Quasi-periodic solutions of a spiral type

for photogravitational restricted three-body problem

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Abstract: Here is presented a new type of exact solutions for photogravitational restricted three-body problem (a case of spiral motion).

A key point is that we obtain the appropriate specific case of *spiral* motions from the Jacobian-type integral of motion for photogravitational restricted three-body problem (when orbit of small 3-rd body is assumed to be like *a spiral*).

Besides, we should especially note that there is a proper restriction to the type of spiral orbital motion of small 3-rd body, which could be possible for choosing as the exact solution of equations for photogravitational restricted three-body problem.

The main result, which should be outlined, is that in a case of *quasi-planar* orbital motion (of the small 3-rd body) the asymptotic expression for component *z* of motion is proved to be given by the proper *elliptical* integral.

Key Words: photogravitational restricted three-body problem, Jacobian-type integral of motion, spiral motion

1. Introduction.

In this contribution, we present a new type of exact solutions for photogravitational restricted three-body problem [1-3], the case of spiral motions.

According to the Bruns theorem [4], there is no other invariants except well-known 10 integrals for three-body problem (including integral of energy, momentum, etc.). But in the case of restricted three-body problem, there is no other invariants except only one, Jacobian-type integral of motion [5-6].

The main idea is to obtain from the Jacobian-type integral of motion the appropriate specific case of spiral motion for photogravitational restricted three-body problem (when orbit of small 3-rd body is assumed to be like a spiral); besides, such a case of spiral motion should be adopted by the structure of the Jacobian-type integral of motion.

In addition we should emphasize the appropriate astrophysical application of the constructed (exact) solutions of a spiral motion: for example, we could consider the Sun-Jupiter system as primaries and assume that only the larger primary (Sun) radiates. Besides, we could consider a small objects such as meteoroids or small asteroids (*about 10 cm to 10 km in diameter*) as the small 3-rd body for such a case.

2. Equations of motion.

Let us consider the system of ODE for photogravitational restricted three-body problem, at given initial conditions [2].

We consider three bodies of masses m_1, m_2 and m such that $m_1 > m_2$ and m is an infinitesimal mass. The two primaries m_1 and m_2 are sources of radiation; q_1 and q_2 are factors of the radiation effects of the two primaries respectively, $\{q_1, q_2\} \in (-\infty, 1]$.

We assume that m_2 is an *oblate* spheroid. The effect of *oblateness* [7-8] is denoted by the factor A_2.

Let r_i ($i = 1, 2$) be the distances between the centre of mass of the bodies m_1 and m_2 and the centre of mass of body m. The unit of mass is chosen so that the sum of the masses of finite bodies is equal to 1.

We suppose that $m_1 = 1 - \mu$ and $m_2 = \mu$, where μ is the ratio of the mass of the smaller primary to the total mass of the primaries and $0 \le \mu \le 0{,}5$. The unit of distance is taken as the distance between the primaries. The unit of time is chosen so that the gravitational constant is equal to 1.

The three dimensional restricted three-body problem, with an *oblate* primary m_2 and both primaries radiating, could be presented in barycentric rotating co-ordinate system by the equations of motion below [7-8]:

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \frac{\partial \Omega}{\partial x}, \\ \ddot{y} + 2n\dot{x} &= \frac{\partial \Omega}{\partial y}, \\ \ddot{z} &= \frac{\partial \Omega}{\partial z}, \end{aligned} \qquad (2.1)$$

$$\Omega = \frac{n^2}{2}\left(x^2+y^2\right) + \frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2}\left[1 + \frac{A_2}{2r_2^2}\cdot\left(1-\frac{3z^2}{r_2^2}\right)\right], \qquad (2.2)$$

- where

$$n^2 = 1 + \frac{3}{2}A_2,$$

- is the angular velocity of the rotating coordinate system and A_2 - is the *oblateness* coefficient. Here

$$A_2 = \frac{AE^2 - AP^2}{5R^2},$$

- where AE is the equatorial radius, AP is the polar radius and R is the distance between primaries. Besides, we should note that

$$r_1^{\,2} = (x+\mu)^2 + y^2 + z^2 \, ,$$

$$r_2^{\,2} = (x-1+\mu)^2 + y^2 + z^2 \, ,$$

- are the distances of infinitesimal mass from the primaries.

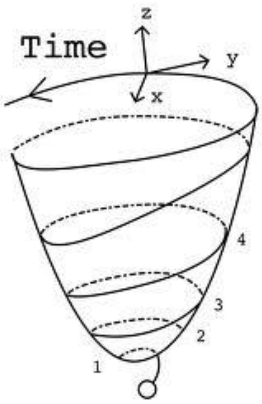
We neglect the relativistic Poynting-Robertson effect [9-10] which may be treated as a perturbation for cosmic dust or for small particles (less than 1 cm in diameter), we neglect the *Yarkovsky* effect of non-gravitational nature [11-13], as well as we neglect the effect of variable masses of 3-bodies [14-15].

The possible ways of simplifying of equations (2.1):

- if we assume effect of *oblateness* is zero, $A_2 = 0$ ($\Rightarrow n = 1$), it means m_2 is *non-oblate* spheroid (we will consider only such a case below);
- if we assume $q_1 = q_2 = 1$, it means a case of restricted three-body problem.

3. Exact solution (a case of spiral motion).

Regarding the orbit of small 3-rd body, let us assume such an orbit to be presented like *a spiral* (Pic.1).



Pic.1. Type of spiral motion.

Besides, let us remind that we could obtain from the equations of system (2.1) a Jacobian-type integral of motion [5-6]:

$$(\dot{x})^2 \ + \ (\dot{y})^2 \ + \ (\dot{z})^2 \ = \ 2\Omega(x, y, z) \ + \ C \qquad (3.1)$$

- where C is so-called Jacobian constant.

As per assumption above, it means that components of solution $\{x_i\} = \{x(t), y(t), z(t)\}$ ($i = 1, 2, 3$) should be presented as below:

$$x = \xi(t)\cdot\cos(w\cdot t), \quad y = \xi(t)\cdot\sin(w\cdot t), \quad z = z(t), \qquad (*)$$

- where the angular velocity is chosen $w = 1$; $\xi(t)$ - is a spiral factor. For example:

1) If $\xi(t) = a{\cdot}t + c$, $z(t) = b{\cdot}t$ - we should obtain the spiral of *screw line* type,
2) If $\xi(t) = a{\cdot}exp(b{\cdot}t)$, $z(t) = c{\cdot}t$ - we should obtain the *3-D logarithmic* spiral,

- here $\{a,b,c\}$ are supposed to be the arbitrary positive real constants.

Thus if we substitute the representation (*) for the components of solution $\{x_i\} = \{x(t), y(t), z(t)\}$ into the Equation (3.1), we should obtain the proper equation below

$$(\dot{\xi}(t)\cdot\cos t - \xi(t)\cdot\sin t)^2 + (\dot{\xi}(t)\cdot\sin t + \xi(t)\cdot\cos t)^2 + (\dot{z})^2 \ = \ 2\Omega(x,y,z) + C\ ,$$

$$\Rightarrow \qquad \dot{\xi}^2(t) + \xi^2(t) \ + \ (\dot{z})^2 \ = \ 2\Omega(x,y,z) + C \tag{3.2}$$

- where the expression for $\Omega\ (t)$ in (2.2) should be simplified in the case of *non-oblateness* $A_2 = 0$ ($n = 1$):

$$\Omega(t) \ = \ \frac{\xi^2(t)}{2} \ + \ \frac{q_1(1-\mu)}{r_1} \ + \ \frac{q_2\mu}{r_2}\ , \tag{3.3}$$

$$r_1^{\,2} \ = \ (\xi(t)\cdot\cos t + \mu)^2 + (\xi(t)\cdot\sin t)^2 + \ z(t)^2\ ,$$

$$r_2^{\,2} \ = \ (\xi(t)\cdot\cos t - 1 + \mu)^2 + (\xi(t)\cdot\sin t)^2 + \ z(t)^2\ .$$

So, taking into consideration the expression (3.3) for $\Omega\,(t)$, we obtain from (3.2)

$$\dot{\xi}^2(t) \ + \ (\dot{z})^2 \ = \ \frac{2q_1(1-\mu)}{r_1} \ + \frac{2q_2\mu}{r_2} + C \tag{3.4}$$

$$r_1^{\,2} \ = \ (\xi(t)\cdot\cos t + \mu)^2 + (\xi(t)\cdot\sin t)^2 + \ z(t)^2\ ,$$

$$r_2^{\,2} \ = \ (\xi(t)\cdot\cos t - 1 + \mu)^2 + (\xi(t)\cdot\sin t)^2 + \ z(t)^2\ .$$

Besides, we should note from (3.4) that the proper restriction below should be valid:

$$\frac{2q_1(1-\mu)}{r_1} + \frac{2q_2\mu}{r_2} + C \ \geq \ 0$$

- here $\{q_1, q_2\} \in (-\infty, 1]$. There are two possibilities to solve the equation (3.4):

- 1) first, we assume z (t) to be given as a proper function of parameter t, then we should obtain a solution of ODE of the 1-st kind for $\xi(t)$;
- 2) or the 2-nd, we assume $\xi(t)$ to be given as a proper function of parameter t, then we should obtain a solution of ODE of the 1-st kind for z (t).

For example, if we choose the 2-nd way of above, we should obtain from (3.4):

$$(\dot{z})^2 = \frac{2\,q_1\,(1-\mu)}{\sqrt{z(t)^2 + r_1^{\,2}(x,y)}} + \frac{2\,q_2\,\mu}{\sqrt{z(t)^2 + r_2^{\,2}(x,y)}} + f \qquad (3.5)$$

$$r_1^{\,2}(x,y) = (\xi(t)\cdot\cos t + \mu)^2 + (\xi(t)\cdot\sin t)^2,$$
$$r_2^{\,2}(x,y) = (\xi(t)\cdot\cos t - 1 + \mu)^2 + (\xi(t)\cdot\sin t)^2,$$
$$f = C - \dot{\xi}^2(t)\ .$$

We should note also that the question ‘Will the spiral (*) converge to a fixed point or diverge to infinity?’ should be researched additionally, depending on initial data of the proper case. So, stability of a spiral motion is an open problem in celestial mechanics.

<u>4. Conclusion</u>.

We have obtained a new type of exact solutions for photogravitational restricted three-body problem [1-3] (the case of spiral motion).

According to the Bruns theorem [4], there is no other invariants except well-known 10 integrals for three-body problem (*including integral of energy, momentum, etc.*). But in the case of *restricted* three-body problem, there is no other invariants except only one, Jacobian-type integral of motion [5-6].

A key point is that we obtain the appropriate specific case of *spiral* motion from the Jacobian-type integral for photogravitational restricted three-body problem (when orbit of small 3-rd body is assumed to be like *a spiral*). Besides, we should especially note that there is a proper restriction to the type of spiral orbital motion of small 3-rd body, which could be possible for choosing as the exact solution of equations for photogravitational restricted three-body problem.

Let us demonstrate the proper asymptotic simplifications of the considered solutions; Eq. (3.5) could be simplified if we consider *a quasi-planar case* of orbital motion:

$$(\dot{z})^2 = \frac{2q_1(1-\mu)}{r_1(x,y)\cdot\sqrt{1+\dfrac{z(t)^2}{r_1^2(x,y)}}} + \frac{2q_2\mu}{r_2(x,y)\cdot\sqrt{1+\dfrac{z(t)^2}{r_2^2(x,y)}}} + f \quad \left\{\frac{z(t)}{r_1}\to 0,\ \frac{z(t)}{r_2}\to 0\right\} \quad \Rightarrow$$

$$(\dot{z})^2 \cong \frac{2q_1(1-\mu)}{r_1(x,y)}\cdot\left(1-\frac{z(t)^2}{2r_1^2(x,y)}\right) + \frac{2q_2\mu}{r_2(x,y)}\cdot\left(1-\frac{z(t)^2}{2r_2^2(x,y)}\right) + C - \dot{\xi}^2(t)\ ,$$

$$\frac{dz}{\sqrt{-\left(\dfrac{q_1(1-\mu)}{r_1^3(x,y)}+\dfrac{q_2\mu}{r_2^3(x,y)}\right)\cdot z(t)^2+\left(\dfrac{2q_1(1-\mu)}{r_1(x,y)}+\dfrac{2q_2\mu}{r_2(x,y)}+C-\dot{\xi}^2(t)\right)}} = dt \qquad (4.1)$$

- where the left side of Equation (4.1) could be transformed to the proper *elliptical* integral [16] in regard to *z*. Indeed, if we assume (for example):

$$\left(\frac{2q_1(1-\mu)}{r_1(x,y)}+\frac{2q_2\mu}{r_2(x,y)}+C-\dot{\xi}^2(t)\right) = A^2\cdot\left(\frac{q_1(1-\mu)}{r_1^3(x,y)}+\frac{q_2\mu}{r_2^3(x,y)}\right), \quad A = const,$$

- it let us obtain the ordinary differential equation of the 1-st order for $\xi(t)$, but the left side of Equation (4.1) should be transformed to the proper *elliptical* integral of a simple kind [16] in regard to *z*, as below

$$\int \frac{dz}{\sqrt{A^2 - z^2}} = \arcsin\left(\frac{z}{A}\right) = \int \sqrt{\left(\frac{q_1(1-\mu)}{r_1^3(x,y)} + \frac{q_2\,\mu}{r_2^3(x,y)}\right)}\,dt \ .$$

Or in other case, taking into consideration the assumption $\{(z/r_1), (z/r_2)\} \rightarrow 0$, we could additionaly assume:

$$z \cdot \left(\frac{2\,q_1(1-\mu)}{r_1(x,y)} + \frac{2\,q_2\,\mu}{r_2(x,y)}\right) \sim \alpha \rightarrow 0\,, \quad \alpha = const\,,$$

- then (4.1) let us obtain as below

$$\frac{(\sqrt{z})\,dz}{\sqrt{\alpha + z\cdot\left(C - \dot{\xi}^2(t)\right)}} = dt\,,$$

- where the left side of expression above could be transformed to the proper *elliptical* integral [16] in regard to z only in case of a spiral of *screw line* type: $\xi(t) = a{\cdot}t + c$.

The case below should be excluded from the variety of possible solutions:

$$-\left(\frac{q_1(1-\mu)}{r_1^3(x,y)} + \frac{q_2\,\mu}{r_2^3(x,y)}\right)\cdot z(t)^2 + \left(\frac{2\,q_1(1-\mu)}{r_1(x,y)} + \frac{2\,q_2\,\mu}{r_2(x,y)} + C - \dot{\xi}^2(t)\right) = 0 \ .$$

Indeed, in such a case we could obtain from Eq. (3.5) that component z is under the linear dependence on the time-parameter t (but we assumed: $\{(z/r_1), (z/r_2)\} \rightarrow 0$).

Besides, the appropriate *restrictions* of meanings of variables should be valid for all meanings of parameter $t \geq 0$ in (4.1) as below:

$$-\left(\frac{q_1(1-\mu)}{r_1^3(x,y)} + \frac{q_2\,\mu}{r_2^3(x,y)}\right)\cdot z(t)^2 + \left(\frac{2\,q_1(1-\mu)}{r_1(x,y)} + \frac{2\,q_2\,\mu}{r_2(x,y)} + C - \dot{\xi}^2(t)\right) > 0 \ .$$

Under *a quasi-planar* assumption above: $\{(z/r_1), (z/r_2)\} \to 0$, it means that the proper restrictions at choosing of the spiral factor $\xi\,(t)$ should be given as below:

$$\dot{\xi}^2(t) \ < \ \frac{2q_1(1-\mu)}{\sqrt{\xi^2(t)+2\xi(t)\cdot\mu\cdot\cos t+\mu^2}} \ + \ \frac{2q_2\,\mu}{\sqrt{\xi^2(t)+2\xi(t)\cdot(\mu-1)\cdot\cos t+(\mu-1)^2}} \ + C \quad .$$

For example, if $\xi\,(t) \gg 1$ we should obtain for the asymptotical final motions $t \to \infty$ (constants are chosen as below):

$$\xi(t)\cdot\dot{\xi}^2(t) \ < \ \{2q_1(1-\mu)+2q_2\,\mu\} \ + \ \xi(t)\cdot C\,,$$

$$C=0 \quad \Rightarrow \quad \xi(t) \ < \ \left(\frac{9}{2}(q_1(1-\mu)+q_2\,\mu)\right)^{\frac{1}{3}}\cdot(t+t_0)^{\frac{2}{3}}\,,$$

$$\left(\frac{9}{2}(q_1(1-\mu)+q_2\,\mu)\right)=1,\ t_0=0, \quad \Rightarrow \quad \xi(t) \ < \ t^{\frac{2}{3}} \quad .$$

It means that we should choose a polinomial function with extent of time-parameter *t* less than < 2/3 as the spiral factor for the modelling of a spiral motion in such a case.

5. Discussions.

We obtain the appropriate specific case of *a spiral* motion for photogravitational restricted three-body problem from the Jacobian-type integral of motion (when orbit of small 3-rd body is assumed to be like *a spiral*).

The main result, which should be outlined, is that in a case of *quasi-planar* orbital motion (of the small 3-rd body) the asymptotic expression for component *z* of motion is proved to be given by the proper *elliptical* integral. But the elliptical integral is known to be a generalization of the class of inverse periodic functions.

Thus, by the proper obtaining of re-inverse dependence of a solution from time-parameter we could present the expression of $z(t)$ as a set of periodic cycles. So, the meaning of component $z(t)$ is proved to be limited in the proper range of values.

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