Analytical Entropy and Prime Number Distribution.

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"Mathematicians have tried in vain to discover some order in the sequence of prime numbers but we have every reason to believe that there are some mysteries which the human mind will never penetrate."
 - L. Euler (1770).

Disorder in the distribution of prime numbers would indicate, as many mathematicians have hypothesized, the presence of entropy in the system of integers. [1] Yet entropy appears to be an empirical concept and not therefore a purely logico-mathematical one. Our aim shall thus be two-fold; to demonstrate that entropy is an *analytical* rather than an empirical concept and also to formalize our intuitions concerning entropy in the prime distribution. The simple implication of this for proofs of the Riemann Hypothesis and also the P versus NP problem may then become clearer.

What is required to achieve these two ends is an *analytical* equation allowing us to precisely measure the exact amount of entropy in the continuum of positive integers at any given point.

The key to deriving this equation lies in exploiting an unnoticed link that exists between Ludwig Boltzmann's well known equation for entropy and the historically contemporaneous Prime Number Theorem of Jacques Hadamard and Charles de la Vallee Poussin. This connection lies in the fact that both formalisms make use of the natural logarithm $\log_e(x)$.

In the case of Boltzmann's statistical interpretation of entropy the natural logarithm amounts to a measurement of entropy (S) – i.e. of disorder or "randomness".

$$\frac{s}{k} = \log_e(x) \tag{1}$$

That is; as $\log_e(x)$ increases disorder S also increases, with k being Boltzmann's constant. The value for Boltzmann's constant is approximately 1.38^{-23} Joules/Kelvin.

It is valid to express the value for entropy so as to extinguish any direct reference to Boltzmann's constant, effectively making our measurement of entropy dimensionless. This also has the substantial benefit of corresponding exactly to Shannon's Information Entropy and may be represented (for example) as

$$S = \ln x \tag{2}$$

[1] Granville R.; *Harald Cramer and the Distribution of Prime Numbers; Scandinavian Actuarial J.* 1 (1995), 12-28).
S.W. Golans; *Probability, Information Theory and Prime Number Theory.* Discrete Mathematics 106-7 (1992) 219-229.
C. Bonnano and M.S. Mega; *Toward a Dynamical Model for Prime Numbers.* Chaos, Solitons and Fractals 20. (2004) 107-118.
Entropy has been reinterpreted (by Boltzmann) as a precise measure of *disorder* in closed dynamical systems. It is this formal interpretation of entropy as a *geometrical* phenomenon which gives it an obvious relevance to mathematics. It indicates the *analytical* rather than the purely empirical relevance of the concept.
Ultimately if entropy is a property of the positive integers this would account for the apriori presence of chaos in the continua of mathematics, including phenomena such as irrational numbers.

This approach allows us to identify entropy squarely with the natural logarithm (lnx). In effect therefore it is valid to treat entropy as an *analytical* rather than a merely empirical concept. Consequently its appearance in mathematics (viz the distribution of the primes) ceases to be anomalous. The dimensionless treatment of the concept of entropy is of course already familiar from Shannon's work[2]

Boltzmann's use of the natural logarithm is obviously different to that of the Prime Number Theorem. Boltzmann intended his equation to be used to measure the disorderliness of dynamical systems. Accordingly the value for x in $\log_e(x)$ is intended to represent the number of potential microstates that a given dynamical system could possibly inhabit. Consequently, the larger the dynamical system is the larger will be the value for x and hence (ipso-facto) the larger will be its entropy.

In the Prime Number Theorem however the x in $\log_e(x)$ refers not to the microstates of a dynamical system but, more specifically, to particular positive integers;

$$\pi(x) \approx \frac{x}{\log_e x} \approx Li(x) \qquad (3)$$

My proposal (which will allow us to blend the above two formalisms) is that the continuum of real numbers be treated as a *geometrical* system and that the positive integers be then interpreted as representing possible microstates within that system, thereby allowing us to measure (in a dimensionless way) the relative magnitude of entropy represented by any given positive integer. For example; the number one will represent one microstate, the number seventy eight will represent seventy eight microstates and the number two thousand and sixty two will represent two thousand and sixty two microstates and so on.

This transposition is legitimate because Boltzmann's equation is famous for treating thermodynamical systems as if they are geometrical systems. In the case of the continuum of real numbers the shift (from dynamics to geometry) is unnecessary since the continuum of real numbers is, to all intents and purposes, already a geometrical system.

If this rationale is granted then it allows us to blend the above two formalisms (Boltzmann's equation for entropy and the Prime Number Theorem). This may occur in the following fashion;

$$\log_e x \approx \frac{x}{\pi(x)} \approx \frac{s}{k} \qquad (4)$$

[2] C.E. Shannon, "A Mathematical Theory of Communication", *Bell System Technical Journal*, vol. 27, pp. 379-423, 623-656, July, October, 1948

Which simplifies to give us;

$$S(x) \approx k \frac{x}{\pi(x)} \tag{5}$$

Expressing this to make it dimensionless so as to correspond with Shannon's information entropy we are left with the following significant result;

$$S(x) \approx \frac{x}{\pi(x)} \tag{6a}$$

In essence; *the* *entropy* *of any positive integer S(x) is equal to the integer itself (x) divided by Gauss'* $\pi(x)$ *(i.e. divided by the number of primes up to x).*

A very simple and elegant result which may alternatively be expressed as;

$$\pi(x) \approx \frac{x}{S(x)} \tag{6b}$$

Consequently, when we plug positive integers into the above equation (6a) as values for *x* what we find is a statistical tendency for entropy to increase as values for *x* get larger – which accords with what we would intuitively expect to find – although there are some interesting statistical anomalies, just as in the empirical manifestations of entropy.

This, finally, is the form of the equation which demonstrates the presence of entropy *across the entire continuum*. It also allows us to measure the *precise* quotient of entropy for every single positive integer – a strikingly original feature by any standards, from which necessarily follows the proof of the Riemann Hypothesis. [3]

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[3] Note also this highly suggestive definition of an integer x;

$$x \approx \frac{s}{k} . \pi(x)$$

which simplifies to;

Expressing the Riemann Hypothesis as a Conjecture about Entropy;

The Riemann hypothesis can itself be interpreted as a conjecture about entropy. This follows because of Helge von Koch's well known version of the Riemann hypothesis which shows the hypothesis to be simply a stronger version of the prime number theorem;

$$\pi(x) = Li_{(x)} + 0(x^{\frac{1}{2}} \log x) \tag{7}$$

Bearing in mind (6) and (7) it follows that;

$$S(x) = k.(\frac{x}{Li_{(x)} + 0(x^{\frac{1}{2}} \log x)}) = k.\frac{x}{\pi_{(x)}} \tag{8}$$

And this too may be expressed so as to make it dimensionless;

$$S(x) = \frac{x}{Li_{(x)} + 0(x^{\frac{1}{2}} \log x)} \tag{9}$$

$$x \approx s.\pi(x)$$

i.e. an integer (x) is defined by its entropy multiplied by $\pi(x)$ (i.e. multiplied by the number of primes up to (x).
This equation, incidentally, may point to the very essence of what number is, i.e. to a shifting balance of order and disorder differentiating each integer from every other.
Another intriguing result to note is that if the integer 1 is treated as a non-prime; $\pi(1) \approx 0$ then (according to equation (6)) this outputs an infinite result – which is clearly anomalous. However, if the integer 1 is treated as a prime; $\pi(1) \approx 1$ then the output according to (6) is *itself* 1 – which is a deeply rational result. These outcomes imply that the number 1 must itself be accounted a prime number, perhaps the cardinal or *base* prime number. But this in turn affects all other calculations using (6).

(7) may also be expressed as;

$$\pi(x) = \int_2^t \frac{dt}{\log t} + 0(\sqrt{x} \log x) \tag{10}$$

from which it follows that;

$$S(x) = k.\frac{x}{\int_2^t \frac{dt}{\log t} + 0(\sqrt{x}} \frac{}{\log x} \tag{11}$$

Which again can be expressed as;

$$S(x) = \frac{x}{\int_2^t \frac{dt}{\log t} + 0(\sqrt{x}} \frac{}{\log x} \tag{12}$$

Reasons Why the Presence of Entropy Proves the Riemann Hypothesis;

It has been noted by mathematicians that proving the Riemann Hypothesis is equivalent to proving that prime number distribution is disorderly (i.e. demonstrates Gaussian randomness):

> "So if Riemann was correct about the location of the zeros, then the error between Gauss's guess for the number of primes less than N and the true number of primes is at most of the order of the square root of N. This is the error margin expected by the theory of probability if the coin is fair, behaving randomly with no bias…
> … To prove that the primes are truly random, one has to prove that on the other side of Riemann's looking-glass the zeros are ordered along his critical line." [4]

And again, more explicitly;

> "*The Riemann hypothesis is equivalent to proving* that the error between Gauss's guess and the real number of primes up to N is never more than the square root of N – the error that one expects from *a random process*. [Italics mine]" [5]

This therefore is the reason why, as asserted earlier, the demonstration of entropy has the almost *incidental* effect of confirming the Riemann Hypothesis.

This is because the error mentioned (the square root of N) is the error that a disorderly distribution of the primes (in effect a "random process") *necessarily* gives rise to. Indeed, the *disorder* (entropy) of the primes is therefore the "random process" mentioned.

Gaussian randomness must logically be considered a *special case* of the type of randomness engendered by entropy, itself the original source of indeterminacy. The type of randomness generated by entropy is exactly of the kind defined by the square root of N, which is also of the type associated with coin tosses.
To reiterate the basic point therefore; Equation (6b) – which we already assume to be valid – would be invalidated if a non-trivial zero ever appeared that was *not* on the critical line. Ergo, all non-trivial complex zeros *must* logically fall on the critical line (in order for (6) to hold).

Requiring additional proof of equation (6) is, as I believe I have already shown, tantamount to requiring proof of the natural logarithm itself; something no mathematician considers necessary because of its intuitive force.

A final observation to make is that the equation (6) associates entropy with *every* positive integer and not merely with the primes alone. However, as I make clear in the prior section the entropy *does* nevertheless relate directly to the distribution of the primes. This is because the only terms in the equation are S(x), x and $\pi(x)$. The reason why there is entropy associated with *every* positive integer is

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[4] M. du Sautoy; *The Music of the Primes.* Harper Collins (2003). P167.

[5] This informed opinion was kindly communicated to me in a private communication from Professor du Sautoy.

because every positive integer is either *itself* a prime or else it is a *composite* of two or more prime factors.[6]
Thus, in short, I feel that both assumptions of the proof - i.e. equation (6) coupled with du Sautoy's obviously informed and explicit insight - are solidly grounded as is the line of reasoning connecting them.

How Entropy Solves the P versus NP Problem[7];

(6) also seems to cast its incidental light on the problem of factorization[8]. It clearly implies that the efficient factorization of very large numbers (of the order of magnitude used to construct R.S.A. encryption codes for example) cannot be achieved in deterministic polynomial time.
This is because if such an efficient factorization algorithm existed it would immediately contradict what we now know from (6) concerning the intrinsically disordered nature of prime number distribution. Consequently, if the factorization problem were in P (i.e. were subject to a "shortcut" algorithmic solution) it would contradict (6).
One can either have disorderly primes *or* one can have an efficient factorization algorithm, *one cannot logically have both.* Thus to prove the inequality we merely need to prove that the primes are inherently entropic or disorderly in their distribution – a situation which is strikingly similar to that concerning the Riemann Hypothesis as laid out above. This proof has already been supplied, viz (6).

If the factorization problem were in P it would mean that prime number distribution is orderly, which, because of (6), we now know is not the case. Therefore, the factorization problem, though in NP *cannot* logically be in P. Ergo;

$$P \neq NP$$

Thus it is not necessary to prove that the factorization problem is NP complete for it to serve as a valid counter-example to P and NP equivalence. Proving that the factorization problem is NP complete is only necessary as part of the process of proving P and NP equivalence, not in-equivalence.
Though not NP complete the factorization problem is nevertheless well known to be in NP since it is a problem whose solutions can at least be checked in P time.

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[6] Incidentally, we should mention at this point that the equally disorderly distribution of non-trivial zeros outputted by the Riemann Zeta-Function indicates the presence of entropy in the continuum of *imaginary* numbers as well. To be more precise; the non-trivial complex zeros outputted by the Riemann Zeta Function are (*for their imaginary part alone)* distributed in a disorderly fashion. Obviously the real parts of these zeros are distributed in an orderly way (if the Riemann hypothesis is true). This effectively retains the strict symmetry between the two continua. It therefore indicates the *presence of entropy in both continua*, presumably for the first time.
[7] For a fairly full introduction to this problem see web reference;
S. Cook; *The P Versus NP Problem; http://www.claymath.org/millennium/P_vs_NP/Official_Problem_Description.pdf*
[8] Given an integer *n* try to find the prime numbers which, when multiplied together, give *n*.

Proving the Twin Primes Conjecture Using Entropy;

This follows because if there were not an infinite number of twin primes then this state of affairs would in turn imply the existence of some sort of “hidden order" in prime number distribution preventing their accidental (in effect “random” or disorderly) recurrence. Since we now know (viz equation 6) that prime numbers *are* inherently disorderly in their distribution it therefore follows that this cannot be the case. Ergo there must be an infinite number of twin primes.

This reasoning should also prove transferable to a treatment of the Mersenne primes conjecture. Put in other terms; as the potential instances of 2-tuple events tends to infinity, so the probability of these events *never* occurring tends to zero (unless prime distribution is orderly).[9]

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