π's representation as an infinite sum

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Abstract: In this article, is developed a π representation as an infinite sum, through a definite integral.

Firstly, let's consider the function:

\[ C(x) = +\sqrt{R^2 - x^2} \]

This function has this graphical shape:

And, through the integral calculus:

\[ \int_a^b f(x) \, dx = \lim_{n \to \infty} \left[ \sum_{i=1}^{n} f\left(\frac{(i-1)(b-a)}{n} + a\right) \right] \frac{(b-a)}{n} \]

Thus, if we integrate the function C(x) on the interval [0,r] we have that:
\[ \int_{0}^{r} C(x) \, d(x) = \lim_{n \to \infty} \left[ \sum_{i=1}^{n} f\left( \frac{(i-1)r}{n} \right) \right] \frac{r}{n} \]

This way:

\[ \int_{0}^{r} C(x) \, d(x) = \lim_{n \to \infty} \left[ \sum_{i=1}^{n} \frac{\left( (i-1)r - \frac{(i-1)^2}{n^2} \right)^{r}}{n} \right] \frac{r}{n} \]

Distributing the exponent in the second member of the square root:

\[ \int_{0}^{r} C(x) \, d(x) = \lim_{n \to \infty} \left[ \sum_{i=1}^{n} \frac{\left( (i-1)^2 - \frac{2(i-1)}{n^2} \right)^{r}}{n} \right] \frac{r}{n} \]

Using the notable product, also in the second member of the root:

\[ \int_{0}^{r} C(x) \, d(x) = \lim_{n \to \infty} \left[ \sum_{i=1}^{n} \frac{\left( (i^2 - 2i + 1) - \frac{2i^2 - r^2}{n^2} \right)^{r}}{n} \right] \frac{r}{n} \]

Making the sum between the two terms of the root and multiplyng \( r^2 \) with \( (i^2 - 2i + 1) \):

\[ \int_{0}^{r} C(x) \, d(x) = \lim_{n \to \infty} \left[ \sum_{i=1}^{n} \frac{\left( (i^2 - 2i^2 + 2i^2 - r^2) - \frac{2i^2 - r^2}{n^2} \right)^{r}}{n} \right] \frac{r}{n} \]

Now, putting the term \( r^2 \) in evidence:

\[ \int_{0}^{r} C(x) \, d(x) = \lim_{n \to \infty} \left[ \sum_{i=1}^{n} \frac{\left( (r^2 - i^2 + 2i - 1) \right)^{r}}{n^2} \right] \frac{r}{n} \]

Extracting the term \( r^2/n^2 \) from the root:

\[ \int_{0}^{r} C(x) \, d(x) = \lim_{n \to \infty} \left[ \sum_{i=1}^{n} \frac{\left( \frac{i^2 - 2i + 1}{n^2} \right)^{r}}{n^2} \right] \frac{r}{n} \]
So, if \( r/n \) is a constant factor on the sum, can we extract from him:

\[
\int_0^r C(x) \, d(x) = \lim_{n \to \infty} \left[ \sum_{i=1}^n \sqrt{n^2 - i^2 + 2i - 1} \right] \frac{r^2}{n^2}
\]

But we know that the formula to the circle area (with \( r \) as a radius) is given by:

\[
A = \pi r^2
\]

Considering that the radius of the semi-circle \( C(x) \) is \( r \), and that the integrated area is only \( \frac{1}{2} \) of the total area from this semi-circle:

\[
A_{[0,R]} = \frac{\pi r^2}{4}
\]

This way, can we equal the "A" area with the definite integral:

\[
\int_0^R C(x) \, d(x) = \frac{\pi r^2}{4}
\]

So, can we conclude from the previous sentence:

\[
\frac{\pi r^2}{4} = \lim_{n \to \infty} \left[ \sum_{i=1}^n \sqrt{n^2 - i^2 + 2i - 1} \right] \frac{r^2}{n^2}
\]

Equaling \( r \) and 2, we’ve that:

\[
\pi = \lim_{n \to \infty} \left[ \sum_{i=1}^n \sqrt{n^2 - (i-1)^2} \right] \frac{4}{n^2}
\]

And then, can we find a “new” infinite sum for \( \pi \). This sum produces a slow convergence, because for \( n=10 \) the difference between the sum and \( \pi \) is approximately 0.1629.