-handed neutrinos. When Higgs fields fluctuate around their VEVs, Dirac-type masses may not be diagonalized in terms of flavors. This causes further mixing of all fermion species.

We propose a general gauge action, which includes both gravity and Yang-Mills gauge fields. The most significant terms are Einstein-Cartan, cosmological constant, and Yang-Mills actions. There is no CP-violating theta term. Along with Yukawa coupling constants, the parameters involve three VEV magnitudes $(1/l, V, v)$ of vierbein and Higgs fields. They are essential in determining mass hierarchies.

Possible extra dimensions of the underlying space-time manifold may deform gravity as seen in 4 dimensions. We consider a specific deformation of Schwarzschild metric, which leads to modified gravity. Compared with MOND, there is an additional logarithmic correction. It may help resolve issues facing MOND beyond galaxy scale.

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